Interpreting the nominal soil production function as inferred from measurements of cosmogenic radionuclides*

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1 Initial remarks

Current descriptions of the rate at which weathered rock is converted to mechanically active soil on soil-mantled hillslopes — the soil production rate — suggest that this rate decreases with increasing soil thickness, although the specific form of this relationship is debated (Harrison et al., 2021). The idea of soil production with downslope transport can be traced to the description of hilltop convexities by Gilbert (1909), elaborated by Carson and Kirkby (1972), wherein it is proposed that the production rate varies as a non-monotonic function of soil thickness, increasing with thickness to a maximum rate at finite thickness then declining with further increasing thickness. Humphreys and Wilkinson (2007) provide historical perspective on this idea.

A standard procedure for inferring the rate of soil production builds on the pioneering work of Lal (1991) concerning the accumulation of cosmogenic radionuclide atoms within earthen materials. For conditions in which the soil thickness remains steady over a long period of time due to removal of soil by surface erosion or by creep, then in principle the rate of soil production can be estimated by measuring the concentration of cosmogenc radionuclide atoms within the top of the immobile saprolite just beneath the mechanically active soil (e.g. Heimsath et al., 1997). This result hinges on the idea that the steady advection of cosmogenic radionuclide atoms within the saprolite toward the soil-saprolite interface is balanced by their steady removal upon reaching this interface in concert with radioactive decay. The cosmogenic radionuclide concentration at the interface is then a steady value that is matched with the soil production rate.

Herein we examine the foundational elements of this procedure for inferring the nominal soil production function relating the rate of production to the soil thickness. We explain why one must be skeptical of the procedure — how it likely leads to spurious results under the transient conditions of varying soil thickness that mostly exist in the wild, yielding empirical curves whose forms are largely determined by the attenuation length of cosmogenic radionuclide production in the soil — regardless of the form of any underlying "true" function relating the soil production rate to soil thickness. In effect the procedure uses values of the independent variable, the soil thickness, to create the values of the dependent variable, the production rate — a statistics no-no. We then show why the soil production rate might be empirically determined only when variations in soil thickness are sufficiently slow that quasi-steady conditions are maintained, and we explain why the procedure is unlikely to reveal a non-monotonic relationship between the production rate and soil thickness, if it exists.

2 Typical data set

Numerous data sets have been published which show plots of estimated soil production rates versus soil thickness based on measurements of the concentrations of cosmogenic radionuclide atoms

*Notes prepared for Emma J. Harrison, Jane K. Willenbring and Gilles Y. Brocard in reaction to their paper *Global rates of soil production independent of soil depth* (2021, EarthArXiv, https://doi.org/10.31223/X5B30J). The basic ideas presented here are theirs. The material thus represents what theoreticians are apt to do: formalize things a bit.

in saprolite. Although the fitted empirical relationship between these quantities varies from one field location to another (Harrison et al., 2021), many appear as approximately straight lines in semi-log plots suggesting an exponential decrease in the soil production rate with increasing soil thickness. One example is that reported by Heimsath et al. (2005) based on measurements from soil pits on convex portions of a topographically undulating hillslope formed on granitic rock at Point Reyes, California, USA.

The measurements involve both ^{10}Be and ^{26}Al sampled from 13 soil pits. Measured soil thicknesses at the site vary from essentially zero to about 1.1 m (see Figure 1 in Heimsath et al., 2005). The data reflect a monotonic decline in the estimated soil production rates with increasing soil thickness; they do not reveal a non-monotonic form as suggested by Carson and Kirkby (1972), although relatively small estimated rates of production are associated with two tors at the field location. As described further below, the variable topography together with the nonuniform soil thickness at this field location — conditions that typically go with such data sets — almost certainly reflect transient conditions rather than the steady conditions examined by Lal (1991).

3 Background

3.1 Land-surface and soil configuration

Within a fixed Cartesian xyz coordinate system let $\zeta(\mathbf{x}, t)$ denote a land-surface elevation field and let $\eta(\mathbf{x}, t)$ denote the associated soil-saprolite interface as these vary with horizontal coordinate position $\mathbf{x} = (x, y)$ and time t. The soil-saprolite interface $\eta(\mathbf{x}, t)$ separates mechanically active soil from immobile saprolite. The local soil thickness $h(\mathbf{x}, t) = \zeta(\mathbf{x}, t) - \eta(\mathbf{x}, t)$.

Consider the derivatives $\partial\zeta(\mathbf{x}, t)/\partial t$ and $\eta(\mathbf{x}, t)/\partial t$. Each consists of two parts. For simplicity let $W(t)$ denote a uniform rate of uplift (or subsidence) and let $E(\mathbf{x}, t)$ denote a local rate of land-surface erosion (or deposition) due to a finite divergence of sediment transport, either at the land surface or involving soil creep. Then $\partial\zeta(\mathbf{x}, t)/\partial t = W(t) + E(\mathbf{x}, t)$. In turn, $\partial\eta(\mathbf{x}, t)/\partial t = W(t) - w(\mathbf{x}, t)$ where by convention $w(\mathbf{x}, t)$ represents the rate of soil production. Note that if $W(t) = 0$ then $\partial\eta(\mathrm{x}, t)/\partial t = -w(\mathbf{x}, t)$. That is, the local rate of lowering of the interface $\eta(\mathbf{x}, t)$ is equal to the negative of the soil production rate with conversion of rock material to soil. If $\partial\eta(\mathrm{x}, t)/\partial t = 0$ such that $\eta(\mathbf{x}, t) \to \eta(\mathbf{x})$ then $w(\mathbf{x}, t) = W(t)$. That is, with $W(t) > 0$ the rate of soil production is equal to the rate at which rock mass moves upward across the steady interface $\eta(\mathbf{x})$ during conversion to soil material. More generally, the most precise interpretation of the soil production rate w is this: it is the velocity of the weathered rock material measured relative to the interface $\eta(\mathbf{x}, t)$.

Consider steady conditions defined as follows. If the derivative $\partial\zeta(\mathbf{x}, t)/\partial t = 0$ for all positions $\mathbf{x}$ then the land-surface elevation field $\zeta(\mathbf{x})$ is steady and independent of time t. If the derivative $\partial\eta(\mathbf{x}, t)/\partial t = 0$ for all positions $\mathbf{x}$ then the soil-saprolite interface field $\eta(\mathbf{x})$ is steady and independent of time t. Either the land-surface elevation field or the soil-saprolite interface field may be unsteady while the other is steady. In this situation the soil thickness field is unsteady. But if $\partial\zeta(\mathbf{x}, t)/\partial t = \partial\eta(\mathbf{x}, t)/\partial t$, regardless of the magnitude of the uplift rate W, then the soil thickness field $h(\mathbf{x}) = \zeta(\mathbf{x}) - \eta(\mathbf{x})$ is steady. Conditions defining steady and unsteady concentrations of accumulated cosmogenic radionuclide atoms are covered in the next section.

If the rate of soil production $w(\mathbf{x}, t)$ functionally decreases with increasing soil thickness $h(\mathbf{x}, t)$, then the conventional view is that a steady soil thickness $h(\mathbf{x})$ implies uniform thickness, that is, $\partial h(\mathbf{x})/\partial\mathbf{x} = 0$. The essential reason is this: a decreasing soil production rate with increasing soil thickness provides a negative feedback that tends to everywhere maintain a uniform, steady soil thickness (Carson and Kirkby, 1972), an idea formalized by Furbish and Fagherazzi (2001) for the case of transport by soil creep. A nonuniform soil thickness then implies the presence of transient conditions. If the rate of soil production functionally increases with soil thickness, then this gives a positive feedback, which, depending on the rate of erosion, may lead to either unchecked thinning or thickening of the soil. If the rate of soil production is independent of soil thickness, then changes in the land-surface elevation and the soil-saprolite interface are decoupled via a mechanism directly involving the soil thickness, although they may be coupled indirectly. In this situation a nonuniform soil thickness implies neither steady nor unsteady conditions.

3.2 Accumulation of cosmogenic radionuclide atoms

3.2.1 General formulation

Let $n(\mathbf{x}, z, t)$ denote the number concentration of cosmogenic radionuclide atoms at the vertical position z within the soil or saprolite. Assuming atom production is primarily due to spallation, let l_s denote the e-folding attenuation length within the soil and let l_r denote the e-folding length within the underlying saprolite, each obtained from the absorption mean free path of cosmic rays and the density of the material. We now appeal to the work of Lal (1991) using the Eulerian formulation of the problem provided by Furbish et al. (2018a, 2018b). We make the standard simplifying assumption of uniform bulk density for both the soil and saprolite.

Let $P_\zeta(\mathbf{x})$ denote the rate of production of cosmogenic radionclide atoms at the soil surface $\zeta(\mathbf{x}, t)$, adjusted for topographic shielding. Then the rate of production of cosmogenic radionuclide atoms at the soil-saprolite interface $\eta(\mathbf{x}, t)$ is

$$P_\eta(\mathbf{x}, t) = P_\zeta(\mathbf{x}) e^{-h(\mathbf{x},t)/l_\mathrm{s}} \,, \tag{1}$$

which may vary with time due to variations in the soil thickness $h(\mathbf{x}, t)$. Within the underlying saprolite, and subject to uncertainty associated with the continuum approximation (Furbish et al., 2018b), the rate of change in the number concentration $n(\mathbf{x}, z, t)$ with respect to time is given by

$$\begin{aligned} \frac{\partial n(\mathbf{x}, z, t)}{\partial t} &= -W \frac{\partial n(\mathbf{x}, z, t)}{\partial z} \\ &+ P_\eta(\mathbf{x}, t) e^{-[\eta(\mathbf{x},t)-z]/l_\mathrm{r}} \\ &- \lambda n(\mathbf{x}, z, t) \,, \qquad z \leq \eta(\mathbf{x}, t) \,, \end{aligned} \tag{2}$$

where λ denotes the decay rate. We are interested in conditions at the interface $\eta(\mathbf{x}, t)$, as these conditions represent what is sampled in order to infer the rate of soil production.

Of particular interest are steady conditions in which the number concentration $n(\mathbf{x}, z, t) \to n(\mathbf{x}, z)$ varies with z but is independent of time t. For the xyz coordinate system associated with (2) this can occur only if $W > 0$ and $\partial\eta(\mathbf{x}, t)/\partial t = \partial\zeta(\mathbf{x}, t)/\partial t = 0$ in which case the production rate $w = W$. Then,

$$\frac{\mathrm{d}n(\mathbf{x}, z)}{\mathrm{d}z} = \frac{P_\eta(\mathbf{x})}{w} e^{-[\eta(\mathbf{x})-z]/l_\mathrm{r}} - \frac{\lambda}{w} n(\mathbf{x}, z) \,. \tag{3}$$

Integrating (3) and using the boundary condition that $n(\mathbf{x}, -\infty) = 0$ yields

$$n(\mathbf{x}, z) = \frac{P_\eta(\mathbf{x})}{\lambda + w/l_\mathrm{r}} e^{-[\eta(\mathbf{x})-z]/l_\mathrm{r}} \,. \tag{4}$$

Setting $z = \eta(\mathbf{x})$ then gives the number concentration at the soil-saprolite interface,

$$n[\mathbf{x}, \eta(\mathbf{x})] = \frac{P_\eta(\mathbf{x})}{\lambda + w/l_\mathrm{r}} \,, \tag{5}$$

which may differ from the number concentration in the soil just above the interface (Appendix A).

More generally we are interested in steady conditions at and beneath the soil-saprolite interface when $\partial\eta(\mathbf{x}, t)/\partial t = \partial\zeta(\mathbf{x}, t)/\partial t \neq 0$. For this situation we define a Galilean-like coordinate z' that moves with the interface $\eta(\mathbf{x}, t)$. In this moving coordinate system we rewrite (2) as

$$\begin{aligned} \frac{\partial n(\mathbf{x}, z', t)}{\partial t} &= -w' \frac{\partial n(\mathbf{x}, z', t)}{\partial z'} \\ &+ P_\eta(\mathbf{x}, t) e^{-[\eta'(\mathbf{x})-z']/l_\mathrm{r}} \\ &- \lambda n(\mathbf{x}, z', t) \,, \qquad z' \leq \eta'(\mathbf{x}) \,, \end{aligned} \tag{6}$$

where w' denotes the rate of soil production viewed with respect to the z' coordinate and the prime on η denotes this as the moving soil-saprolite interface viewed with respect to the fixed z coordinate. That is, $\eta'(\mathbf{x}) \to \eta(\mathbf{x}, t)$. With steady conditions,

$$\frac{\mathrm{d}n(\mathbf{x}, z')}{\mathrm{d}z'} = \frac{P_\eta(\mathbf{x})}{w'} e^{-[\eta'(\mathbf{x})-z']/l_\mathrm{r}} - \lambda n(\mathbf{x}, z') \,. \tag{7}$$

Integrating (7) and using the boundary condition that $n(\mathbf{x}, -\infty) = 0$ yields

$$n(\mathbf{x}, z') = \frac{P_\eta(\mathbf{x})}{\lambda + w'/l_\mathrm{r}} e^{-[\eta'(\mathbf{x})-z']/l_\mathrm{r}} . \qquad (8)$$

Setting $z' = \eta'(\mathbf{x})$ then gives the number concentration at the soil-saprolite interface,

$$n[\mathbf{x}, \eta'(\mathbf{x})] = \frac{P_\eta(\mathbf{x})}{\lambda + w'/l_\mathrm{r}} . \qquad (9)$$

Notice that (8) and (9) match (4) and (5). Viewed with respect to the fixed z coordinate, $w' = W - \partial\eta(\mathbf{x}, t)/\partial t$. When $\partial\eta(\mathbf{x}, t)/\partial t = 0$ then $z' = z$ and $w' = W = w$.

For simplicity we hereafter omit the functional notation showing the dependence of various quantities on the coordinate position $\mathbf{x} = (x, y)$. Nonetheless it is important to keep in mind that this dependence is implied.

3.2.2 Implications of the formulation of Lal (1991)

Before considering descriptions of the rate of soil production, there is value in revisiting the formulation of Lal (1991) and examining its implications. The starting point is (6), although Lal (1991) casts this in its Lagrangian form. The essential point is that the coordinate system is defined with respect to the surface η' (which may be moving in a global reference frame). We now let $\psi = \eta' - z'$. That is, the coordinate ψ is positive downward with $\psi = 0$ at the surface η'. Then (6) becomes

$$\frac{\partial n(\psi, t)}{\partial t} = -w' \frac{\partial n(\psi, t)}{\partial \psi} + P_\eta e^{-\psi/l_\mathrm{r}} - \lambda n(\psi, t) , \qquad \psi \geq 0 . \qquad (10)$$

As a reminder, P_η denotes the cosmogenic radionuclide production rate at the surface η. In the absence of a soil layer above this surface, the problem that Lal (1991) examines, then $P_\eta = P_\zeta$. Thus P_η now is independent of time t. (Note that this rate actually varies with Earth's magnetic field over a time scale of 100 000 years (Gosse and Phillips, 2001)).

For steady erosion of the earthen surface η at a rate $E = w' = -w$ the solution of (10) is

$$n(\psi, t) = n(\psi, 0)e^{-\lambda t} + \frac{P_\eta}{\lambda + w/l_\mathrm{r}} \left[1 - e^{-(\lambda + w/l_\mathrm{r})t}\right] e^{-\psi/l_\mathrm{r}} , \qquad (11)$$

with e-folding time $T_\mathrm{r} = 1/(\lambda + w/l_\mathrm{r})$. The initial condition $n(\psi, 0) = 0$ for $\psi \gg l_\mathrm{r}$, so with erosion of long duration the first term on the right side of (11) may be neglected. Then in the limit of $t \to \infty$ (or in practical terms, $t \gg T_\mathrm{r}$) we have

$$n(\psi) = \frac{P_\eta}{\lambda + w/l_\mathrm{r}} e^{-\psi/l_\mathrm{r}} . \qquad (12)$$

At the surface ($\psi \to 0$) this becomes

$$n(0) = \frac{P_\eta}{\lambda + w/l_\mathrm{r}} . \qquad (13)$$

If $\lambda \ll w/l_\mathrm{r}$ then this reduces to

$$n(0) = \frac{P_\eta l_\mathrm{r}}{w} , \qquad (14)$$

and $T_\mathrm{r} = l_\mathrm{r}/w$. As a point of reference, $\lambda \approx 5.0 \times 10^{-7}$ yr^{-1} for ^{10}Be and $\lambda \approx 9.6 \times 10^{-7}$ yr^{-1} for ^{26}Al. Assuming $l_\mathrm{r} \approx 0.5$ m (e.g. Lal, 1991), this means that decay can be neglected when $w \gg 2.5 \times 10^{-7}$ m yr^{-1} for ^{10}Be and when $w \gg 4.8 \times 10^{-7}$ m yr^{-1} for ^{26}Al. In turn we rewrite (13) and (14) as

$$w = \frac{P_\eta l_\mathrm{r}}{n(0)} - \lambda l_\mathrm{r} \quad \text{and} \qquad (15)$$

$$w = \frac{P_\eta l_\mathrm{r}}{n(0)} . \qquad (16)$$

These steady-state solutions provide the starting point of the procedure for inferring the steady erosion rate $E = -w$ of an earthen (e.g. rock) surface, or for inferring the steady soil production rate $w = W$ (or $w = -E$) with $\partial h/\partial t = 0$ from measurements of the concentration $n(0)$ within the top of the immobile saprolite just beneath the mechanically active soil.

Lal (1991) provides guidance on assessing the steady-state condition using measurements of more than one cosmogenic radionuclide. Here we take a different approach that becomes particularly useful when transient effects of a soil layer are added to the problem (Section 4.3), where the production rate $P_\eta \to P_\eta(t)$ then depends on the soil thickness and may vary with time according to (1). Specifically, here we consider variations in the erosion rate $w(t) = -E(t)$ specified as a sinusoidal function representing a harmonic of a Fourier spectrum of a more complex signal.

We define zeroth-order (basic) states associated with steady conditions and first-order fluctuations about these basic states, denoted by the subscripts 0 and 1. Namely,

$$w(t) = w_0 + w_1(t) \quad \text{and}$$
$$n(\psi, t) = n_0(\psi) + n_1(\psi, t)\,. \tag{17}$$

It is then possible to show (Appendix B) that the frequency response function relating first-order variations in the cosmogenic radionuclide concentration $n_1(0,t)$ to first-order variations in the erosion rate $E_1(t) = -w_1(t)$ is

$$F_{\hat{n}_1(0),\hat{w}_1}(\hat{\omega}) = \frac{\lambda T_\mathrm{r} - 1}{1 + i\hat{\omega}}\,. \tag{18}$$

Here the dimensionless quantities $\hat{n}_1(0,\hat{t}) = n_1(0,t)/n_0(0)$ and $\hat{w}_1(\hat{t}) = w_1(t)/w_0$ with dimensionless time $\hat{t} = t/T_\mathrm{r}$. The dimensionless frequency $\hat{\omega} = T_\mathrm{r}\omega = T_\mathrm{r}2\pi/T_\mathrm{t}$ with period T_t, and the imaginary number is defined as $i^2 = -1$. In turn the gain function (Figure 1) shows that (18) represents a low-pass filter. That is, high-frequency variations in the erosion rate $E_1(t) = -w_1(t)$ are attenuated in the response signal $n_1(0,t)$, and the lowest frequencies are "passed" without attenuation. This means that low-frequency variations yield responses with high fidelity such that quasi-steady conditions are maintained. High-frequency variations in the erosion rate do not produce similar variations in the number concentration $n_1(0,t)$. Here is the significance of this result.

High-frequency fluctuations in the erosion rate $E_1(t) = -w_1(t)$ are not recorded as variations in the concentration $n_1(0,t)$ at the earthen surface. That is, the concentration $n(0,t)$ remains relatively steady and equal to the basic-state value $n_0(0)$. Using (15) or (16), a measurement

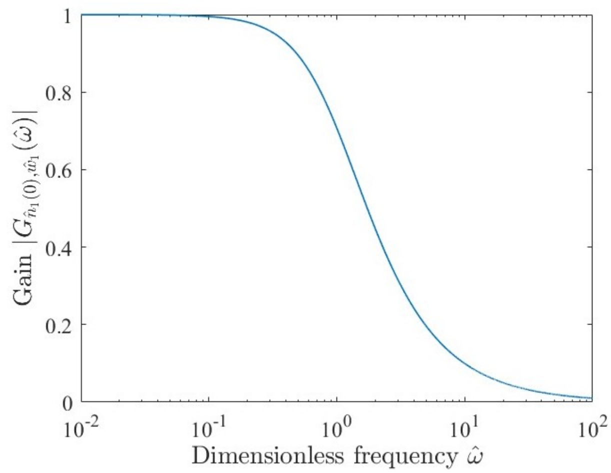

Figure 1: Gain function $|G_{\hat{n}_1(0),\hat{w}_1}(\hat{\omega})|$ versus dimensionless frequency $\hat{\omega}$ showing low-pass quality of the frequency response function $F_{\hat{n}_1(0),\hat{w}_1}(\hat{\omega})$.

of $n(0,t) \approx n_0(0)$ at any instant in the presence of high-frequency variations in the erosion rate yields the basic-state erosion rate w_0 rather than the actual rate. At the other extreme, low-frequency fluctuations in the erosion rate record with high fidelity the variations in the concentration $n_1(0,t)$ commensurate with the extant erosion rate. Quasi-steady conditions are maintained, so using (15) or (16), a measurement of $n(0,t)$ at any instant in the presence of low-frequency variations in the erosion rate yields the actual rate $w(t)$. Measurements of $n(0,t)$ at any instant in the presence of moderate-frequency fluctuations in the erosion rate yield underestimates of the magnitude of the actual rate using (15) or (16).

The e-folding time $T_\mathrm{r} = 1/(\lambda + w/l_\mathrm{r})$ has special significance in this problem. Lal (1991, p. 431) is quite clear in pointing out that the steady-state relationships described by (15) and (16) hinge on the presence of steady erosion for a period of time at least as long as $\sim 4T_\mathrm{r}$ with fixed production rate P_η. (The e-folding time T_r is denoted as T_eff in Lal (1991).) As a point of reference, and neglecting radioactive decay, with $w = 0.0001$ m yr^{-1} and $l_\mathrm{r} = 0.5$ m, then $T_\mathrm{r} = 5\,000$ yr and $4T_\mathrm{r} = 20\,000$ yr, or nearly twice the duration of the Holocene epoch. With $w = 0.00001$ m

yr^{-1} then $4T_\mathrm{r} = 200\,000$ yr, and with $w = 0.001$ m yr^{-1} then $4T_\mathrm{r} = 2\,000$ yr. This idea is directly embodied in the gain function depicted in Figure 1. Namely, for a varying erosion rate a quasi-steady condition ($|G_{\hat{n}_1(0),\hat{w}_1}(\hat{\omega})| \to 1$) is maintained only when the dimensionless frequency $\hat{\omega} \leq 0.1$. This means that with $w = 0.0001$ m yr^{-1} and $l_\mathrm{r} = 0.5$ m, the period $T_\mathrm{r} \geq 300\,000$ yr. With $w = 0.00001$ m yr^{-1} then $T_\mathrm{t} \geq 3\,000\,000$ yr, and with $w = 0.001$ m yr^{-1} then $T_\mathrm{t} \geq 30\,000$ yr.

To summarize, if the erosion rate is strictly steady, then the steady-state condition is satisfied only after a period $\sim 4T_\mathrm{r}$. If the erosion rate is unsteady, then the quasi-steady condition is satisfied only if the rate of change is at least as slow as that of a sinusoidal variation with dimensionless frequency $\hat{\omega} \leq 0.1$ and is maintained for the associated period T_t.

We now turn to the topic of soil production, where we add effects of a soil layer. A key element of this is that the cosmogenic radionuclide production rate $P_\eta \to P_\eta(t)$ now generally depends on time t due to variations in soil thickness $h(t)$. For reference below, note that the time scales described above in relation to achieving steady (or quasi-steady) conditions with respect to cosmogenic radionuclide concentrations have little to do with time scales associated with hillslope and soil dynamics, for example, the response time of a soil-mantled hillslope to variations in its boundary conditions, or the relaxation time of creeping soil following disturbance (e.g. Furbish and Fagherazzi, 2001). Further note that hereafter we neglect radioactive decay, so the e-folding time $T_\mathrm{r} = l_\mathrm{r}/w$.

4 Soil production function

4.1 Conceptual framework

The current conceptualization of a soil production function assumes that the rate of production $w(t)$ depends on soil thickness $h(t)$. Namely,

$$w(t) = f[h(t); \mathbf{F}(t)]\,, \tag{19}$$

where $\mathbf{F}$ denotes a vector of factors representing effects of the rock type and the climate and associated hydrologic and biotic conditions. Note that this function in principle is independent of the uplift rate $W(t)$. Current approaches for determining this function empirically from measurements of cosmogenic radionuclide concentrations require assuming time independent conditions so that (19) becomes

$$w = f(h; \mathbf{F})\,. \tag{20}$$

For a given set of factors $\mathbf{F}$ and time independent conditions, the production rate w must equal the steady uplift rate, $w = W$ (with $E = 0$), or it must equal the steady erosion rate, $w = -E$ (with $W = 0$), in which case a one-to-one correspondence between the soil thickness h and either W or E must exist. Current conceptualizations typically propose an empirical relationship of the form

$$w = P_0 e^{-h/l_w}\,, \tag{21}$$

where P_0 is interpreted as a maximum production rate in the limit of $h \to 0$ and l_w is an empirically determined e-folding length.

For a given setting we know (or assume) the surface production rate P_ζ, the e-folding lengths l_s and l_r, the soil thickness h and the cosmogenic radionuclide concentration $n(\eta)$ at the interface η. We also may have an independent measure of the uplift rate W. We must assume that the soil thickness has been steady ($\partial h/\partial t = 0$) for a sufficiently long period of time ($\sim 4T_\mathrm{r}$) that the concentration $n(\eta)$ has achieved a steady value. We now combine (1) and (16) with $n(0) \to n(\eta)$ to give

$$w = \frac{P_\zeta l_\mathrm{r}}{n(\eta)} e^{-h/l_\mathrm{s}}\,. \tag{22}$$

This says that the product $n(\eta)w$ is a unique function of h. If in the steady case described above where $w = W$, then this rate is satisfied by an infinite set of values of $n(\eta)$ and h. Or, for a specified thickness h, an infinite set of values of $n(\eta)$ and w satisfy (22).

If the production rate w varies as a single-valued function of the soil thickness h according to (21), then because (21) must be consistent with (22) the concentration $n(\eta)$ must be a fixed value where $P_0 = P_\zeta l_\mathrm{r}/n(\eta)$ and $l_w = l_\mathrm{s}$ (or, for example, $n(\eta) \sim e^{-h/l_n}$ with e-folding length l_n

such that $1/l_w = 1/l_s - 1/l_n$.) Moreover, because the production rate $w = W$ is unique to h according to (21), there is only one soil thickness h that is compatible with a specific uplift rate $w = W$, so the function (21) in this situation is in fact not independent of W. In turn, estimates of the soil production rate from a set of soil pits with different thicknesses h at the same location presumably involve the same uplift rate. But this is incompatible with the idea that a specific soil thickness is associated with a specific uplift rate. Alternatively, if the rate of soil production $w = W - \partial\eta(t)/\partial t$ in each pit is different, then with the same cosmogenic radionuclide concentration $n(\eta)$ (i.e. for the same soil production curve), the soil thickness h associated with each pit must be independently steady for a period of time at least as long as $\sim 4T_\mathrm{r}$ (or satisfy the quasi-steady condition; see below) as the surface is nonuniformly lowered despite a uniform uplift rate W.

Note that (22) is akin to an equation of state, for example, the ideal gas law,

$$\rho = \frac{p}{RT}\,, \tag{23}$$

where ρ denotes the density of the gas, p denotes its pressure, T denotes its temperature and R is the specific gas constant. For a specified gas at thermodynamic equilibrium an infinite set of values of the state variables ρ, p and T satisfy this expression. One cannot express the density ρ as a single-valued function of the pressure p without first specifying a fixed temperature T. A plot of density ρ versus pressure p in fact consists of an infinite set of ρ–p curves, each associated with a specific temperature T. The ideal gas law (23) merely represents a condition that must be satisfied as the $\rho p T$ state of a system changes. As written it says nothing about the dynamics of a system (or its phase trajectory) leading to a specific set of state values.

Similarly, (22) merely expresses a condition that must be satisfied under steady-state conditions. A plot of production rate w versus soil thickness h in fact consists of an infinite set of w–h curves, each associated with a specific concentration $n(\eta)$. As written, (22) says nothing about the dynamics of a system leading to a specific set of values of $n(\eta)$, w and h. This expression only describes the outcome of the systematics of accumulation of cosmogenic radionuclide atoms in an infinite half-space with finite motion of the material in the half-space. Despite its exponential form, it does *not* suggest any particular form of the function $f(h;\mathbf{F})$ in (20) — just as the ideal gas law (23) does not claim that the density ρ varies linearly with the pressure p for any given system, or a set of systems, unless isothermal conditions are maintained.

4.2 Steady-state conditions

Consider an idealized soil mantled landscape at steady state with respect to soil production. The uniform rate of uplift is matched by the rate of channel incision and the land-surface elevation $\zeta(\mathbf{x})$ is everywhere fixed in a global reference frame. Alternatively the elevation is everywhere lowered at a fixed rate equal to the soil production rate. Assuming the soil production rate varies as a single-valued function of soil thickness — exponential or otherwise — then such a landscape by definition possesses a uniform soil thickness. Further assuming that the analysis of Lal (1991) is correct, then the cosmogenic radionuclide concentration at the soil-saprolite interface is a fixed value everywhere, and reflects the uniform soil production rate. In this idealized situation a plot of the soil production rate versus soil thickness as measured in one or a great number of soil pits involves a single point. For this reason an empirical soil production function — exponential or otherwise — cannot be inferred from a single (idealized) steady-state landscape.

Continuing with this scenario, to obtain an empirical soil production function using the steady-state condition imposed by the analysis of Lal (1991) requires sampling different landscapes, or different parts of a landscape, each in a steady-state condition with a different soil thickness that has been steady for a period much greater than the e-folding time $T_\mathrm{r} = l_\mathrm{r}/w$. Moreover, to be empirically meaningful this would require the same geological and environmental conditions $\mathbf{F}$ across such locations. Imagining that this is possible, the outcome of this effort would be a plot of $w = W$ versus h — an empirical soil production function for the specified rock type and climate conditions — such that (22) is used only to determine the rate w assuming the steady-state condition is satisfied. Unfortunately, such idealized conditions are unlikely to be realized. It is difficult if not impossible to control for rock type and climate conditions for varying uplift rates while also satisfying the condition of uniform, steady soil thickness.

One can also envision an idealized situation involving steady, uniform lowering of the land surface (or a steady land surface with uniform soil production matching uplift) in the presence of a nonuniform soil thickness. The uniform soil production rate might arise from hydrochemical processes in concert with effects of soil thickness. But then this situation would be inconsistent with the idea of a single-valued relationship between the soil production rate and soil thickness, instead requiring a multivariable function $w = f(h, \ldots; \mathbf{F})$ that explicitly involves relevant quantities in addition to the soil thickness h.

4.3 Transient conditions

4.3.1 Qualitative assessment

We start with a simple example to illustrate the essence of the consequences of using (22) under transient conditions. We then systematically increase the complexity of the analysis.

Consider a steady condition in which the soil production rate $w = W$ with $\partial h/\partial t = 0$, where we denote the steady soil thickness as h_0. The associated fixed radionuclide concentration at the soil-saprolite interface η is $n(\eta; h_0)$. Now suppose that the production rate $w = w_0$ is in fact independent of soil thickness. That is, $w = f(h; \mathbf{F}) \to w = f(\mathbf{F}) = w_0$. (This may or may not have a physical basis; but this does not matter, as we need only to envision initially steady conditions.) With unknown uplift rate W, under these conditions measurements of h_0 and $n(\eta; h_0)$ yield the correct value $w_0 = W$. Namely,

$$w_0 = \frac{P_\zeta l_\mathrm{r}}{n(\eta; h_0)} e^{-h_0/l_\mathrm{s}} \,. \qquad (24)$$

But now suppose, following the scenario suggested by Harrison et al. (2021), that the soil thickness at some location suddenly decreases to a value h_1 due to erosion. This location inherits the number concentration $n(\eta; h_0)$. Letting an asterisk denote an estimate, if we take measurements soon (i.e. $\ll T_\mathrm{r}$) after the decrease in thickness then we obtain $w^* = w_1 > w_0$ as

$$w_1 = \frac{P_\zeta l_\mathrm{r}}{n(\eta; h_0)} e^{-h_1/l_\mathrm{s}} \,. \qquad (25)$$

Further suppose that at another location the soil thickness suddenly increases to a value h_2. This location also inherits the concentration $n(\eta; h_0)$. If we again take measurements soon after the increase in thickness then we estimate $w^* = w_2 < w_0$ as

$$w_2 = \frac{P_\zeta l_\mathrm{r}}{n(\eta; h_0)} e^{-h_2/l_\mathrm{s}} \,. \qquad (26)$$

The leading coefficient $C = P_\zeta l_\mathrm{r}/n(\eta; h_0)$ in (24), (25) and (26) is fixed. Upon taking logarithms we have

$$\ln w_0 = C_1 - \frac{1}{l_\mathrm{s}} h_0 \,,$$

$$\ln w_1 = C_1 - \frac{1}{l_\mathrm{s}} h_1 \qquad \text{and}$$

$$\ln w_2 = C_1 - \frac{1}{l_\mathrm{s}} h_2 \,, \qquad (27)$$

with $C_1 = \ln C$. These estimates w^* of the production rate associated with three measurements of soil thickness h have the same intercept C_1 and semi-log slope $1/l_\mathrm{s}$ so they fall on the same curve. More generally,

$$\ln w^* = C_1 - \frac{1}{l_\mathrm{s}} h \,. \qquad (28)$$

Thus, despite the fact that the soil production rate is *independent* of soil thickness (in fact, in this example it could have any functional relationship with thickness; see below), we nonetheless have created an empirical soil production function whose form is exponential with semi-log slope equal to the negative of the reciprocal of the attenuation length l_s — merely because the rate of production P_η of cosmogenic radionuclide atoms at the soil-saprolite interface η involves an exponential function for steady-state conditions that we have, perhaps unwittingly, misapplied to transient conditions.

Consider a second example in which the soil production rate $w = W = w_0$ with steady soil thickness h_0. The associated fixed radionuclide concentration at the soil-saprolite interface is $n(\eta; h_0)$. Now suppose that the actual production rate w varies according to an exponential function with semi-log slope of $-\alpha \neq -1/l_\mathrm{s}$ (Figure 2). We again imagine locations that experience a sudden decrease and increase in the soil thickness to h_1 and h_2. As in the preceding example these locations inherit the concentration $n(\eta; h_0)$ leading to the estimated production rates w_1^* and w_2^* on a curve with a semi-log slope of $-1/l_\mathrm{s}$. But the actual production rates become w_1 and w_2 if the new thicknesses h_1 and h_2 are sustained. This would represent an unsteady configuration with nonuniform soil production. Further note that this argument does not depend on the initial state. That is, the initial soil thickness h_0 and associated soil production rate need not fall on the actual soil production curve as shown in Figure 2. A different initial state would lead to essentially the same conclusions, albeit involving different response trajectories than those shown in the figure.

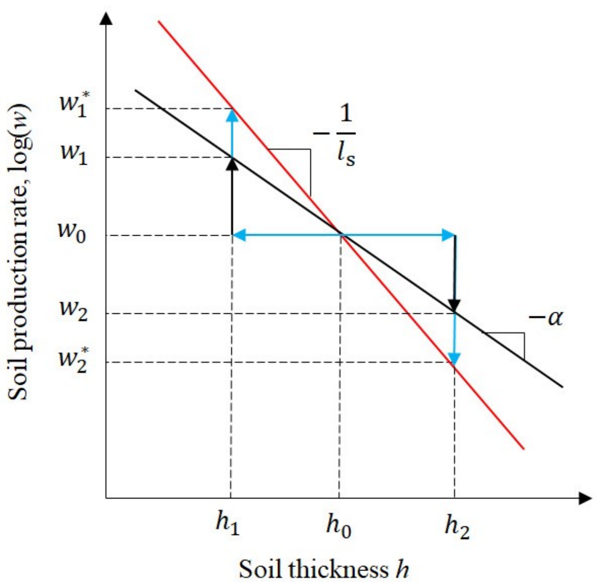

Figure 2: Schematic diagram of actual soil production curve (black line) with semi-log slope of $-\alpha$ and apparent production curve (red line) with semi-log slope of $-1/l_\mathrm{s}$ showing initial steady soil thickness h_0. Horizontal blue arrows represent sudden thinning and thickening to h_1 and h_2. Vertical blue arrows represent changes to apparent soil production rates w_1^* and w_2^* and black arrows represent changes to actual production rates w_1 and w_2 if new thicknesses h_1 and h_2 are sustained.

Of course conditions immediately following a sudden decrease or increase in soil thickness h are not likely to be sustained. A decreased thickness sees an increased exposure to cosmic rays and production of cosmogenic nuclide atoms at the soil-saprolite interface η, and the radionuclide concentration at this interface changes from its inherited concentration $n(\eta; h_0)$. Such a location may also experience relaxation to an increased soil thickness due to transport as well as continued soil production (e.g. Furbish and Fagherazzi, 2001). Similarly an increased thickness sees a decreased exposure to cosmic rays and production of cosmogenic nuclide atoms at the soil-saprolite interface, and the radionuclide concentration at this interface changes from its inherited concentration $n(\eta; h_0)$. Such a location may also experience relaxation to a decreased soil thickness due to transport in the presence of continued soil production. Effects of these changes still give incorrect estimates w^* of the fixed (time independent) production rate $w = w_0$.

Again assuming for illustration that the rate of soil production $w = w_0$ is independent of the soil thickness h, consider the response to a square-wave variation in soil thickness (Figure 3). Following a transient response to the onset of the wave train the number concentration $n(\eta, t)$ at the interface appears as a series of step responses centered about the expected value $n(\eta; h_0)$, each individual response asymptotically approaching a value compatible with the extant soil thickness. In turn the estimated production rate $w^*(t)$ appears as a mirror image of the response $n(\eta, t)$, each individual response asymptotically approaching a value compatible with the extant soil thickness. That is, during each period following a change in soil thickness, the estimated rate $w^*(t)$ varies with time independently of the

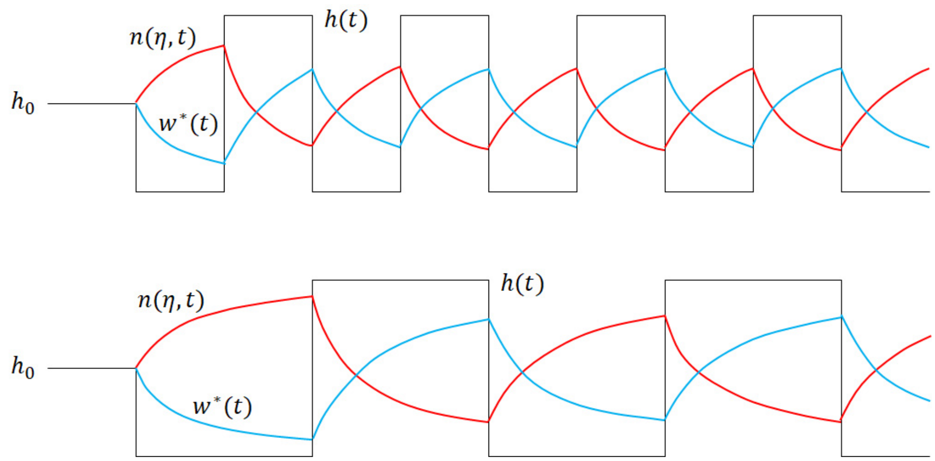

Figure 3: Idealized square-wave variations in soil thickness (black lines) and responses of number concentration $n(\eta, t)$ (red lines) and estimated production rate $w^*(t)$ (blue lines), assuming the rate of soil production $w = w_0$ is independent of thickness h. Vertical positions and magnitudes of signals $n(\eta, t)$ and $w^*(t)$ are only relative, chosen for visual clarity.

actual production rate $w = w_0$. As the period of the waves increases the responses of $n(\eta, t)$ and $w^*(t)$ more closely approach their final values relative to the responses associated with shorter periods. This indicates that with increasing period the phase shift between the thickness waveforms and the response waveforms decreases and the response waveforms experience less attenuation. This information formally emerges from the quantitative assessment provided in the next section, and reflects the competing time scales in this problem.

If instead of a square-wave the soil thickness varies as an Ornstein-Uhlenbeck (mean reverting) process in the sense described by Furbish and Fagherazzi (2001) for creeping soil, then whether the soil production rate is or is not coupled with the soil thickness, values of $n(\eta, t)$ and $w^*(t)$ to which these responses asymptotically approach are "moving targets." (One conceptualization of this signal is provided by Sweeney et al. (2020) who describe the response to fluctuations in the erosion rate about a mean rate as a mean-reverting process.) Nonetheless the outcome is qualitatively similar. Values of $n(\eta, t)$ and $w^*(t)$ fluctuate about expected (average) states with phase shifts that decrease with increasing periods of the variations in thickness.

Before turning to a more detailed assessment of the effects of transient conditions, a comment on the statistics of induced correlations is merited. This comment in effect views the preceding examples through a probabilistic lens. Without reference to the dynamics of a soil-mantled hillslope or landscape, suppose that at any instant the soil thickness h is a random variable. Further assume that the measured concentration $n(\eta; h)$ also is a random variable whose mean value coincides with the expected value associated with the mean soil thickness h_0 and production rate w_0 according to (24). The values of h and $n(\eta; h)$ are independent (as is the case for the data of Heimsath et al. (2005)). In this situation the production rate w defined by (22) is a new random variable that is strongly correlated with the thickness h. Indeed, this new random variable is created as the product of a noise given by the reciprocal of the random variable $n(\eta; h)$ and the exponential function of the thickness h. The outcome is an induced exponential correlation between the production rate w and the thickness h whose semilog slope is equal to $-1/l_{\mathrm{s}}$ (Figure 4). Moreover, in the limit of vanishing variance of the concentration $n(\eta; h)$ the exponential correlation becomes exact. Thus, variability in soil thickness h combined with uncorrelated variability in measurements of concentration $n(\eta; h)$ — which may be attributable to transient conditions or due to

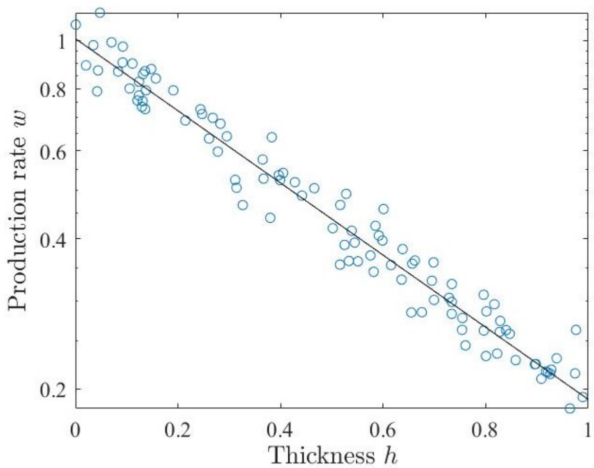

Figure 4: Plot of an induced exponential correlation between the production rate w (normalized) and the soil thickness h. In this example the thickness h is drawn from a uniform distribution and the concentration $n(\eta; h)$ is drawn from a Gaussian distribution, where $n(\eta; h)$ and h are uncorrelated. The line has semi-log slope of $-1/l_\mathrm{s}$.

unrelated effects — yield the appearance of an exponential production function when the thickness h is transformed as an exponential function to create the variable w. Normally in statistics the values of the dependent variable are not created from the values of the independent variable; these quantities are measured independently.

4.3.2 Quantitative assessment

The next step in this line of reasoning involves changing the land-surface elevation $\zeta(t)$ as a sinusoidal function representing a harmonic of a Fourier spectrum of a more complex signal. Here we consider the situation in which the soil production rate $w(t)$ may be coupled with the soil thickness $h(t)$. To do this we need to solve the unsteady problem involving changes in the land-surface elevation $\zeta(t)$, the soil-saprolite interface $\eta(t)$, the soil thickness $h(t)$, the production rate $P_\eta(t)$, the radionuclide concentration $n(\eta, t)$ and the estimated soil production rate $w^*(t)$. This includes solving for the changing concentration $n(z', t)$ beneath the soil-saprolite interface $\eta(t)$ in order to obtain the concentration $n(\eta', t)$ at this interface.

The sinusoidal form of the transient is not essential, but it is convenient. We choose this form because it nicely provides insight regarding the time scales involved in the process. In addition we approach this problem both analytically and numerically. Our analytical analysis (Appendix C) involves obtaining the frequency response functions relating variations in soil thickness to the rate of soil production, the rate of production of cosmogenic radionuclide atoms at the soil-saprolite interface, and the number concentration of cosmogenic radionuclide atoms at this interface, assuming a fixed uplift rate W. For this we linearize the governing equations as needed. The value of the analytical analysis is that it provides clarity regarding key dimensionless quantities that cannot be obtained from numerical analysis, it provides clarity on the nature of the response in relation to different transient time scales, and it serves as a check for consistency with the numerical analyses.

For comparison our numerical work involves solving the full nonlinear equations using an explicit finite-difference scheme. Beneath the soil-saprolite interface, cosmogenic radionuclide atoms are advected upward toward the interface. Solving the advection equation can be notoriously challenging when using finite-difference schemes. Nonetheless, because $n(z', t)$ varies smoothly and monotonically beneath the interface, and because information is lost at the interface as saprolite is converted to soil, a simple upwind scheme performs nicely. The algorithm is vetted with analytical solutions.

For simplicity we treat the problem using the differential equations presented in Section 3.2 and associate the outcome with a particular Eulerian position. The more realistic situation involves coupling these equations with appropriate partial differential equations describing soil transport in response to initial and boundary conditions on a hillslope (e.g. Furbish and Fagherazzi, 2001) or a landscape. Nonetheless we may envision the sinusoidal forcing of the land-surface elevation $\zeta(t)$ with period T_t as being specified externally to the system. Then T_t is the time scale over which a perturbation from the steady-state condition is envisioned to persist. At a relatively local scale, it might be the time scale of relaxation of a soil pit-mound couplet due to tree throw (Doane et al., 2021). More generally the time scale T_{t} is merely a way to specify the persistence of a perturbation regardless of its specific physical form. Thus, it could represent the relaxation time scale of an entire hillslope in response to changes in the lower hillslope boundary condition (e.g. Mudd and Furbish, 2007). Or it could represent the time scale of adjustment of a massif to rearrangements of the massif-scale channel network with adjacent hillslope responses. Or it could represent the time scale of variations in the overall erosion rate due to climate change and associated environmental factors, including biological conditions.

We assume the possibility of a soil production function with the form

$$w = P_0 e^{-h/l_w} \,, \tag{29}$$

where, as described above, P_0 is interpreted as a maximum production rate in the limit of $h \to 0$ and l_w is a specified e-folding length. We now write

$$h(t) = h_0 + h_1(t) \,, \tag{30}$$

where h_0 is the basic-state thickness under steady conditions and $h_1(t)$ is a fluctuation about the basic state. Then (29) may be linearized to

$$w = P_0 e^{-\alpha h_0}(1 - \alpha h_1) \,, \tag{31}$$

with $\alpha = 1/l_w$. This provides a very good approximation of the exponential function (29) near h_0 for small h_1, and more generally it represents a linear production function for arbitrarily large h_1. For reference below the leading coefficient in (31) is just a constant $P(h_0) = P_0 e^{-\alpha h_0}$. We may thus rewrite (31) as

$$w = P(h_0)(1 + \alpha h_1) \,, \tag{32}$$

with arbitrary coefficient α and without reference to an exponential form as in (29). Now (32) represents a linear production function that either decreases or increases with soil thickness about the basic state thickness h_0 depending on the sign of α. If $\alpha = 0$ then (32) simply becomes

$$w = P(h_0) \,, \tag{33}$$

which represents a fixed soil production rate that is independent of soil thickness.

We now define three additional zeroth-order (basic) states and first-order fluctuations denoted by the subscripts 0 and 1. Namely,

$$E(t) = E_0 + E_1(t) \,,$$

$$w(t) = w_0 + w_1(t) \quad \text{and}$$

$$n(\psi, t) = n_0(\psi) + n_1(\psi, t) \,, \tag{34}$$

with $\psi = \eta' - z'$. It is then possible to show (Appendix C) that the frequency response function relating first-order variations in the cosmogenic radionuclide concentration $n_1(0, t)$ at the soil-saprolite interface to first-order variations in the erosion rate $E_1(t)$ is given by

$$F_{\hat{n}_1(0),\hat{E}_1}(\hat{\omega}) = \frac{(1 - \alpha l_{\mathrm{s}})(l_{\mathrm{r}}/l_{\mathrm{s}})}{\alpha l_{\mathrm{s}} + \hat{\omega}^2 - i(1 - \alpha l_{\mathrm{s}})\hat{\omega}} \,, \tag{35}$$

where with $w_0 = W$ the dimensionless values $\hat{n}_1(0, \hat{t}) = n_1(0, t)/n_0(0)$ and $\hat{E}_1(\hat{t}) = E_1(t)/w_0$ with dimensionless time $\hat{t} = (w_0/l_{\mathrm{r}})t$, and the dimensionless frequency $\hat{\omega} = (l_{\mathrm{r}}/w_0)\omega = (l_{\mathrm{r}}/w_0)2\pi/T_{\mathrm{t}}$ with period T_{t}. In turn the gain function (Figure 5) shows that (35) represents a

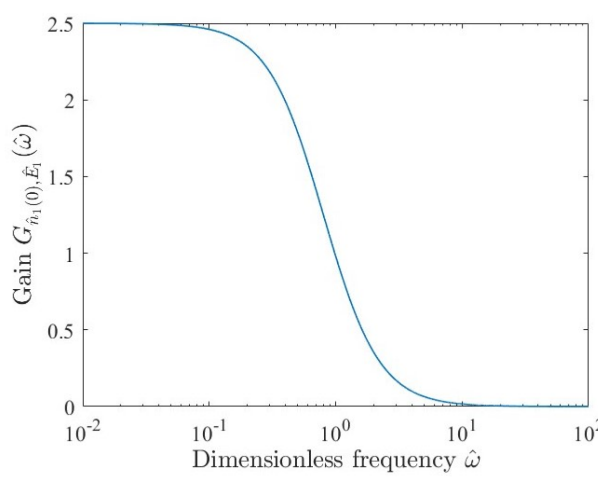

Figure 5: Example gain function $G_{\hat{n}_1(0),\hat{E}_1}(\hat{\omega})$ versus dimensionless frequency $\hat{\omega}$ showing low-pass quality of the frequency response function $F_{\hat{n}_1(0),\hat{E}_1}(\hat{\omega})$.

low-pass filter. Here is the significance of this result.

High frequency fluctuations in the erosion rate $E_1(t)$ and associated fluctuations in the soil thickness $h_1(t)$ are not recorded as variations in the concentration $n_1(0,t)$ at the soil-saprolite interface. That is, this concentration remains relatively steady. As a consequence the fluctuations in soil thickness give fluctuations in the estimates of the soil production rate $w_1^*(t)$ that fall on a curve with a semi-log slope of $-1/l_\mathrm{s}$. This situation is most like the qualitative analysis above where sudden changes in soil thickness inherit the preexisting (fixed) concentration $n(\eta; h_0)$ yielding an empirical curve (28) that merely reflects a $-1/l_\mathrm{s}$ relationship.

At the other extreme, low-frequency fluctuations in the erosion rate and associated fluctuations in the soil thickness record with high fidelity the variations in the concentration $n_1(0,t)$ commensurate with the extant soil thickness. The concentration $n_1(0,t)$ and the thickness $h_1(t)$ track together, so estimates of the production rate $w_1^*(t)$ are close to those defined by the specified soil production relationship. In the example depicted in Figure 5 the frequency $\hat{\omega}$ must be less than about $\hat{\omega} \sim 0.1$ in order to approach this high-fidelity condition. With $w_0 = 0.001, 0.0001, 0.00001$ m yr^{-1} this frequency coincides respectively with periods $T_\mathrm{t} > 25\,000, 250\,000, 2\,500\,000$ yr.

We can use the gain function and associated phase function (Appendix C) to map w^*–h trajectories in the w–h state space. The analytical solutions reasonably mimic the full numerical solutions. Nonetheless, for completeness we use the numerical solutions next.

We start with the situation in which the soil production rate w is specified as an exponential function (29) with $P_0 = 0.0001$ m yr^{-1}, $l_w = 0.6$ m, and $h_0 = 0.5$ m so that $w_0 = 0.00004$ m yr^{-1}. For illustration we use the attenuation lengths $l_\mathrm{s} = l_\mathrm{r} = 0.4$ m, and we choose $T_\mathrm{t} = 10\,000$ yr. Following an initial transient the estimated rate $w^*(t)$ follows a trajectory that loops about the line with a semi-log slope of $-1/l_\mathrm{s}$, not $-1/l_w$ (Figure 6). The loop reflects the effect of the phase shift associated with this period. When $T_\mathrm{t} = 100\,000$ yr the trajectory bends toward the specified exponential relation with semi-log slope of $-1/l_w$. When $T_\mathrm{t} = 1\,000\,000$ yr the trajectory begins to converge to this specified exponential relationship.

Similar results are obtained for the situations in which the soil production rate w is specified as a linear function (32) (Figure 7) and as a constant w_0 independent of soil thickness h (Figure 8). Note that we can change the values of the various parameters listed above, but in all cases the essence of the outcome is the same.

We may imagine a great number of hillslope or landscape locations that intermittently experience disturbances from a nominal steady-state condition, representing a range of scales and transient periods. This gives a family of trajectories in w–h the phase plot. Suppose that we then measure the cosmogenic radionuclide concentration at a great number of such locations at one instant. Such a plot is unlikely to reflect the true soil production function that would be obtained under steady state conditions as described above. Rather, it reflects one instant in the history of disturbances (thinning and thickening of soils with recovery) on the particular landscape.

4.3.3 Competing time scales

Recall that the time scale T_t characterizes the persistence of transient variations in erosion that give thinning or thickening of the soil. This time scale is associated with the horizontal component of the trajectory of a system in a w–h phase plot. In turn $T_\mathrm{r} = l_\mathrm{r}/w$ is an advective time scale that determines the interface number concentration $n(\eta)$ (Appendix C). This time scale characterizes the rate at which the radionuclide concentration at the soil-saprolite interface approaches a steady condition in concert with radioactive decay and it therefore controls the vertical component of the trajectory of a system in a w–h phase plot.

We now define the dimensionless Lal number[1] as

$$La = \frac{T_\mathrm{r}}{T_\mathrm{t}} = \frac{1}{T_\mathrm{t}(\lambda + w/l_\mathrm{r})} \approx \frac{l_\mathrm{r}}{T_\mathrm{t} w}\,, \tag{36}$$

[1] This number is named in honor of Devendra Lal (14 February 1929 – 01 December 2012) for his pioneering work on the accumulation of cosmogenic radionuclide atoms in earthen materials and the implications thereof.

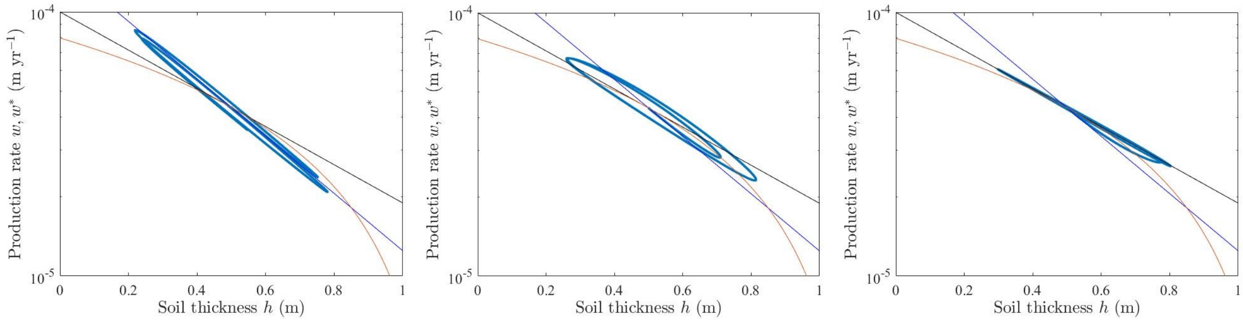

Figure 6: Example phase plot of production rate w, w^* versus soil thickness h assuming true soil production rate is exponential function with $P_0 = 0.0001$ m yr^{-1} and $l_w = 0.6$ m (black line). Also shown is linear function (red line), exponential function with $l_\mathrm{s} = 0.4$ m (blue line), and simulated estimates of production rate (blue dots) for periods $T_\mathrm{t} = 10\,000, 100\,000, 1\,000\,000$ yr.

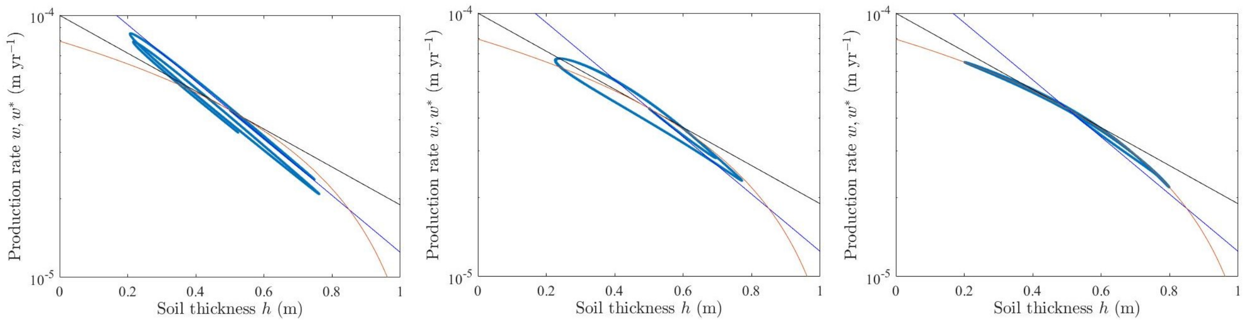

Figure 7: Example phase plot of production rate w, w^* versus soil thickness h assuming true soil production rate is linear function with $\alpha = -1/l_w$ and $l_w = 0.6$ m (red line). Simulated estimates of production rate (blue dots) are for periods $T_\mathrm{t} = 10\,000, 100\,000, 1\,000\,000$ yr.

which differs from the dimensionless frequency $\hat{\omega}$ by the factor 2π. A small value of La implies that the approach to a steady number concentration $n(\eta)$ is rapid relative to the rate of change in the soil thickness. A large value of La implies that this approach to a steady concentration does not keep pace with the rate of change in the soil thickness. This means that the steady-state approximation with respect to cosmogenic radionuclide concentrations is recovered in the limit of $La \to 0$.

In each of the examples depicted in Figures 6, 7 and 8, the Lal number is $La = 0.9, 0.09, 0.009$ for the periods $T_\mathrm{t} = 10\,000, 100\,000, 1\,000\,000$ yr with $w_0 = 0.0001$ m yr^{-1} and $l_\mathrm{r} = 0.4$ m. If the zeroth-order soil production rate w_0 is increased by an order of magnitude then the Lal number decreases by an order of magnitude for these periods. This implies that with increasing uplift rates, assuming steady zeroth-order conditions, the w^*–h phase trajectories begin to converge to the specified soil production curve over shorter transient periods T_t, a point supported by numerical simulations.

4.4 Absence of non-monotonic empirical functions

The procedure outlined above for estimating the soil production rate w likely precludes the possibility of empirically discovering a non-monotonic form of the soil production function under transient conditions, if such a form exists. To illustrate this point we start with the first example in Section 4.3.1.

Consider a steady condition in which the production rate $w = W$ with $\partial h/\partial t = 0$, where we denote the steady soil thickness as h_0. The as-

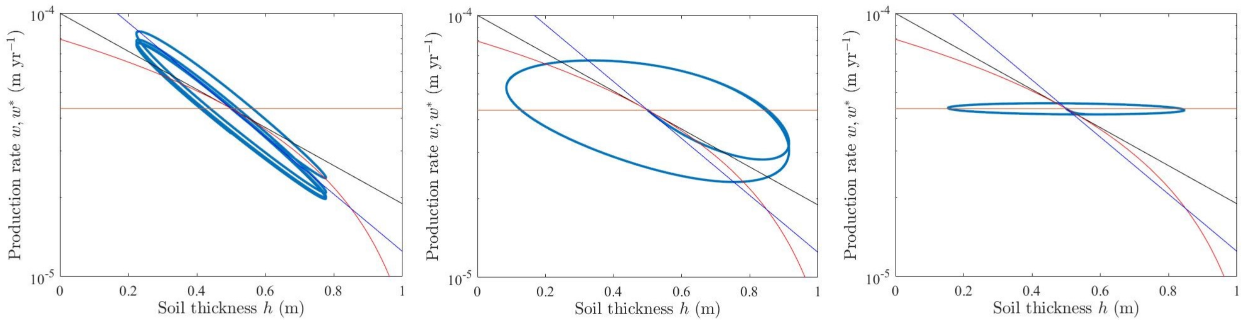

Figure 8: Example phase plot of production rate w, w^* versus soil thickness h assuming true soil production rate is a constant (horizontal line). Simulated estimates of production rate (blue dots) are for periods $T_\mathrm{t} = 10\,000, 100\,000, 1\,000\,000$ yr.

sociated fixed radionuclide concentration at the soil-saprolite interface η is $n(\eta; h_0)$. Now suppose that the production rate w is given by (32) with positive sign, that is, the production rate increases with increasing soil thickness h. This represents conditions to the left of the maximum production rate in a non-monotonic production function (e.g. Carson and Kirkby, 1972), where now the soil thickness h_0 coincides with a metastable state. With unknown uplift rate W, under these conditions measurements of h_0 and $n(\eta; h_0)$ yield the correct value $w_0 = W$ given by (24). But now suppose that the soil thickness at some location suddenly decreases to a value h_1 due to erosion. This location inherits the number concentration $n(\eta; h_0)$. And, suppose that at another location the soil thickness suddenly increases to a value h_2. This location also inherits the concentration $n(\eta; h_0)$. If we take measurements soon after the decrease or increase in thickness, then following the same reasoning presented in Section 4.3.1, we end up producing an empirical function having the form of (28). Thus, despite the fact that the true soil production rate *increases* with increasing soil thickness, we nonetheless again have created an empirical soil production function whose form is exponential by misapplying a steady-state formulation to transient conditions.

Of course conditions immediately following a sudden decrease or increase in soil thickness h are not likely to be sustained. As described in Section 4.3.1, changes in thickness are accompanied by changes in exposure to cosmic rays and production of cosmogenic radionuclide atoms at the soil-saprolite interface. But there also is a response to the metastable configuration represented by h_0. A decrease in soil thickness, unless matched by a decrease in the erosion rate combined with continued soil production, leads to unchecked thinning of the soil. An increase in soil thickness combined with continued soil production, unless matched by an increase in the erosion rate, leads to unchecked thickening of the soil (e.g. Furbish and Fagherazzi, 2001). Measurements of the concentration of cosmogenic radionuclides at the soil-saprolite interface during such periods of response could give estimates of the soil production rate that are confusing in relation to a conceptualized single-valued function of soil thickness.

Using the same conditions of the example presented in Figure 6 but with a soil production rate w that increases linearly with increasing soil thickness h, simulations reveal examples of unchecked thickening and thinning (Figure 9). In this figure unchecked thickening follows from an initial thickening of the soil from its metastable configuration, and unchecked thinning follows from an initial thinning of the soil from its metastable configuration. Notice that these phase trajectories have little to do with the underlying linear soil production function, and instead the oscillations locally mimic a relationship with negative semi-log slope. Also note that, unlike the examples in the preceding section, long

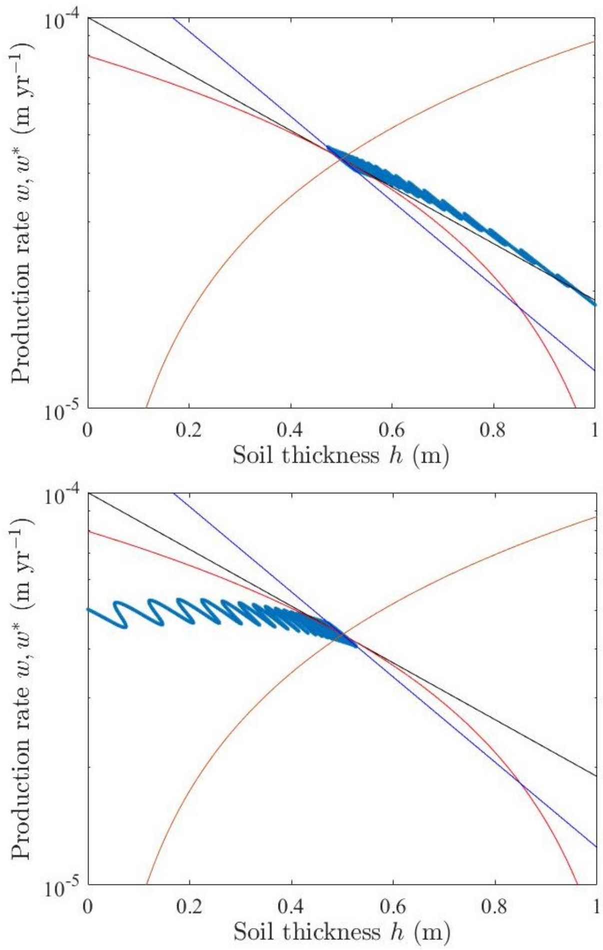

Figure 9: Example phase plots of production rate w, w^* versus soil thickness h assuming the true soil production rate (orange line) increases linearly with soil thickness, showing unchecked thickening (top) and unchecked thinning (bottom) from initial metastable configuration.

transient periods T_{t} do not involve convergence to the specified soil production curve. As the soil thickens or thins a metastable configuration is maintained; the initial condition does not matter.

4.5 Extant rate of soil production

A question lingers. Aside from its use for inferring the form of a relationship between the rate of soil production w and soil thickness h, does the procedure provide an accurate measure of the extant rate of soil production under moderate-frequency to high-frequency transient conditions?

The veracity of the steady-state relationship, (15) or (16), hinges on specifying the correct cosmogenic radionuclide production rate at the soil-saprolite interface, P_η. This rate only can be inferred from (1), which depends on the soil thickness h. Thus the historical state of the soil thickness must be independently known such that P_η can be correctly related to the cosmogenic radionuclide concentration $n(0)$ — just as the rate of production of cosmogenic radionuclide atoms at the soil surface, P_ζ, must be independently known. Unless the historical state of the soil thickness is known to satisfy the steady-state or quasi-steady condition, then the value of P_η used in (15) or (16) based on the extant soil thickness is just a guess.

5 Summary of key results

Here we summarize key results from our analysis pertaining to the procedure for inferring the rate of soil production. For clarity we reproduce several equations.

The procedure starts with the equation,

$$n(\eta) = \frac{P_\eta}{\lambda + w/l_{\mathrm{r}}} \approx \frac{P_\eta l_{\mathrm{r}}}{w}\,. \tag{37}$$

This equation is satisfied only if quasi-steady conditions have existed for a period $T_{\mathrm{t}} \gg T_{\mathrm{r}}$ (i.e. $La \ll 1$) such that the number concentration $n(\eta)$ at the interface η is a fixed value for a steady atom production rate P_η and a steady soil production rate w. Then, neglecting decay and rearranging (37),

$$w = \frac{P_\eta l_{\mathrm{r}}}{n(\eta)}\,. \tag{38}$$

Like (37), (38) states a condition that must be satisfied if and only if the quasi-steady condition is satisfied. It says nothing about whether this condition is actually satisfied. That is, if one uses (38) to estimate the soil production rate w, one cannot know if this estimate of w is correct unless it is independently verified that the quasi-steady condition is satisfied. Thus, if for a specified value l_{r} one has a set of paired values of P_η

and $n(\eta)$ and uses (38) to create a set of values w, then unless one independently verifies that the quasi-steady condition is satisfied for each pair, the numbers w are just guesses.

To use (38) requires measuring $n(\eta)$ and estimating the value P_η. To estimate P_η requires specifying the soil thickness h and using

$$P_\eta = P_\zeta e^{-h/l_\mathrm{s}} \, . \tag{39}$$

Combining (38) and (39) then gives

$$w = \frac{P_\zeta l_\mathrm{r}}{n(\eta)} e^{-h/l_\mathrm{s}} \, . \tag{40}$$

Like (37) and (38), (40) states a condition that must be satisfied if and only if the quasi-steady condition is satisfied. Thus, if for a specified value l_r one has a set of paired values of h and $n(\eta)$ and uses (40) to create a set of values w, then unless one independently verifies that the quasi-steady condition is satisfied for each pair, the numbers w are just guesses. Moreover, for a fixed value of $n(\eta)$ or for a set of values of $n(\eta)$ that reflect transient variations in saprolite and soil properties, including the thickness h, then (40) yields a set of values of w which by definition are correlated exponentially with the thicknesses h — an induced correlation — as values of w are created from values of h rather than being determined independently. And, in practice, whether one does or does not algebraically combine (38) and (39) to give (40), the outcome is the same.

Momentarily setting aside effects of transient conditions and other sources of uncertainty, it is statistically indefensible to create values of the dependent variable using values of the independent variable, then turn around and use curve fitting (e.g. regression) to "discover" the form of the function used to create the dependent values. Moreover, it is immaterial whether quantities estimated from such an analysis do or do not closely match expected values (e.g. $1/l_w \sim 1/l_\mathrm{s}$), as the exercise, at the outset, is not statistically meaningful.

To form an empirical relationship between the soil production rate w and the soil thickness h requires knowing what the production rate is independently of the soil thickness, that is, without reference to the equations above. But how does one independently determine the soil production rate? Perhaps one could do this by determining the uplift rate W independently then assume based on separate evidence that quasi-steady conditions have existed for a period of time where $La \ll 1$ such that the soil production rate w is equal to the uplift rate W. Then the use of (40) mostly becomes a check on what one is already assuming to be true, that quasi-steady conditions exist and $w = W$.

6 Postscript

The current procedure for estimating soil production rates in relation to soil thickness using measurements of cosmogenic radionulide concentrations likely leads to spurious outcomes from the misapplication of what Lal (1991) explained three decades ago regarding the accumulation of radionucide atoms within earthen materials in the presence of steady erosion. Due to the transient conditions that mostly exist in natural landscapes this procedure, which is based on the assumption of steady conditions, cannot reveal the form of the soil production function in relation to soil thickness, if such a function exists, in the presence of moderate to high-frequency variations in soil thickness. Moreover, that the magnitude of the semi-log slope of an empirical soil production function generated from this procedure is of the same order as the reciprocal of the attenuation length l_s is likely not a coincidence (Harrison et al., 2021).

The analyses presented herein suggest a straightforward lesson. If the semi-log slope of an empirical soil production curve obtained by the method described above is similar to the reciprocal of the soil attenuation length l_s — making allowances for uncertainty in soil properties over long times, any covariance between the concentration $n(\eta)$ and the thickness h (Appendix D), and measurement and analytical uncertainties — then the parsimonious interpretation is that this situation likely represents the presence of moderate to high-frequency transient conditions. If instead the slope markedly differs from the reciprocal of the soil attenuation length scale, or the empirical relationship is altogether different from an exponential curve (e.g. Heimsath et al., 2020; Harrison et al., 2021), then the relationship might merit closer examination.

In exploring the dynamics of soil-mantled hillslopes in the presence of soil production, it is entirely reasonable to assume that the rate of production varies as a function, exponential or otherwise, of soil thickness. But this must be viewed as an hypothesis, not a confirmed relationship. Moreover, a genuine soil production function is unlikely to involve a single-valued relationship with soil thickness for a specified rock type in view of natural variability in climate and associated hydrologic and biotic conditions (e.g. Heimsath et al., 2020).

The basic idea that the rate of soil production might systematically vary with soil thickness is compelling, and merits closer examination. In particular the negative feedback associated with a decreasing production rate with increasing soil thickness (Carson and Kirkby, 1972; Furbish and Fagherazzi, 2001) provides a sensible hypothesis for the seeming stability in the mantling of soils on hillslopes in the presence of soil transport. Clarifying how the rate of soil production might be related to soil thickness will require an explanation of the mechanics, in concert with chemical effects, that produce the transition from immobile to mobile conditions at the soil-saprolite interface, giving way to active transport above — likely representing a set of stochastic processes rather than a continuum-like behavior.

We end with an observation. As mentioned above, Lal (1991) is quite clear in pointing out that the steady-state relationships described by (13) and (14) hinge on the presence of steady erosion for a period of time at least as long as $\sim 4T_\mathrm{r}$ with fixed production rate P_η — a rather rigid constraint imposed on the procedure. Heimsath et al. (1997) clearly are aware of this constraint, but seem to justify the procedure by appealing to the time scale of hillslope relaxation as a measure of steadiness rather than providing an assessment of steadiness with respect to the relevant e-folding time T_r. In their review paper on cosmogenic radionuclides, Granger and Riebe (2014) show the unsteady solution (11) of the concentration $n(\psi, t)$ with fixed erosion rate, but only note the steady-state constraint in passing without further comment when applied to the problems of rock erosion and the soil production rate. In their recent paper, Schaller and Ehlers (2022) simply give the steady-state formula for calculating the cosmogenic radionuclide concentration without reference to its applicability to only steady-state conditions nor to the constraint imposed by the e-folding time T_r. The progression in such examples offers the impression that the procedure is now disconnected from the basis of the work that Lal (1991) presented, and simply assumes that (13) and (14) yield the correct result with P_η determined by the extant soil thickness without independently constraining its history.

Appendixes

A: Discontinuity of the radionuclide concentration at the soil-saprolite interface

The rate of production $P_\eta(t)$ of cosmogenic radionuclide atoms at the soil-saprolite interface is set by the surface production rate P_ζ, the soil thickness h and the attenuation length l_s. However, different values of the attenuation lengths l_s and l_r due to differences in density can lead to a discontinuity in the cosmogenic radionuclide concentrations $n(\eta, t)$ measured just above and just below the soil-saprolite interface. The conversion of weathered rock (saprolite) to mechanically active soil means that radionuclide atoms within the saprolite particles are "handed" to the base of the soil and are thus subject to mixing. The mathematics used herein give the impression that this is a piecewise continuous process, whereas in reality the conversion (with a change in bulk density) and entrainment of weathered material likely is a complex stochastic process. One perhaps could perform matching of the concentrations across the soil-saprolite interface with additional description of the physics involved, but this level of mathematics is not warranted. Indeed, we may be using mathematics that are far more precise than might be justified by the stochastic ingredients of the natural process.

B: Frequency response function without soil

In order to examine effects of unsteady erosion, which Lal (1991) does not consider, here we derive the frequency response function relating the rate of erosion $E(t) = -w(t)$ to the cosmogenic radionuclide concentration $n(0,t)$. The starting point is the statement of conservation (10). As in the main text we define zeroth-order and first-order quantities denoted by the subscripts 0 and 1. Namely,

$$n(\psi, t) = n_0(\psi) + n_1(\psi, t) \quad \text{and} \tag{41}$$

$$w(t) = w_0 + w_1(t)\,. \tag{42}$$

Upon substituting (37) and (38) into (10) we have at zeroth order,

$$0 = w_0 \frac{\partial n_0(\psi)}{\partial \psi} + P_\eta e^{-\psi/l_\mathrm{r}} - \lambda n_0(\psi)\,. \tag{43}$$

From this we obtain

$$n_0(0) = \frac{P_\eta}{\lambda + w_0/l_\mathrm{r}}\,, \tag{44}$$

which in effect is a restatement of (13). At linearized first order,

$$\frac{\partial n_1(\psi, t)}{\partial t} = w_0 \frac{\partial n_1(\psi, t)}{\partial \psi} + w_1(t) \frac{\partial n_0(\psi)}{\partial \psi} - \lambda n_1(\psi, t)\,. \tag{45}$$

We now define the following dimensionless quantities denoted by circumflexes:

$$n = n_0(0)\hat{n}\,, \quad w = w_0 \hat{w}\,,$$

$$\psi = l_\mathrm{r}\hat{\psi} \quad \text{and} \quad t = T_\mathrm{r}\hat{t}\,, \tag{46}$$

with $T_\mathrm{r} = 1/(\lambda + w_0/l_\mathrm{r})$. Substituting these expressions into (41) then gives

$$\frac{\partial \hat{n}_1(\hat{\psi}, \hat{t})}{\partial \hat{t}} - \frac{w_0 T_\mathrm{r}}{l_\mathrm{r}} \frac{\partial \hat{n}_1(\hat{\psi}, \hat{t})}{\partial \hat{\psi}} + \lambda T_\mathrm{r} \hat{n}_1(\hat{\psi}, \hat{t}) = -\hat{w}_1(\hat{t}) e^{-\hat{\psi}} + \lambda T_\mathrm{r} \hat{w}_1(\hat{t}) e^{-\hat{\psi}}\,. \tag{47}$$

We now assume that

$$\hat{n}_1(\hat{\psi}, \hat{t}) = \hat{A}_{\hat{n}_1} e^{i\hat{\omega}\hat{t}} e^{-\hat{\psi}} \quad \text{and}$$

$$\hat{w}_1(\hat{t}) = \hat{A}_{\hat{w}_1} e^{i\hat{\omega}\hat{t}}\,, \tag{48}$$

with dimensionless frequency $\hat{\omega} = T_\mathrm{r}\omega$ and imaginary number defined by $i^2 = -1$. Substituting these expressions into (43) then leads to the frequency response function,

$$F_{\hat{n}_1(0), \hat{w}_1}(\hat{\omega}) = \frac{\lambda T_\mathrm{r} - 1}{1 + i\hat{\omega}}\,, \tag{49}$$

where the first subscript denotes the response and the second subscript denotes the forcing quantity. This has the gain and phase functions,

$$G_{\hat{n}_1(0), \hat{w}_1}(\hat{\omega}) = \frac{\lambda T_\mathrm{r} - 1}{\sqrt{1 + \hat{\omega}^2}} \quad \text{and} \tag{50}$$

$$\phi_{\hat{n}_1(0), \hat{w}_1}(\hat{\omega}) = \tan^{-1}(-\hat{\omega})\,. \tag{51}$$

The gain function (46) revels that (45) is a low-pass filter (Figure 1). That is, high-frequency variations in the erosion rate $E_1(t) = -w_1(t)$ are attenuated in the response signal $n_1(0,t)$, and the lowest frequencies are "passed" without attenuation. This means that low-frequency variations yield responses with high fidelity such that quasi-steady conditions are maintained. High-frequency variations in the erosion rate do not produce similar variations in the number concentration $n_1(0,t)$.

C: Frequency response functions with soil

Here we obtain the frequency response functions relating variations in soil thickness to the rate of soil production, the rate of production of cosmogenic radionuclide atoms at the soil-saprolite interface, and the number concentration of cosmogenic radionuclide atoms at this interface. Throughout we assume a fixed uplift rate W. We start by collecting the essential equations from the main text.

Variations in the land-surface elevation are related to the erosion rate as

$$\frac{\partial \zeta(t)}{\partial t} = W + E(t)\,. \tag{52}$$

Variations in the soil-saprolite interface are given by

$$\frac{\partial \eta(t)}{\partial t} = W - w(t)\,. \tag{53}$$

Variations in the soil thickness therefore are given by

$$\frac{\partial h(t)}{\partial t} = E(t) + w(t)\,. \tag{54}$$

We assume a linear soil production function at the outset. Namely,

$$w(t) = w_0(1 + \alpha h_1)\,. \tag{55}$$

If this is viewed as a linearized form of an exponential function then $w_0 = P_0 e^{-h_0/l_w}$ with $\alpha = -1/l_w$ where h_1 denotes a fluctuation about the basic state soil thickness h_0. Otherwise it may be considered a linear function that decreases ($\alpha < 0$) or increases ($\alpha > 0$) with increasing soil thickness h. If $\alpha = 0$ then the rate of soil production $w(t)$ is independent of soil thickness. The rate of production of cosmogenic radionuclide atoms at the soil-saprolite interface is given by

$$P_\eta(t) = P_\zeta e^{-h(t)/l_\mathrm{s}}\,. \tag{56}$$

Conservation of cosmogenic radionuclide atoms beneath the soil-saprolite interface is described by an advection equation with source term,

$$\frac{\partial n(\psi,t)}{\partial t} = w\frac{\partial n(\psi,t)}{\partial \psi} + P_\eta(t)e^{-\psi/l_\mathrm{r}}\,, \tag{57}$$

where $\psi = \eta' - z'$ with $\mathrm{d}\psi = -\mathrm{d}z'$. The estimated rate of soil production assuming steady-state conditions is given by

$$w^* = \frac{P_\zeta l_\mathrm{r}}{n(\eta)} e^{-h/l_\mathrm{s}}\,. \tag{58}$$

Before continuing, here we note an important solution of (53), namely, the response $n(\psi,t)$ to a step change in the production rate $P_\eta(t) = P_\eta$ with fixed velocity $w(t) = w$. Namely, upon neglecting radioactive decay (11) becomes

$$n(\psi,t) = \frac{P_\eta l_\mathrm{r}}{w}\left[1 - e^{-(w/l_\mathrm{r})t}\right] e^{-\psi/l_\mathrm{r}}\,, \tag{59}$$

such that $n(0,\infty) = P_\eta l_\mathrm{r}/w$. This reveals that the advective time scale $T_\mathrm{r} = l_\mathrm{r}/w$ coincides with the e-folding response time of the number concentration $n(\psi,t)$. Moreover, as a step response this solution reveals that the concentration $n(0,t)$ at the soil-saprolite interface may be described in terms of variations in the production rate $P_\eta(t)$ without reference to concentrations beneath the interface ($\psi > 0$).

We now define zeroth-order states and first-order fluctuations denoted by the subscripts 0 and 1. Namely,

$$\begin{aligned} h(t) &= h_0 + h_1(t)\,, \\ E(t) &= E_0 + E_1(t)\,, \\ w(t) &= w_0 + w_1(t)\,, \\ P_\eta(t) &= P_{\eta 0} + P_{\eta 1}(t) \quad \text{and} \\ n(\psi,t) &= n_0(\psi) + n_1(\psi,t)\,. \end{aligned} \tag{60}$$

Substituting these expressions into (50), (51), (52) and (53) leads to the zeroth-order identity $w_0 = w_0$ plus the following conditions:

$$0 = E_0 + w_0\,, \tag{61}$$

$$P_{\eta 0} = P_\zeta e^{-h_0/l_\mathrm{s}} \quad \text{and} \tag{62}$$

$$0 = w_0 \frac{\partial n_0(\psi)}{\partial \psi} + P_{\eta 0} e^{-\psi/l_\mathrm{r}}\,. \tag{63}$$

From these we obtain

$$\frac{\partial n_0(\psi)}{\partial \psi} = \frac{P_{\eta 0}}{w_0} e^{-\psi/l_\mathrm{r}} \quad \text{and} \tag{64}$$

$$n_0(0) = \frac{P_{\eta 0} l_\mathrm{r}}{w_0}\,. \tag{65}$$

At linearized first order,

$$\frac{\partial h_1(t)}{\partial t} = E_1(t) + w_1(t)\,, \tag{66}$$

$$w_1(t) = w_0 \alpha h_1(t)\,, \tag{67}$$

$$P_{\eta 1}(t) = -\frac{P_{\eta 0}}{l_\mathrm{s}} h_1(t) \quad \text{and} \tag{68}$$

$$\frac{\partial n_1(\psi,t)}{\partial t} = w_0 \frac{\partial n_1(\psi,t)}{\partial \psi} + w_1(t)\frac{\partial n_0(\psi)}{\partial \psi} + P_{\eta 1}(t) e^{-\psi/l_\mathrm{r}}\,. \tag{69}$$

We then combine (62), (63), (64) and (65) to give

$$\frac{\partial h_1(t)}{\partial t} = E_1(t) + w_0 \alpha h_1(t) \quad \text{and} \tag{70}$$

$$\frac{\partial n_1(\psi,t)}{\partial t} = w_0 \frac{\partial n_1(\psi,t)}{\partial \psi} + \frac{w_0 n_0(0)}{l_\mathrm{r}} \left(\alpha - \frac{1}{l_\mathrm{s}}\right) h_1(t) e^{-\psi/l_\mathrm{r}} . \quad (71)$$

We now define the following dimensionless quantities denoted by circumflexes:

$$h = l_\mathrm{s}\hat{h} ,$$
$$E = W\hat{E} , \quad w = W\hat{w} ,$$
$$n = n_0(0)\hat{n} ,$$
$$\psi = l_\mathrm{r}\hat{\psi} \quad \text{and} \quad t = \frac{l_\mathrm{r}}{W}\hat{t} \quad (72)$$

Substituting these expressions into (66) and (67) and recognizing that $\hat{w}_0 = 1$ leads to

$$\frac{\partial \hat{h}_1(\hat{t})}{\partial \hat{t}} - \alpha l_\mathrm{r} \hat{h}\cdot(\hat{t}) = \frac{l_\mathrm{r}}{l_\mathrm{s}} \hat{E}_1(\hat{t}) \quad \text{and} \quad (73)$$

$$\frac{\partial \hat{n}_1(\hat{\psi},\hat{t})}{\partial \hat{t}} = \frac{\partial \hat{n}_1(\hat{\psi},\hat{t})}{\partial \hat{\psi}} + (\alpha l_\mathrm{s} - 1)\hat{h}_1(\hat{t}) e^{-\hat{\psi}} . \quad (74)$$

Inspired by the form of the step response given by (55) we assume that

$$\hat{E}_1(\hat{t}) = \hat{A}_{\hat{E}_1} e^{i\hat{\omega}\hat{t}} ,$$
$$\hat{h}_1(\hat{t}) = \hat{A}_{\hat{h}_1} e^{i\hat{\omega}\hat{t}} \quad \text{and}$$
$$\hat{n}_1(\hat{\psi},\hat{t}) = \hat{A}_{\hat{n}_1} e^{i\hat{\omega}\hat{t}} e^{-\hat{\psi}} , \quad (75)$$

with dimensionless frequency $\hat{\omega} = (l_\mathrm{r}/W)\omega$ and imaginary number defined by $i^2 = -1$. Substituting these expressions into (69) and (70) then leads to the frequency response functions,

$$F_{\hat{h}_1,\hat{E}_1}(\hat{\omega}) = \frac{l_\mathrm{r}/l_\mathrm{s}}{-\alpha l_\mathrm{s} + i\hat{\omega}} \quad \text{and} \quad (76)$$

$$F_{\hat{n}_1(0),\hat{h}_1}(\hat{\omega}) = \frac{\alpha l_\mathrm{s} - 1}{1 + i\hat{\omega}} , \quad (77)$$

where the first subscript denotes the response and the second subscript denotes the forcing quantity. Moreover we can combine (72) and (73) to give the frequency response function relating the cosmogenic radionuclide concentration to the erosion rate. Namely,

$$F_{\hat{n}_1(0),\hat{E}_1}(\hat{\omega}) = \frac{(1-\alpha l_\mathrm{s})(l_\mathrm{r}/l_\mathrm{s})}{\alpha l_\mathrm{s} + \hat{\omega}^2 - i(1-\alpha l_\mathrm{s})\hat{\omega}} . \quad (78)$$

These have the gain and phase functions

$$G_{\hat{h}_1,\hat{E}_1}(\hat{\omega}) = \frac{l_\mathrm{r}/l_\mathrm{s}}{\sqrt{\alpha^2 l_\mathrm{s}^2 + \hat{\omega}^2}} , \quad (79)$$

$$\phi_{\hat{h}_1,\hat{E}_1}(\hat{\omega}) = \tan^{-1}\left(\frac{\hat{\omega}}{\alpha l_\mathrm{s}}\right) , \quad (80)$$

$$G_{\hat{n}_1(0),\hat{h}_1}(\hat{\omega}) = \frac{\alpha l_\mathrm{s} - 1}{\sqrt{1 + \hat{\omega}^2}} , \quad (81)$$

$$\phi_{\hat{n}_1,\hat{h}_1}(\hat{\omega}) = \tan^{-1}(-\hat{\omega}) , \quad (82)$$

$$G_{\hat{n}_1(0),\hat{E}_1}(\hat{\omega}) = \frac{(1-\alpha l_\mathrm{s})(l_\mathrm{r}/l_\mathrm{s})}{\sqrt{(\alpha l_\mathrm{s} + \hat{\omega}^2)^2 + (1-\alpha l_\mathrm{s})^2 \hat{\omega}^2}} \quad \text{and} \quad (83)$$

$$\phi_{\hat{n}_1(0),\hat{E}_1}(\hat{\omega}) = \tan^{-1}\left[\frac{(1-\alpha l_\mathrm{s})\hat{\omega}}{\alpha l_\mathrm{s} + \hat{\omega}^2}\right] . \quad (84)$$

The frequency response, gain and phase functions representing the situation in which the rate of soil production is independent of soil thickness are obtained by setting $\alpha = 0$.

The forms of the gain functions (75), (77) and (79) reveal that each of the frequency response functions, (72), (73) and (74), represents a low-pass filter. That is, high-frequency variations in the forcing quantity are attenuated in the response signal, and the lowest frequencies are “passed” without attenuation. This means that low-frequency variations yield responses with high fidelity such that quasi-steady conditions are maintained.

D: Effect of covariance between the concentration $n(\eta)$ and the thickness h

From a statistics perspective, the expression (22), reproduced as (40), may involve an important effect due to a finite covariance between the radionuclide atom concentration $n(\eta)$ and the soil thickness h. To illustrate this point, suppose on purely empirical grounds that $n(\eta)$ varies exponentially with h as

$$n(\eta) \sim e^{-h/l_n} , \quad (85)$$

where l_n is an e-folding length. Now (22) may be expressed as

$$w \sim e^{-(1/l_\mathrm{s} - 1/l_n)h} . \quad (86)$$

Taking logarithms,

$$\ln w \sim -\left(\frac{1}{l_\mathrm{s}} - \frac{1}{l_n}\right) h\,. \tag{87}$$

This means that the magnitude of the expected empirical slope $1/l_\mathrm{s}$ is reduced by $1/l_n$ due to the negative covariance between $n(\eta)$ and h. More generally, without reference to the specific form of the relationship between $n(\eta)$ and h, a negative covariance between these quantities reduces the magnitude of the expected slope, and a positive covariance increases the magnitude of the expected slope. As a point of reference the data of Heimsath et al. (2005) for Point Reyes, California, indicate that $n(\eta)$ and h are uncorrelated. Similarly, the expected slope can be influenced by a finite covariance between the surface production rate P_ζ and the thickness h.

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