Elusive problems in extremal graph theory

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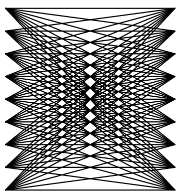
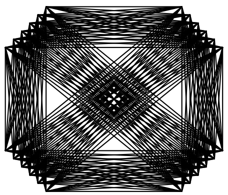
Overview of talk

- uniqueness of extremal configurations
  motivation and formulation of problem
- graph limits
  representation of large graphs
- finitely forcible graph limits
  large graphs with assymptotically unique structures
- main result, proof tools and extensions

Turán Problems

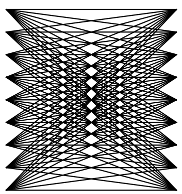
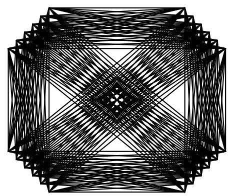
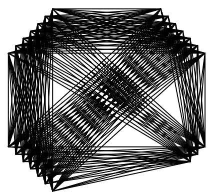
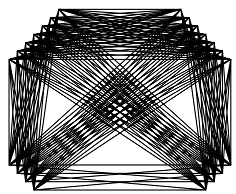
- Maximum edge-density of H-free graph
- Mantel's Theorem (1907): $\frac{1}{2}$ for $H = K_3$ $(K_{\frac{n}{2},\frac{n}{2}})$
- Turán's Theorem (1941): $\frac{\ell-2}{\ell-1}$ for $H = K_\ell$ $(K_{\frac{n}{\ell-1},\dots,\frac{n}{\ell-1}})$
- Erdős-Stone Theorem (1946): $\frac{\chi(H)-2}{\chi(H)-1}$
- extremal examples unique up to $o(n^2)$ edges





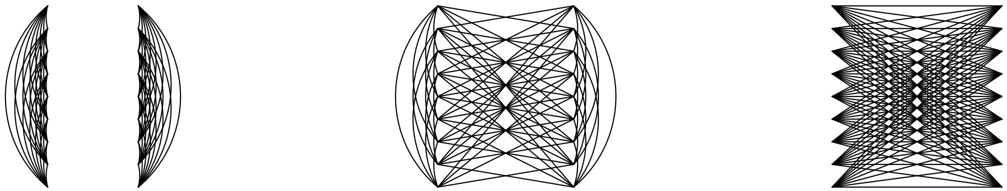
Edge vs. Triangle Problem

- Minimum density of K_3 for a specific edge-density
- determined by Razborov (2008), $K_{\alpha n, \ldots, \alpha n, (1-k\alpha)n}$
- extensions by Nikiforov (2011) and Reiher (2016) for K_ℓ
- Pikhurko and Razborov (2017) gave extremal examples generally not unique, can be made unique by $\overline{K_n} = 0$

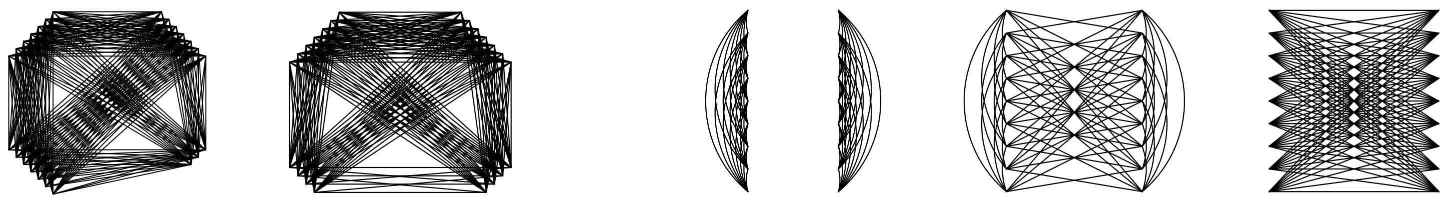
Another example

- Minimum sum of densities of K_3 and $\overline{K_3}$
- Goodman's Bound (1959): $K_3 + \overline{K_3} \geq \frac{1}{4}$
  every $n/2$-regular graph is a minimizer
- minimizer can be made unique $K_3 = 0$, or $\overline{K_3} = 0$, or $C_4 = 1/16$ (Erdős-Rényi random graph $G_{n,1/2}$)



This talk

- Conjecture (Lovász 2008, Lovász and Szegedy 2011)
  Every finite feasible set $H_i = d_i$, $i = 1, \ldots, k$,
  can be extended to a finite feasible set
  with an asymptotically unique structure.
- Every extremal problem has a finitely forcible optimum.
- Theorem (Grzesik, K., Lovász Jr.): FALSE

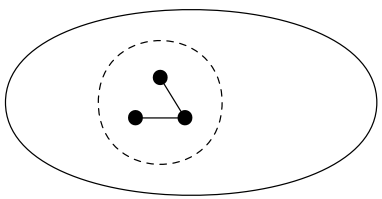
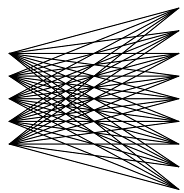
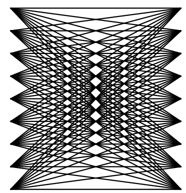


Graph limits

- large networks $\approx$ large graphs
  how to represent? how to model? how to generate?
- concise (analytic) representation of large graphs
  we implicitly use limits in our considerations anyway
- mathematics motivation – extremal graph theory
  What is a typical structure of an extremal graph?
  calculations avoiding smaller order terms
- in this talk: dense graphs ($|E| = \Omega(|V|^2)$)
  Borgs, Chayes, Lovász, Sós, Szegedy, Vesztergombi, ...
- convergence vs. analytic representation

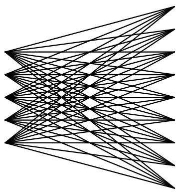
Convergent graph sequence

- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- a sequence $(G_n)_{n \in \mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H
- examples: K_n, $K_{\alpha n, n}$, blow ups $G[K_n]$
  Erdős-Rényi random graphs $G_{n,p}$, planar graphs
- extendable to other discrete structures

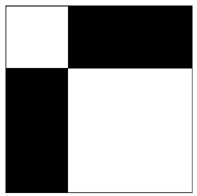




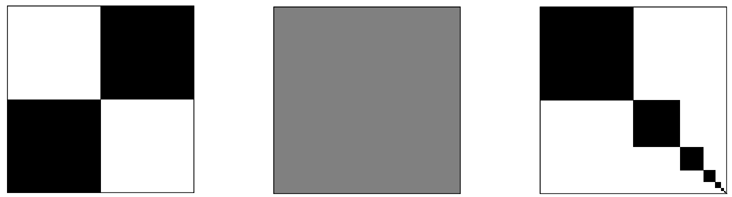
Limit object: graphon

- $\text{graphon } W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- W-random graph of order n
  random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$
- $d(H,W)$ = prob. $|H|$-vertex W-random graph is H
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$



$$\begin{matrix} 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0 \end{matrix}$$





LIMIT OBJECT: GRAPHON

- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$
- W-random graph of order n
  random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$
- $d(H,W)$ = prob. $|H|$-vertex W-random graph is H
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)$
- every convergent sequence of graphs has a limit
- W-random graphs converge to W with probability one

Applications of graph limits

- extremal combinatorics

  flag algebras of Razborov

  density calculations, computer search

- computer science

  property and parameter testing

  cover of the space of all graphons

- structure of typical graphs

  graphon entropy, number of graphs $\approx c^{\binom{n}{2}}$

Statement of problem

- Conjecture (Lovász 2008, Lovász and Szegedy 2011):
  Every finite feasible set $H_i = d_i$, $i = 1, \ldots, k$,
  can be extended to a finite feasible set that
  is satisifed by a unique graphon.
- uniqueness of graphons (Borgs, Chayes, Lovász 2010)
  $W(x, y)$ and $W^\varphi(x, y) := W(\varphi(x), \varphi(y))$ are the same
- A graphon W is finitely forcible if there exist
  $H_1, \ldots, H_k$ and $d_1, \ldots, d_k$ such that W is the only
  graphon with the density of H_i equal to d_i.

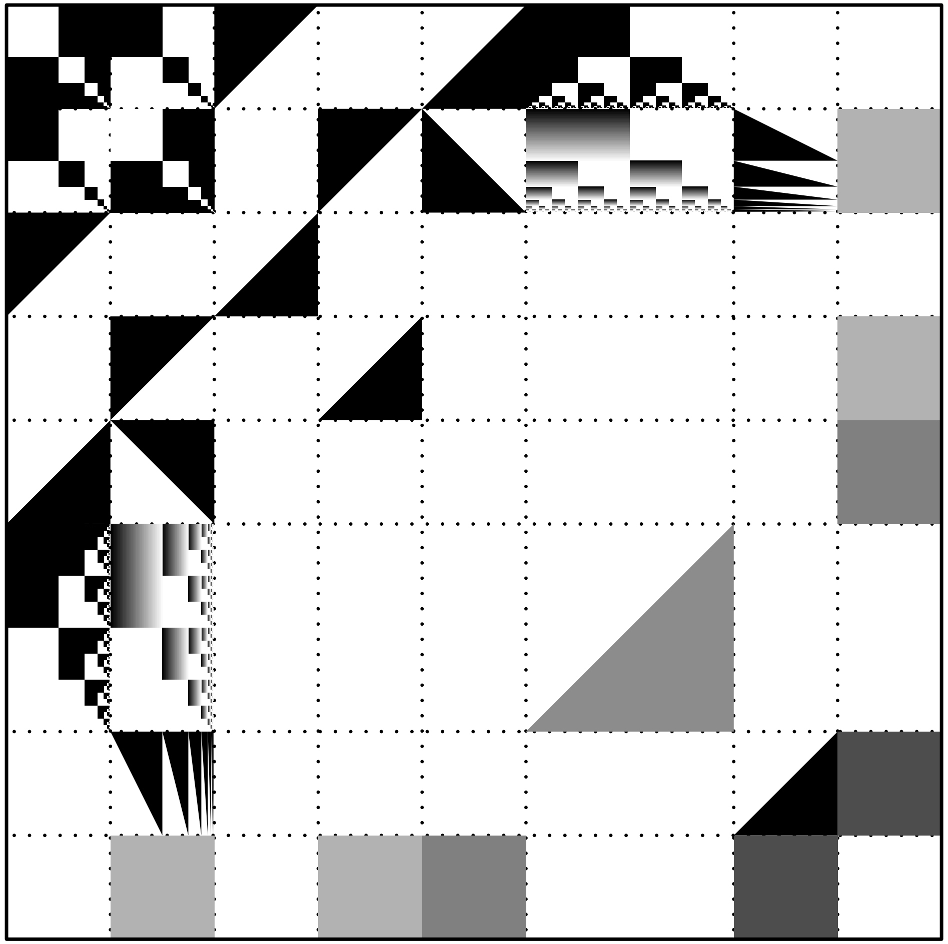
Finitely forcible graph limits

- Lovász, Sós (2008): Step graphons are finitely forcible.
- extremal graph theory problem $\rightarrow$
  finitely forcible optimal solution $\rightarrow$
  "simple structure" gives new bounds on old problems
- Conjectures (Lovász and Szegedy):
  The space $T(W)$ of a finitely forcible W is compact.
  The space $T(W)$ has finite dimension.

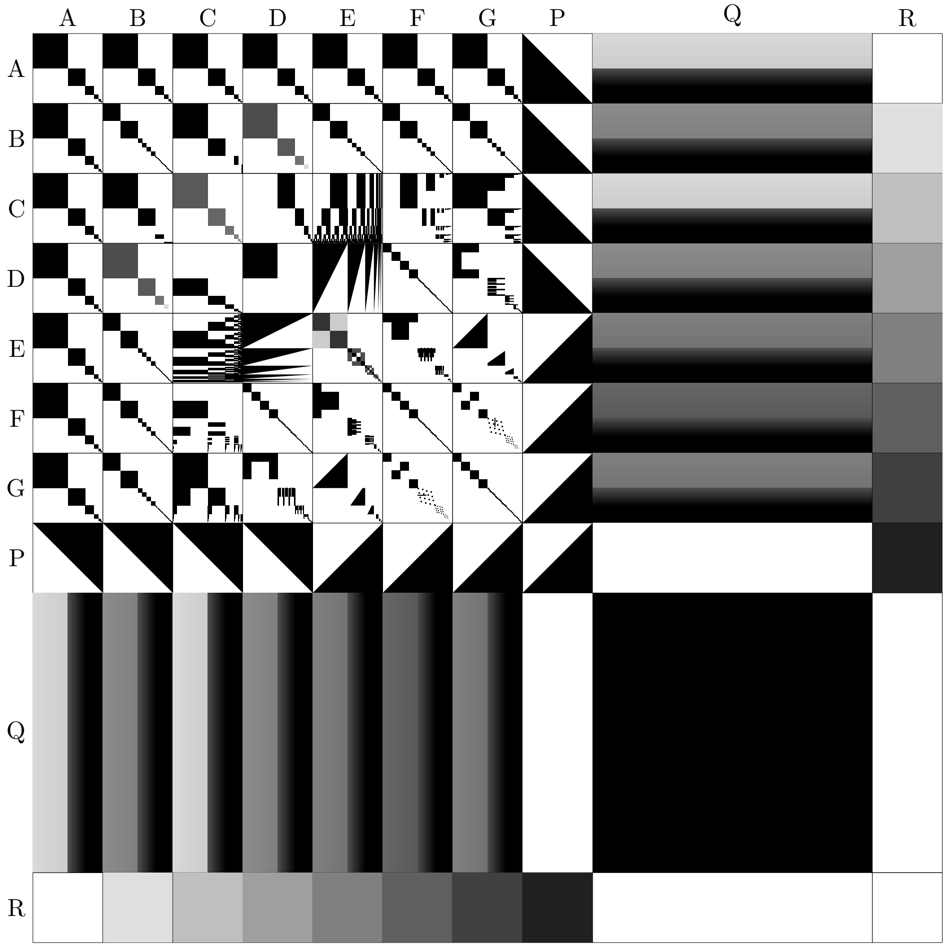
Finitely forcible graphons

- Theorem (Glebov, K., Volec):
  $T(W)$ can fail to be locally compact
- Theorem (Glebov, Klimošová, K.):
  $T(W)$ can have a part homeomorphic to $[0,1]^\infty$
- Theorem (Cooper, Kaiser, K., Noel):
  $\exists$ finitely forcible W such that every ε-regular partition has at least $2^{\varepsilon^{-2}/\log\log\varepsilon^{-1}}$ parts (for inf. many $\varepsilon \to 0$).
- Theorem (Cooper, K., Martins):
  Every graphon is a subgraphon of a finitely forcible graphon.

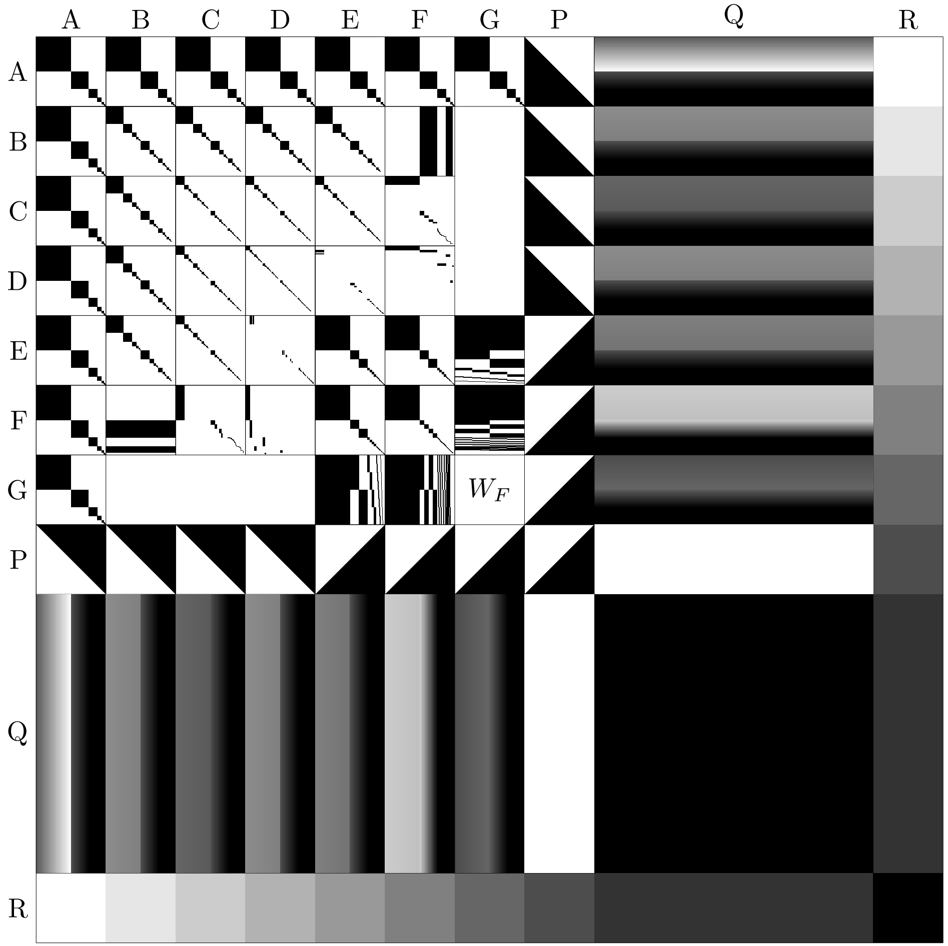
Rademacher graphon



Non-regular graphon

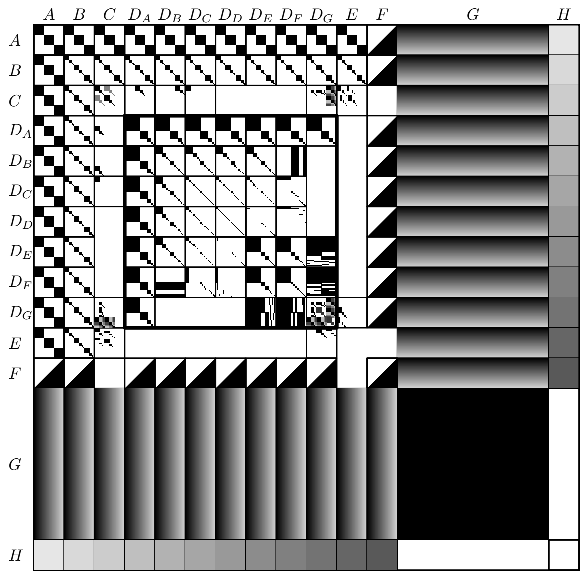


Universal construction



Main result

- Theorem (Grzesik, K., Lovász Jr.)
  $\exists$ graphon family $\mathcal{W}$, graphs H_i, reals d_i, $i = 1, \ldots, m$
  $W \in \mathcal{W} \Leftrightarrow d(H_i, W) = d_i$ for $i = 1, \ldots, m$
  no graphon in $\mathcal{W}$ is finitely forcible



Some details of the proof

- graphons $W_P(\vec{z})$, $\vec{z} \in [0,1]^{\mathbb{N}}$
  $\vec{z}$ satisfies polynomial inequalities in P (e.g. $z_1 + z_2^2 \le 1$)
- $\vec{z}$ constrained to be from $Z \subseteq [0,1]^{\mathbb{N}}$ such that
  $d(H_1, W_P(\vec{z})) = f_1(z_1, z_2)$
  $d(H_2, W_P(\vec{z})) = f_2(z_1, z_2, z_3, z_4, z_5)$
  $d(H_3, W_P(\vec{z})) = f_3(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9)$
- the set Z is non-trivial
  there exists a bijective map from $[0,1]^{\mathbb{N}}$ to Z such that
  $(x_1) \to (z_1, z_2)$, $(x_1, x_2) \to (z_1, z_2, z_3, z_4, z_5)$, etc.

Some details of the proof

- graphons $W_P(\vec{z})$, $\vec{z} \in [0,1]^{\mathbb{N}}$
  $\vec{z}$ satisfies polynomial inequalities in P (e.g. $z_1 + z_2^2 \leq 1$)
- independent of P: there exist graphs $H_1, \ldots, H_k$
  there exist polynomials $q_1, \ldots, q_\ell$ in $d(H_i, W)$
- for every P: there exist reals $\alpha_1, \ldots, \alpha_\ell$
  $W_P(\vec{z})$ are precisely graphons satisfying $q_i = \alpha_i$
- analysis of the dependance of $d(H_i, W_P(\vec{z}))$ on P
  approximation of inverse maps by polyn. inequalities

Possible extensions

- techniques universal to prove more general results
  equalize other functions than subgraph densities
- Theorem (Grzesik, K., Lovász Jr.)
  $\exists$ graphon family $\mathcal{W}$, graphs H_i, reals d_i, $i = 1, \ldots, m$
  $W \in \mathcal{W} \Leftrightarrow d(H_i, W) = d_i$ for $i = 1, \ldots, m$
  no graphon in $\mathcal{W}$ is finitely forcible
  all graphons in $\mathcal{W}$ have the same entropy
- extremal problems with no typical structure

Thank you for your attention!