Edge-coloring Multigraphs

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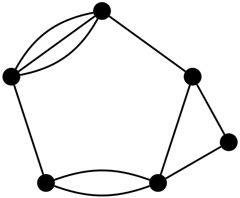

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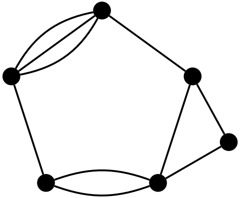

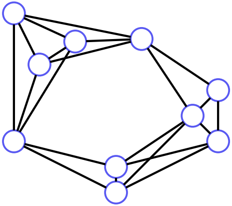

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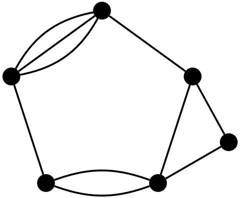

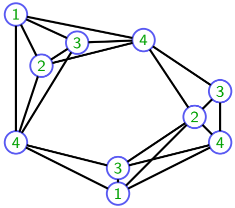

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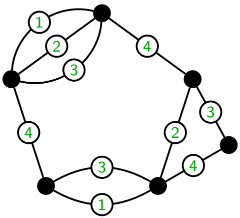

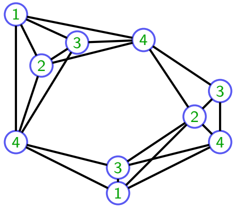

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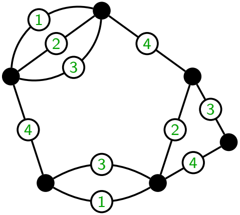

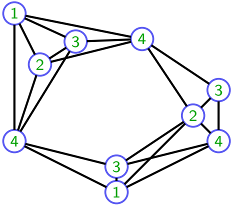

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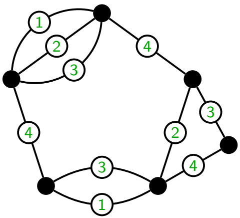

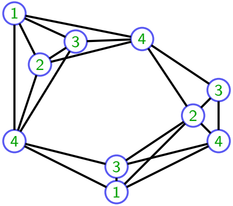

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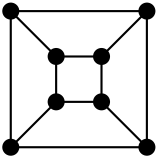

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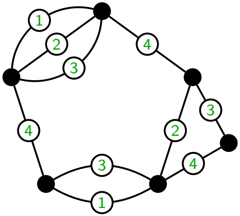

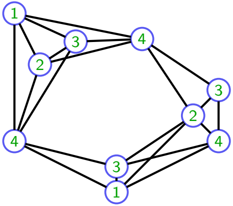

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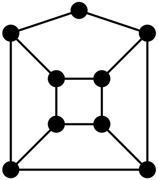

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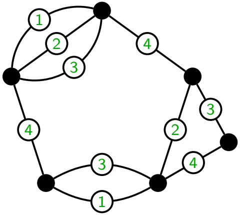

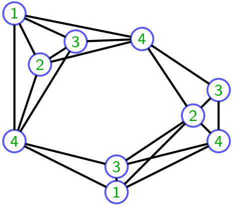

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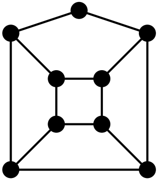

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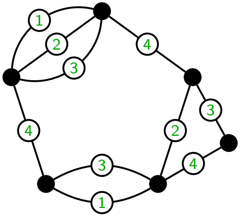

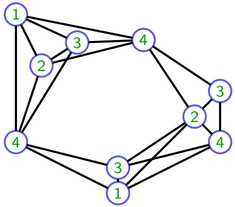

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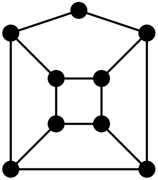

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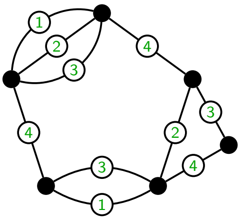

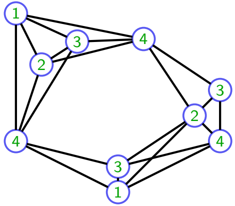

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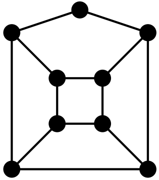

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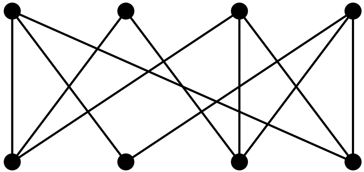

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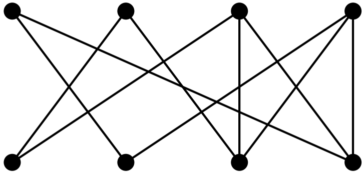

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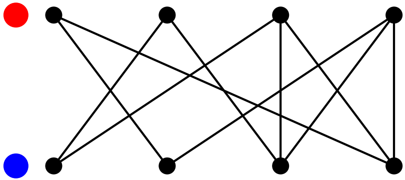

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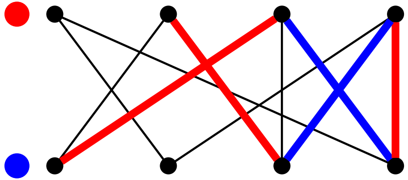

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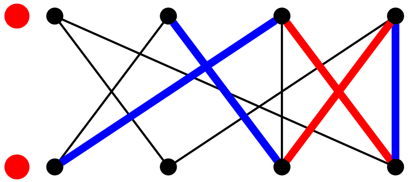

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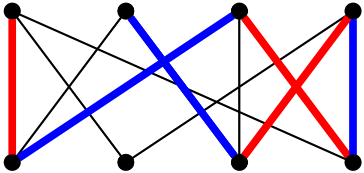

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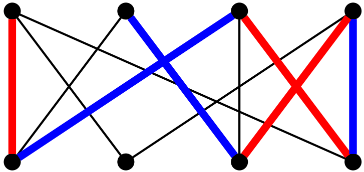

Rem: Kempe swaps are fundamental tool for edge-coloring.

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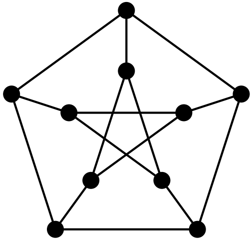

Tutte's Edge-coloring Conj (proved!):
If G is 3-regular, has no overfull subgraph, and has no subdivision of the Petersen graph, then $\chi'(G) = 3$.

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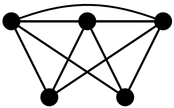

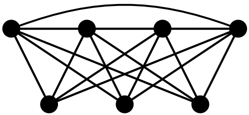

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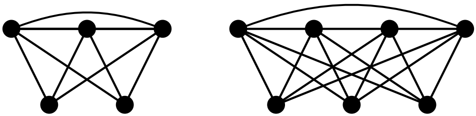

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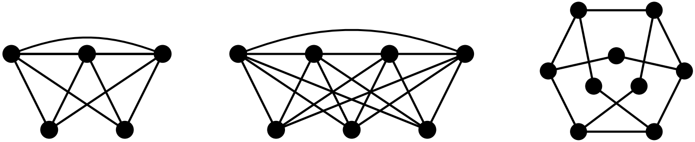

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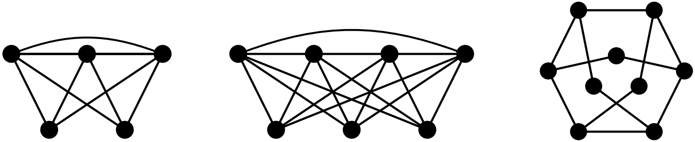

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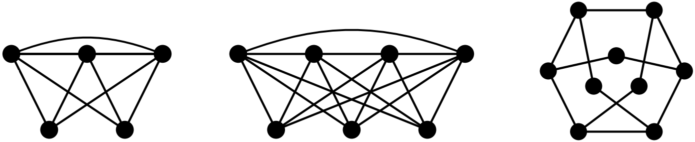

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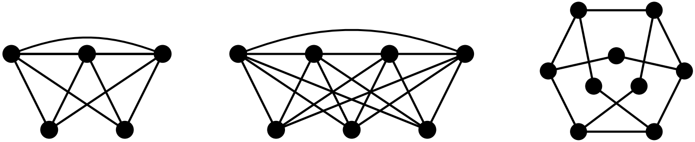
- Does $\Delta(G_\Delta) \leq 2$ imply $\chi'(G) = \Delta$?
- No. G could be overfull.



- Does $\Delta(G_\Delta) \leq 2$ imply $\chi'(G) = \Delta$ if G is not overfull? No.

Hilton–Zhao Conjecture:
If $\Delta(G_\Delta) \leq 2$ and $G \neq P^*$, then $\chi'(G) > \Delta$ iff G is overfull.
Cariolaro–Cariolaro: True for $\Delta = 3$.
C.–Rabern: True for $\Delta = 4$.

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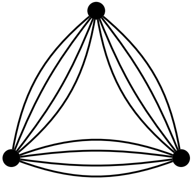

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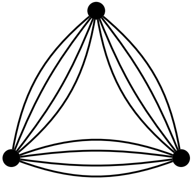

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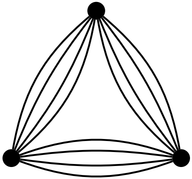

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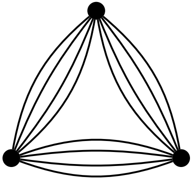

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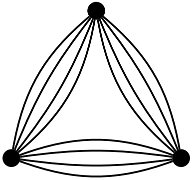

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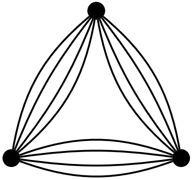

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Always $\chi'(G) \le \max\{\Delta + \sqrt[3]{\Delta/2}, \lceil \mathcal{W}(G) \rceil\}$.

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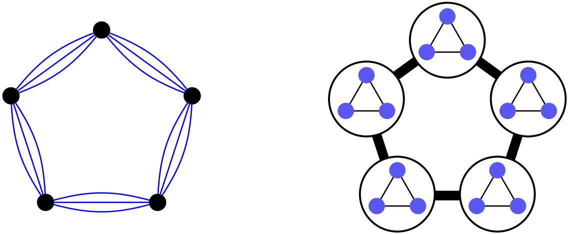

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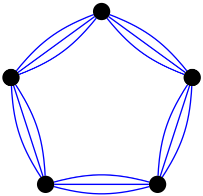

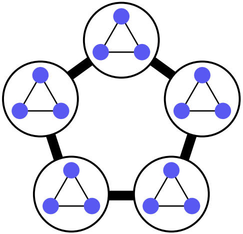

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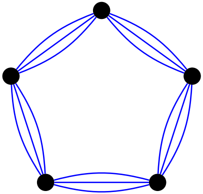

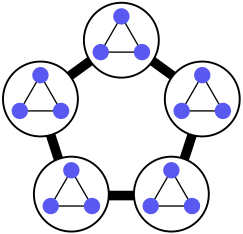

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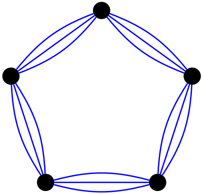

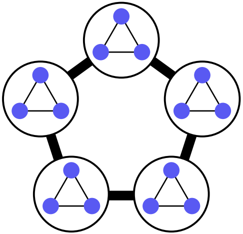

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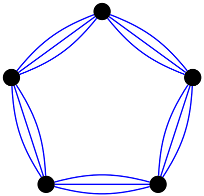

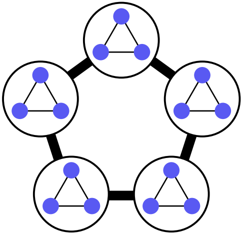

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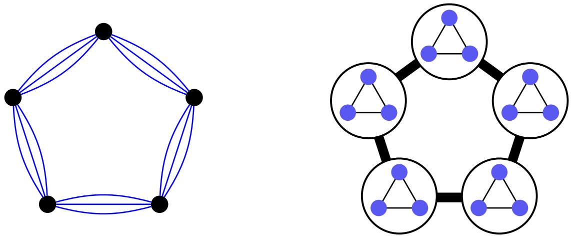

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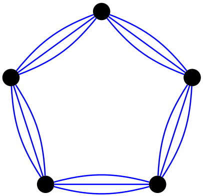

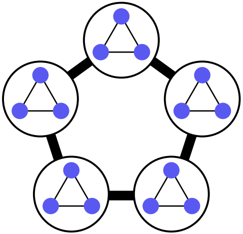

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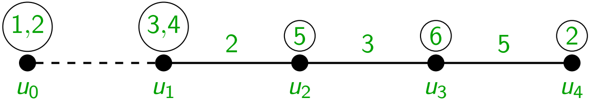

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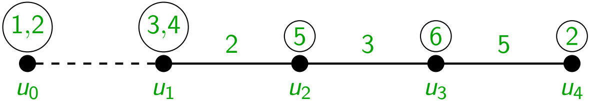

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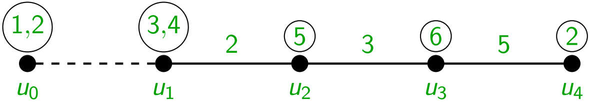

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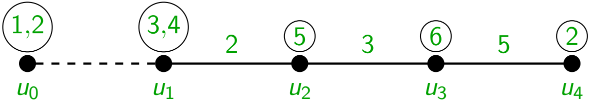

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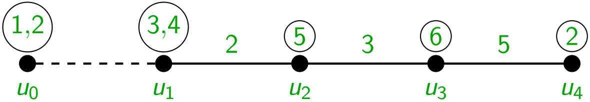

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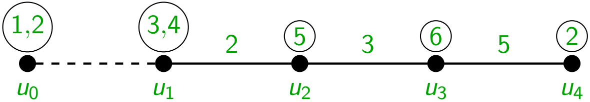

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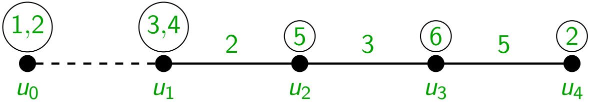

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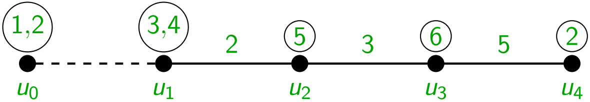

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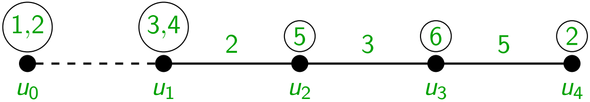

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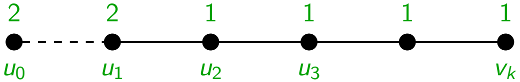

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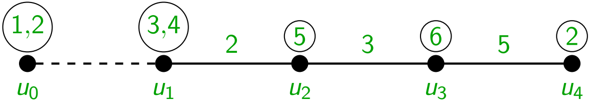

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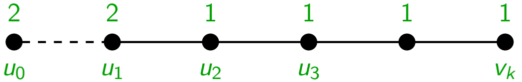
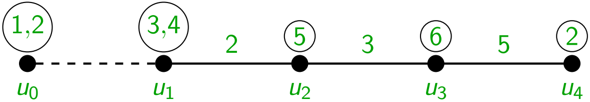


By Pigeonhole, two vertices miss the same color.

Kierstead Paths

Def: Fix G, $u_0u_1 \in E(G)$, $k \geq \Delta(G)+1$, and φ a k-edge-coloring of $G - u_0u_1$. A Kierstead Path is a path $u_0, u_1, \ldots, u_\ell$ where for each i, $\varphi(u_iu_{i-1})$ is *missing* at u_j for some $j < i$.



Key Lemma: If a Kierstead Path has distinct u_i and u_j with color α missing at both, then G has a k-coloring.

Vizing's Theorem: If G is simple, then $\chi'(G) \leq \Delta(G)+1$.
Pf (using Key Lemma): Induction on $|E(G)|$. Let $k = \Delta(G)+1$.
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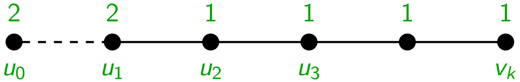

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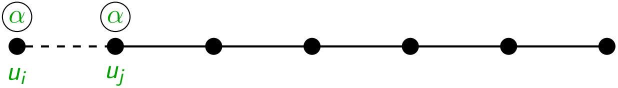

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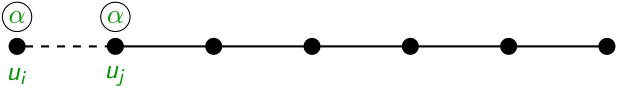

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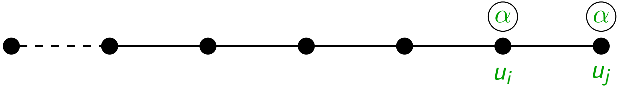

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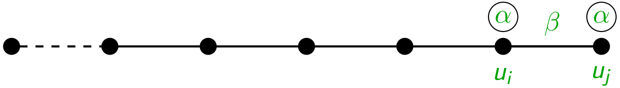

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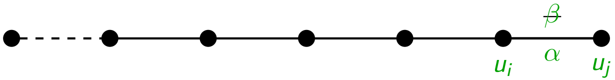

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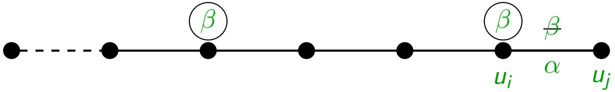

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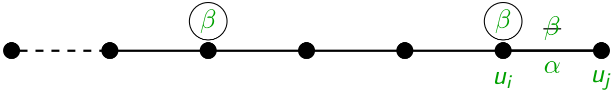

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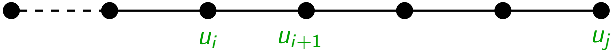

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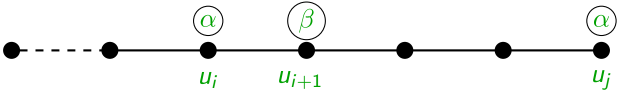

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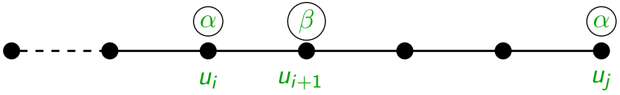

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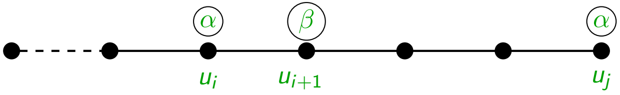

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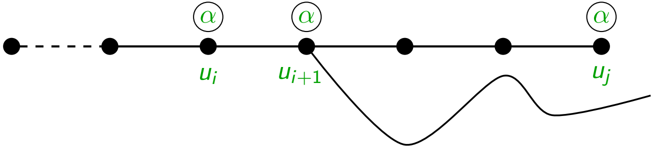

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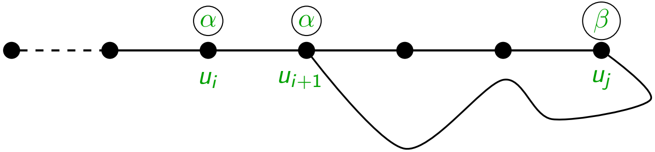

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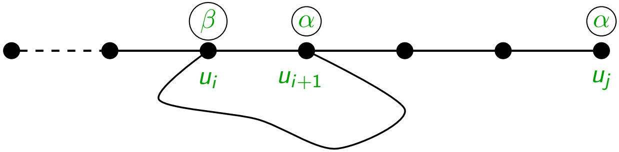

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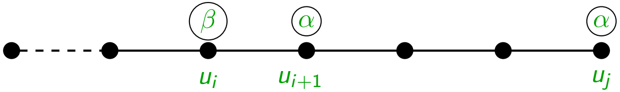

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In each case, win by induction hypothesis. ■

Tashkinov Trees

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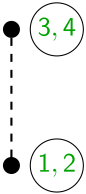

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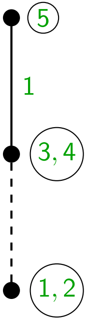

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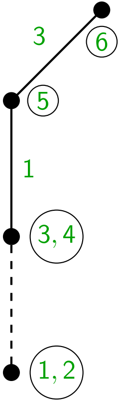

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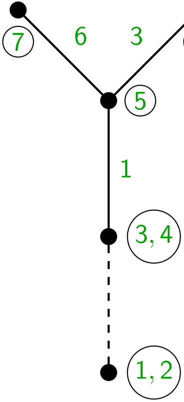

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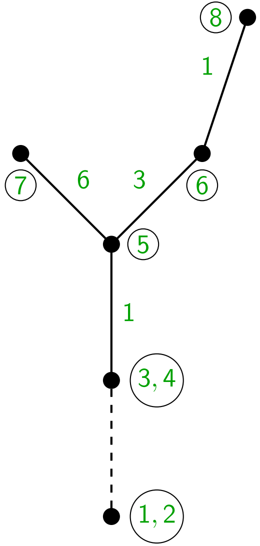

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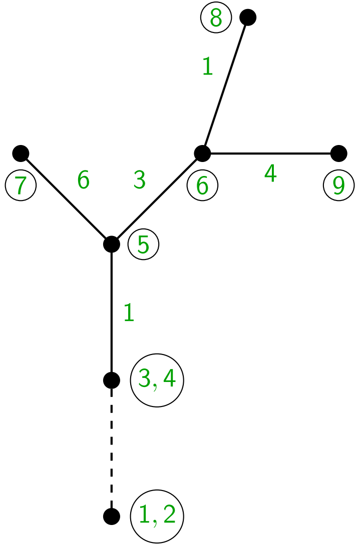

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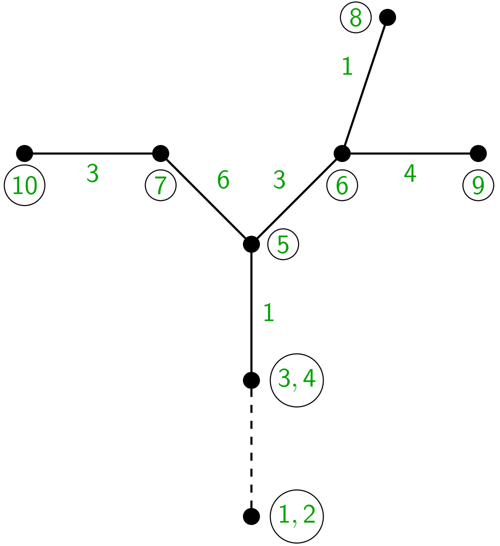

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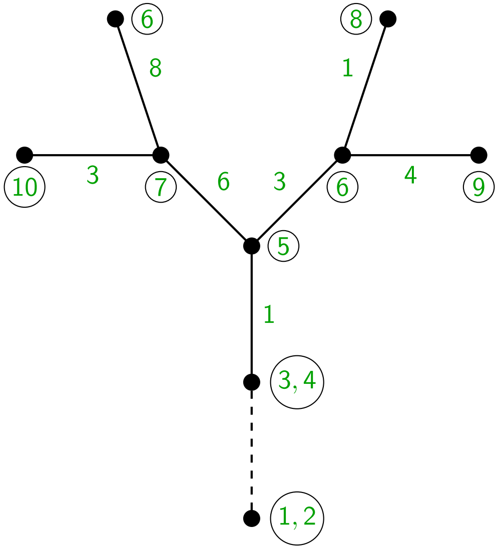

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