The maximum size of a partial spread in a finite vector space

Esmeralda Năstase
Xavier University

Joint work with Papa Sissokho

June 12, 2017
- \( V = V(n, q) \) the \textit{vector space} of dimension \( n \) over GF\((q)\).

- A \textit{partial} \( t \)-\textit{spread} of \( V \) is a collection of subspaces \( \{W_1, \ldots, W_k\} \subseteq V \) s.t.
  
  - \( \text{dim}(W_1) = \ldots \text{dim}(W_k) = t \)
  
  - \( W_i \cap W_j = \{0\} \) for \( i \neq j \).
- $V = V(n, q)$ the vector space of dimension $n$ over $\text{GF}(q)$.

- A partial $t$-spread of $V$ is a collection of subspaces $\{W_1, \ldots, W_k\} \subseteq V$ s.t.
  - $\dim(W_1) = \ldots \dim(W_k) = t$
  - $W_i \cap W_j = \{0\}$ for $i \neq j$.

- If $W_1 \cup \cdots \cup W_k$ contains all the vectors of $V$, then it is called a $t$-spread.
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If \( W_1 \cup \ldots \cup W_k \) contains all the vectors of \( V \), then it is called a \( t \)-spread.

André (1954), Segre (1964)- A \( t \)-spread of \( V \) exists if and only if \( t \mid n \).
Question

- What is the maximum size of a partial $t$-spread of $V$?
Applications

- error-correcting codes
- orthogonal arrays
- \((s, k, \lambda)\)-nets
- subspace codes
Let $\mu_q(n, t) =$ maximum size of a partial $t$-spread of $V$, and let

$$\ell_q(n, t) = \frac{q^n - q^{t+r}}{q^t - 1}.$$

Conjecture (Hong and Patel, 1972)

Let $n = kt + r$, and $0 \leq r < t$. Then

$$\mu_q(n, t) = \frac{q^n - q^{t+r}}{q^t - 1} + 1 = \ell_q(n, t) + 1.$$
Theorem (Hong and Patel, 1972; Beutelspacher, 1975)

Let \( n = kt + r \), and \( 0 \leq r < t \). Then

\[
\mu_q(n, t) \geq \frac{q^n - q^{t+r}}{q^t - 1} + 1 = \ell_q(n, t) + 1.
\]
Theorem (Drake and Freeman, 1979)

Let $n = kt + r$, and $0 \leq r < t$. Then

$$\mu_q(n, t) \leq \frac{q^n - q^r}{q^t - 1} - \lfloor \omega \rfloor - 1 = \ell_q(n, t) + q^r - \lfloor \omega \rfloor - 1,$$

where

$$2\omega = \sqrt{4q^t(q^t - q^r) + 1} - (2q^t - 2q^r + 1).$$
Theorem (André, 1954, Segre, 1964 \((r = 0)\); Hong and Patel, 1972 \((r = 1, q = 2)\); Beutelspacher, 1975 \((r = 1, q > 2)\))

Let \(n = kt + r\), and \(r \in \{0, 1\}\). Then

\[
\mu_q(n, t) = \frac{q^n - q^{t+r}}{q^t - 1} + 1 = \ell_q(n, t) + 1.
\]
Theorem (El-Zanati, Jordon, Seelinger, Sissokho, and Spence, 2012)

If \( r = 2, \ t = 3, \) and \( q = 2, \) then

\[
\mu_2(n, 3) = \frac{2^n - 2^5}{7} + 2 = \ell_2(n, 3) + 2.
\]
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If \( r = 2, \ t = 3, \) and \( q = 2, \) then

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\]

Theorem (Kurz, 2016)

If \( t > 3 \) and \( q = r = 2, \) then

\[
\mu_2(n, t) = \frac{2^n - 2^{t+2}}{2t - 1} + 1 = \ell_2(n, t) + 1.
\]
Theorem (N., Sissokho, 2017+)

Let \( n = kt + r \), and \( 0 \leq r < t \). If \( t > \frac{q^r - 1}{q - 1} \), then

\[
\mu_q(n, t) = \frac{q^n - q^{t+r}}{q^t - 1} + 1 = \ell_q(n, t) + 1.
\]
A **subspace partition** or **partition** \( \mathcal{P} \) of \( V \), is a collection of subspaces \( \{ W_1, \ldots, W_k \} \) s.t.

- \( V = W_1 \cup \cdots \cup W_k \)
- \( W_i \cap W_j = \{0\} \) for \( i \neq j \).

Let \( \mathcal{P} \) be any partition of \( V \). We say \( \mathcal{P} \) has **type** \([t^{n_t}, \ldots, 1^{n_1}]\), if there are \( n_i > 0 \) subspaces of \( \dim i \) in \( \mathcal{P} \), and \( 1 < \cdots < t \).
Lemma 1 - (Heden and Lehmann, 2013)
Let $\mathcal{P}$ be a partition of $V$ of type $[t^{n_t}, \ldots, 1^{n_1}]$ and let $b_{H,i}$ be the number of $i$-subspaces in $\mathcal{P}$ that are in a hyperplane $H$ of $V$. Then

$$|\mathcal{P}| = 1 + \sum_{i=1}^{t} b_{H,i} q^i$$
Lemma 2 - (Heden and Lehmann, 2013)

Let $P$ be a partition of $V$ of type $[t^{n_t}, \ldots, 1^{n_1}]$ and let $b_{H,i}$ be the number of $i$-subspaces in $P$ that are in a hyperplane $H$ of $V$. Then

1. $\sum_{H \in \mathcal{H}} H = \frac{q^n - 1}{q - 1}$

2. $\sum_{H \in \mathcal{H}} b_{H,i} H = n_i \frac{q^{n-i} - 1}{q - 1}$
Main Lemma - (N. and Sissokho, 2017+)

Let $n = kt + r$, and $0 \leq r < t$. If $t > \frac{q^r - 1}{q - 1}$, then

$$\mu_q(n, t) \leq \frac{q^n - q^{t+r}}{q^t - 1} + 1 = \ell_q(n, t) + 1.$$
For any integer $i \geq 1$, let $\Theta_i = \frac{q^i - 1}{q - 1}$ and $\delta_i = \frac{q^i - 2q^{i-1} + 1}{q - 1}$. Then

$\blacktriangleright$ $0 < \delta_i < q^{i-1}$ and $\frac{\delta_i}{q} < \delta_{i-1}$. 

E. Năstase

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Then

- $0 < \delta_i < q^{i-1}$ and $\frac{\delta_i}{q} < \delta_{i-1}$.

**Proof Sketch:**

- Assume that $\mu_q(n, t) > \ell_q(n, t) + 1$. Then $V$ has a partial $t$-spread of size $\ell_q(n, t) + 2$. 
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$\nabla$ $0 < \delta_i < q^{i-1}$ and $\frac{\delta_i}{q} < \delta_{i-1}$.

Proof Sketch:

$\nabla$ Assume that $\mu_q(n, t) > \ell_q(n, t) + 1$. Then $V$ has a partial $t$-spread of size $\ell_q(n, t) + 2$.

$\nabla$ $\exists P_0$ of $V(n, q)$ of type $[t^{n_t}, 1^{n_1}]$, where

\[
\begin{align*}
n_t &= \ell_q(n, t) + 2 \\
n_1 &= \left(\frac{q^r - 1}{q - 1} - 1\right)q^t + \frac{q^{t+1} - 2q^t + 1}{q - 1} = (\Theta_r - 1)q^t + \delta_{t+1}.
\end{align*}
\]
We use induction on $j$, for $0 \leq j \leq \Theta_r - 1$:

- **Base Case:** $H_0 = V(n, q)$ and $\mathcal{P}_0 = \{H_0\}$
We use induction on $j$, for $0 \leq j \leq \Theta_r - 1$:

- **Base Case:** $H_0 = V(n, q)$ and $\mathcal{P}_0 = \{H_0\}$
- **Inductive Step:** there exists $\mathcal{P}_j$ of $H_j \cong V(n - j, q)$ of type

$$[t^{m_{j,t}}, (t - 1)^{m_{j,t-1}}, \ldots, (t - j)^{m_{j,t-j}}, 1^{m_{j,1}}],$$

where $m_{j,t}, \ldots, m_{j,t-j}, m_{j,1}$, and $c_j$ are nonnegative integers s.t.

$$\sum_{i=t-j}^{t} m_{j,i} = n_t = \ell_q(n, t) + 2,$$

and

$$m_{j,1} = c_j q^{t-j} + \delta_{t+1-j}, \text{ and } 0 \leq c_j \leq \Theta_r - 1 - j.$$
Using Lemma 2,

\[ b_{avg,1} = \frac{\sum_{H \in \mathcal{H}} b_{H,1}}{\sum_{H \in \mathcal{H}} H} \] < \frac{c_j q^{t-j} + \delta_{t+1-j}}{q} < c_j q^{t-j-1} + \delta_{t-j}. \]
Using Lemma 2,

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\beta_{avg,1} = \frac{\sum_{H \in \mathcal{H}} b_{H,1} H}{\sum_{H \in \mathcal{H}} H} < \frac{c_j q^{t-j} + \delta_{t+1-j}}{q} < c_j q^{t-j-1} + \delta_{t-j}.
\]

\[\implies \exists \text{ a hyperplane } H_{j+1} \text{ of } H_j \text{ with }\]

\[
b_{H_{j+1},1} \leq \beta_{avg,1} < c_j q^{t-j-1} + \delta_{t-j},
\]
Using Lemma 2,

\[ b_{\text{avg},1} = \frac{\sum_{H \in \mathcal{H}} b_{H,1} H}{\sum_{H \in \mathcal{H}} H} < \frac{c_j q^{t-j} + \delta_{t+1-j}}{q} < c_j q^{t-j-1} + \delta_{t-j}. \]

\[ \implies \exists \text{ a hyperplane } H_{j+1} \text{ of } H_j \text{ with } \]

\[ b_{H_{j+1},1} \leq b_{\text{avg},1} < c_j q^{t-j-1} + \delta_{t-j}, \]

and applying Lemma 1,

\[ m_{j+1,1} = b_{H_{j+1},1} = c_{j+1} q^{t-j-1} + \delta_{t-j}, \text{ and } 0 \leq c_{j+1} \leq \Theta_r - 2 - j. \]
Thus, we let $\mathcal{P}_{j+1}$ be the subspace partition of $H_{j+1}$ defined by:

$$\mathcal{P}_{j+1} = \{ W \cap H_{j+1} : W \in \mathcal{P}_j \}.$$
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Therefore, $\mathcal{P}_{j+1}$ is a subspace partition of $H_{j+1}$ of type

$$[t^{m_{j+1,t}},(t-1)^{m_{j+1,t-1}}, \ldots, (t-j-1)^{m_{j+1,t-j-1}}, 1^{m_{j+1,1}}], \quad (1)$$

where $m_{j+1,t}, m_{j+1,t-1}, \ldots, m_{j+1,t-j-1}$ satisfy

$$\sum_{i=t-j-1}^{t} m_{j+1,i} = \sum_{i=t-j}^{t} m_{j,i} = n_t = \ell_q(n,t) + 2, \quad \text{and} \quad (2)$$

$$m_{j+1,1} = b_{H_{j+1,1}} = c_{j+1} q^{t-j-1} + \delta_{t-j}, \quad \text{and} \quad 0 \leq c_{j+1} \leq \Theta_r - 2 - j.$$
Final Step, \( j = \Theta_r - 1 \):

- \( c_{\Theta_r-1} = 0 \) and thus, \( m_{\Theta_r-1,1} = \delta_{t+2-\Theta_r} \) in \( \mathcal{P}_{\Theta_r-1} \) of \( H_{\Theta_r-1} \).
Final Step, $j = \Theta_r - 1$:

- $c_{\Theta_r - 1} = 0$ and thus, $m_{\Theta_r - 1, 1} = \delta_{t+2-\Theta_r}$ in $P_{\Theta_r - 1}$ of $H_{\Theta_r - 1}$.

- Using the averaging argument, there exists a hyperplane $H^*$ of $H_{\Theta_r - 1}$ with

$$b_{H^*, 1} \leq b_{\text{avg}, 1} < \delta_{t+1-\Theta_r}.$$
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- Using Lemma 1 on the partition $\mathcal{P}_{\Theta_r-1}$ and the hyperplane $H^*$ of $H_{\Theta_r-1}$ we obtain that 
  $$b_{H^*,1} \geq \delta_{t+1-\Theta_r},$$

which is a contradiction.
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- $c_{\Theta_r - 1} = 0$ and thus, $m_{\Theta_r - 1,1} = \delta_{t+2-\Theta_r}$ in $P_{\Theta_r - 1}$ of $H_{\Theta_r - 1}$.
- Using the averaging argument, there exists a hyperplane $H^*$ of $H_{\Theta_r - 1}$ with
  \[ b_{H^*,1} \leq b_{\text{avg},1} < \delta_{t+1-\Theta_r}. \]

- Using Lemma 1 on the partition $P_{\Theta_r - 1}$ and the hyperplane $H^*$ of $H_{\Theta_r - 1}$ we obtain that
  \[ b_{H^*,1} \geq \delta_{t+1-\Theta_r}, \]
  which is a contradiction.

\[ \implies \mu_q(n, t) \leq \ell_q(n, t) + 1 \quad \square. \]
Theorem (N., Sissokho, 2017+)

Let $n = kt + r$, and $0 \leq r < t$. If $t > \frac{q^r - 1}{q - 1}$, then

$$\mu_q(n, t) = \frac{q^n - q^{t+r}}{q^t - 1} + 1 = \ell_q(n, t) + 1.$$
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Proof.

- \( r = 0 \): André’s, Segre’s Theorem.
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**Proof.**

- \( r = 0 \): André’s, Segre’s Theorem.
- \( r = 1 \): Hong and Patel’s, and Beutelspacher’s Theorem.
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Let $n = kt + r$, and $0 \leq r < t$. If $t > \frac{q^r - 1}{q - 1}$, then

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Proof.

- $r = 0$: André’s, Segre’s Theorem.
- $r = 1$: Hong and Patel’s, and Beutelspacher’s Theorem.
- $r \geq 2$: LB for $\mu_q(n, t)$, by Hong and Patel’s and Beutelspacher’s Theorem and UB for $\mu_q(n, t)$, by Main Lemma, are equal.
Theorem (N., Sissokho, 2017+)

Let \( n = kt + r \), and \( 0 \leq r < t \). If \( r \geq 2 \) and \( t = \Theta_r \), then

\[
\mu_q(n, t) \leq \ell_q(n, t) + q.
\]
Hypergraphs

Let $\mathcal{H}_q(n, t)$ be the hypergraph whose vertices are the 1-subspaces of $V(n, q)$ and whose edges are its $t$-subspaces.
Hypergraphs

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Then $\mathcal{H}_q(n, t)$ is a $(q^t - 1)/(q - 1)$-uniform hypergraph.
Let $\mathcal{H}_q(n, t)$ be the hypergraph whose vertices are the 1-subspaces of $V(n, q)$ and whose edges are its $t$-subspaces.

Then $\mathcal{H}_q(n, t)$ is a $(q^t - 1)/(q - 1)$-uniform hypergraph.

By our Theorem, if $n = kt + r$, $0 \leq r < t$, and $t > \frac{q^r - 1}{q - 1}$, then the maximum size of a matching in $\mathcal{H}_q(n, t)$ is

$$\frac{q^n - q^{t+r}}{q^t - 1} + 1 = \ell_q(n, t) + 1.$$
Question

What is the maximum size of a partial $t$-spread of $V$ if $r \geq 2$ and $t \leq \frac{q^r - 1}{q - 1}$?
Thank you!