The Edge-Independent Spanning Tree Conjecture for $k = 4$

Alex Hoyer

Robin Thomas

School of Mathematics
Georgia Institute of Technology

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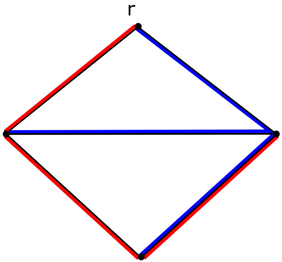
Edge-Independence

Definition (Edge-Independence)

Two subtrees $T_1, T_2 \subset G$ are *edge-independent with root r* if $r \in V(T_1) \cap V(T_2)$, and for every other $v \in V(T_1) \cap V(T_2)$, the unique paths between r and v in T_1 and T_2 are edge-disjoint. This definition applies pairwise to larger sets of trees.



The Edge Conjecture

Edge Conjecture (Itai and Rodeh, 1984)

If G is a k-edge-connected graph and $r \in V(G)$, then there exists a set of k edge-independent spanning trees of G rooted at r.

Known Cases:

- $k = 1$: Trivial
- $k = 2$: Itai and Rodeh, 1984
- $k = 3$: Gopalan and Ramasubramanian, 2011 (Alternate proof by Schlipf and Schmidt, 2016)
- $k = 4$: H. and Thomas, 2017+

Application

Network Redundancy:

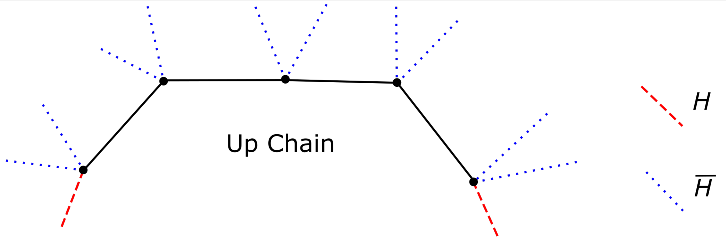
- Suppose G is a network, r is a server, and all other vertices are clients.
- k-edge-connectivity means that r can communicate with individual clients, while withstanding up to $k-1$ edge failures.
- k edge-independent spanning trees mean that r can broadcast to all clients simultaneously, while withstanding up to $k-1$ edge failures.

Chain Definitions

Definition (Up Chain)

An *up chain* of G with respect to the pair of subgraphs $(H, \overline{H})$ is a subgraph which is either:

1. A path such that every vertex is either r or has degree at least two in $\overline{H}$, and the ends are either r or are in H, OR
2. A cycle such that every vertex is either r or has degree at least two in $\overline{H}$, and some vertex is either r or has degree at least two in H.



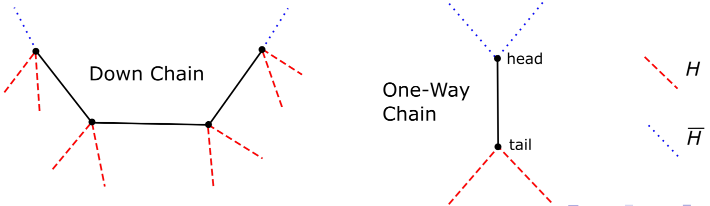
Chain Definitions

Definition (Down Chain)

A *down chain* of G with respect to the pair of subgraphs $(H, \overline{H})$ is an up chain with respect to $(\overline{H}, H)$.

Definition (One-Way Chain)

A *one-way chain* of G with respect to the pair of subgraphs $(H, \overline{H})$ is a subgraph induced by an edge uv, such that u is either r or has degree at least two in H, and v is either r or has degree at least two in $\overline{H}$. We call u the *tail* of the chain and v the *head*.



The Chain Decomposition

Definition (Chain Decomposition)

Let $G_0, G_1, \ldots, G_m$ be a sequence of subgraphs of G. Denote $H_i = G_0 \cup G_1 \cup \cdots \cup G_{i-1}$ and $\overline{H_i} = G_{i+1} \cup G_{i+2} \cup \cdots \cup G_m$, so that H_0 and $\overline{H_m}$ are the null graph. We say that the sequence $G_0, G_1, \ldots, G_m$ is a *chain decomposition* of G rooted at r if:

1. The sets $E(G_0), E(G_1), \ldots, E(G_m)$ partition $E(G)$, AND
2. For $i = 0, \ldots, m$, the subgraph G_i is either an up chain, a down chain, or a one-way chain with respect to the subgraphs $(H_i, \overline{H_i})$.

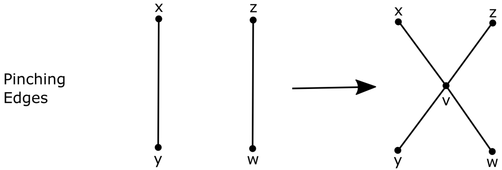
The Chain Decomposition is analogous to the Planar Chain Decomposition of Curran, Lee, and Yu.

Definitions

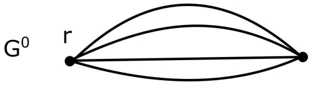
Definition (Mader Operation)

A *Mader operation* is one of the following operations:

- Add an edge between two (not necessarily distinct) vertices.
- Consider two distinct edges, say e_1 with ends x, y and e_2 with ends z, w, and "pinch" them as follows:
  - Delete the edges e_1 and e_2.
  - Add a new vertex v.
  - Add the new edges e_x, e_y, e_z, e_w with one end v and the other end x, y, z, w respectively.



The Mader Construction



Theorem (Special Case of Mader, 1978)

A graph G is 4-edge-connected if and only if, for any $r \in V(G)$, one can construct G from G^0 using Mader operations.

Mader's full result applies to all edge connectivities, but the general form of the Mader operations is more complex.

Finding a Chain Decomposition

Theorem (H. and Thomas, 2017+)

Suppose G is a 4-edge-connected graph and $r \in V(G)$. Then G has a chain decomposition rooted at r.

Proof Idea:

- Consider a Mader construction of G.
- Show that the chain decomposition can be maintained through each step of the construction.
- Requires case analysis based on chain types.

Finding Edge-Independent Spanning Trees

Theorem (H. and Thomas, 2017+)

Suppose G is a 4-edge-connected graph, $r \in V(G)$, and G has a chain decomposition rooted at r. Then there is a set of four edge-independent spanning trees of G rooted at r.

Proof idea:

- Use the chain structure to define two edge numberings of G.
- For each vertex, assign an incident edge to each tree, using the numberings and chain index.
- Each tree is monotonic in chain index and strictly monotonic in one of the edge numberings, which gives independence.

The Vertex Conjecture

There is an analogous problem relating k-connectivity to independent trees, in which paths back to r are internally vertex-disjoint, rather than edge-disjoint.

Vertex Conjecture (Itai and Rodeh, 1984)

If G is a k-connected graph and $r \in V(G)$, then there exists a set of k independent spanning trees of G rooted at r.

Known Cases:

- $k = 1$: Trivial
- $k = 2$: Itai and Rodeh, 1984
- $k = 3$: Independently by Cheriyan and Maheshwari, 1988, and Zehavi and Itai, 1989
- G planar, any k: Huck, 1994
- $k = 4$: Curran, Lee, Yu, 2005-2006

Other Problems

Implication:

- Does the Vertex Conjecture imply the Edge Conjecture?
  - Attempted proof (Khuller and Schieber; 1992) replaced vertices with cliques to transform a k-edge-connected graph into a k-connected graph.
  - Proof was incorrect, but technique inspired a proof for $k = 3$ (Gopalan and Ramasubramanian; 2011) by replacing vertices with paths, rather than cliques.
  - General case still unknown.
- Does the Edge Conjecture imply the Vertex Conjecture?

Generalization:

Can we span a subset of $V(G)$ with k (edge)-independent trees, if that subset is k-(edge)-connected to r?

- Vertex version is true for subsets of size at most 2.

Thank you