What is a Laminar Matroid?

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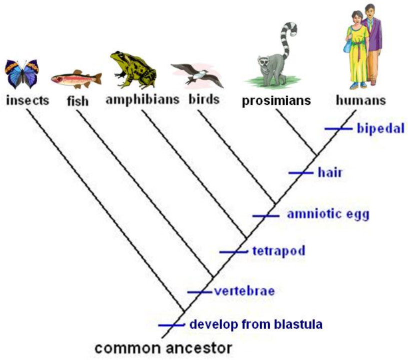
Department of Mathematics
Louisiana State University
Baton Rouge Louisiana

Cumberland Conference, May, 2017

Laminar Family



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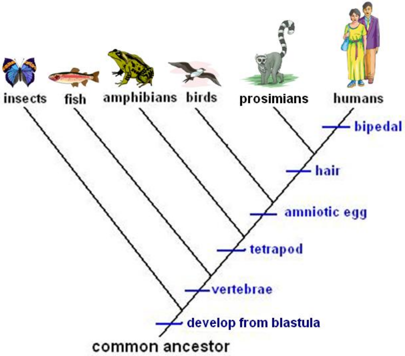
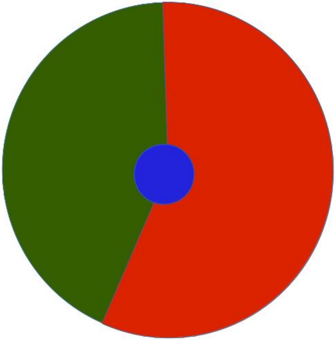
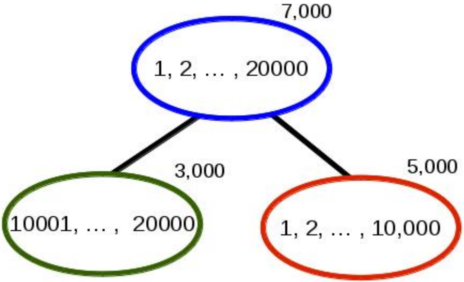


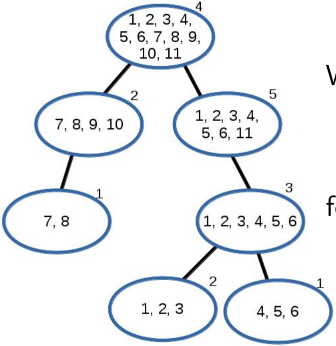
A family $\mathscr{A}$ of sets is laminar if for all $A_1, A_2 \in \mathscr{A}$, either $A_1 \cap A_2 = \emptyset$ or $A_i \subseteq A_j$, for distinct $i, j \in \{1, 2\}$.

Island Example





Independence

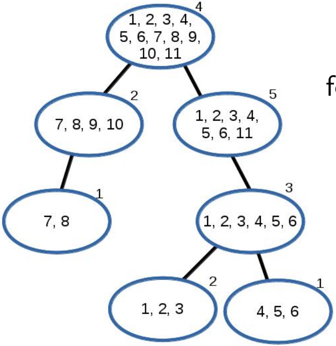


We call a set I independent If

$$|I \cap A| \leq c(A)$$

for each $A \in \mathscr{A}$.

Independence



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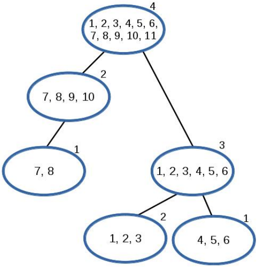

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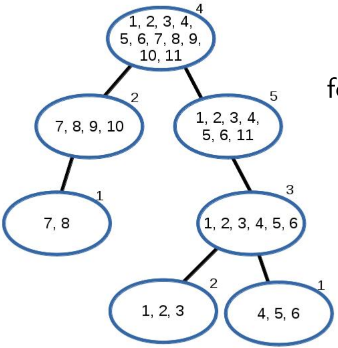

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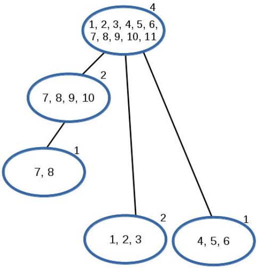

Geometric Presentation

The following are dependent sets.

- Two dots on a point.

Geometric Presentation

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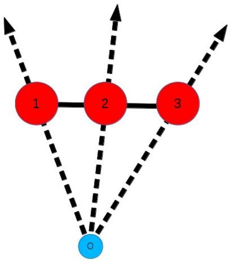
- Three dots on a line.



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- Four dots on a plane.

Geometric Presentation

The following are dependent sets.

- Two dots on a point.
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- Four dots on a plane.
- Five dots in space.

What is a Minor?

Delete e: Remove e.

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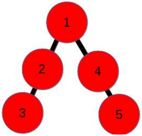
Delete 1



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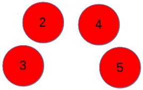

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Delete e: Remove e.

Contract e: Project from e onto a hyperplane that does not contain e.

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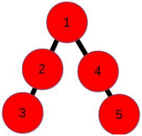

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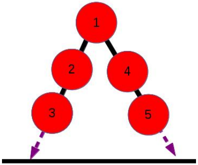

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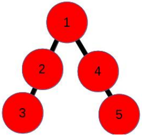

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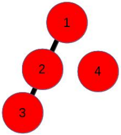

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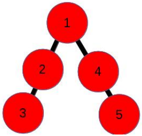

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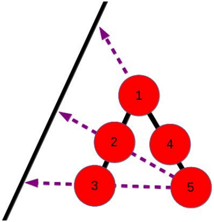

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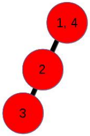
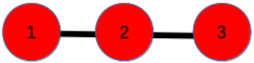
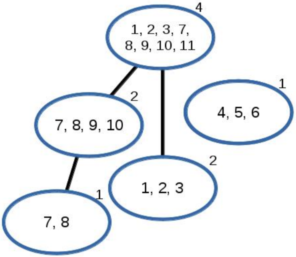


Matroids

A *matroid* is a nice notion of independence and dependence in a finite set.

Geometric Representation of a Laminar Matroid





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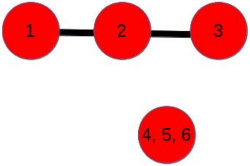

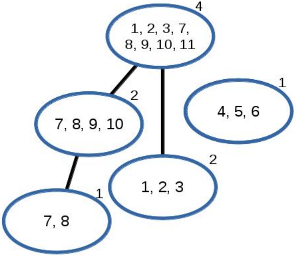

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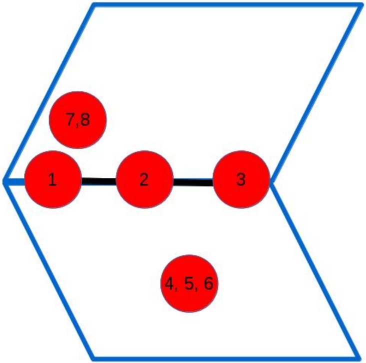

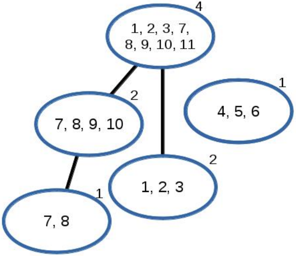

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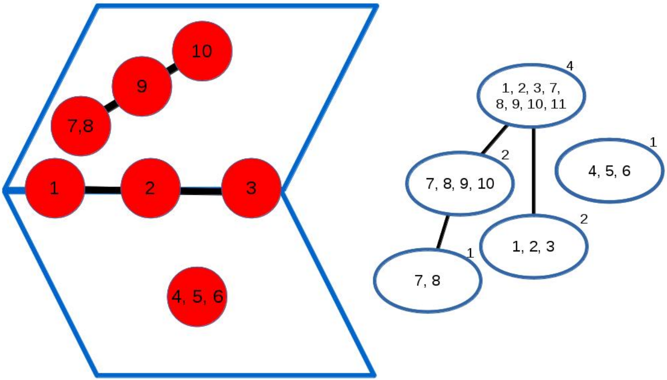

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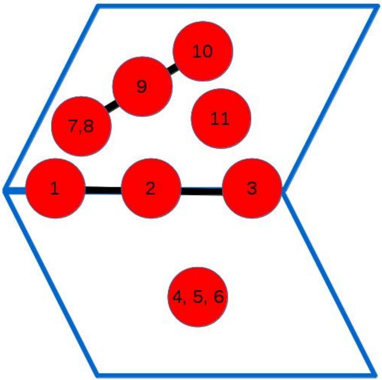
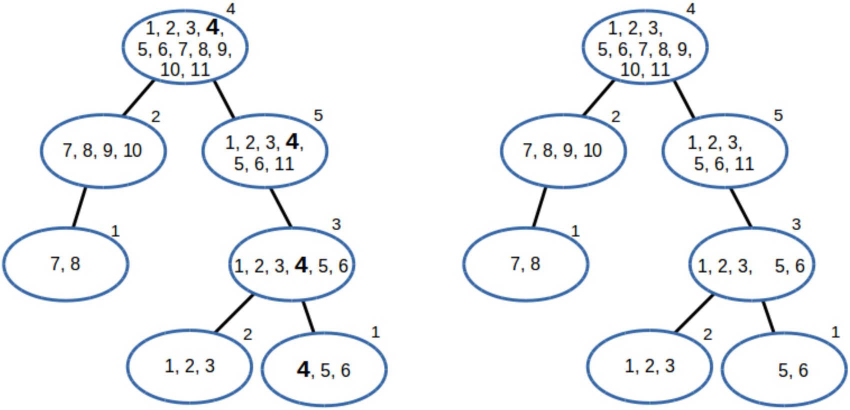


Minors of Laminar Matroids



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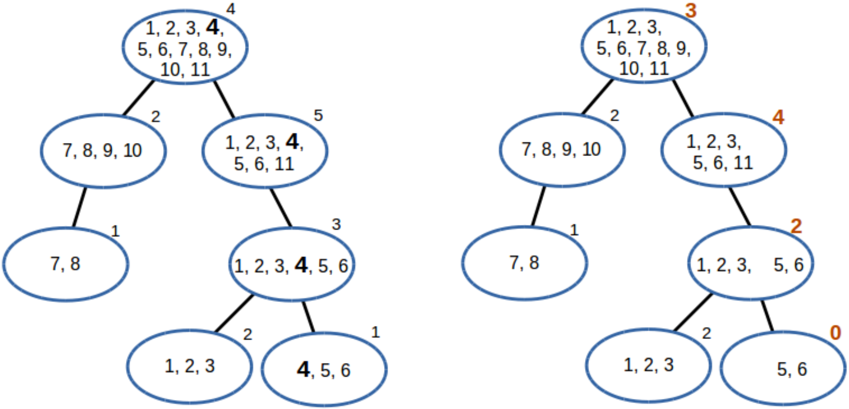
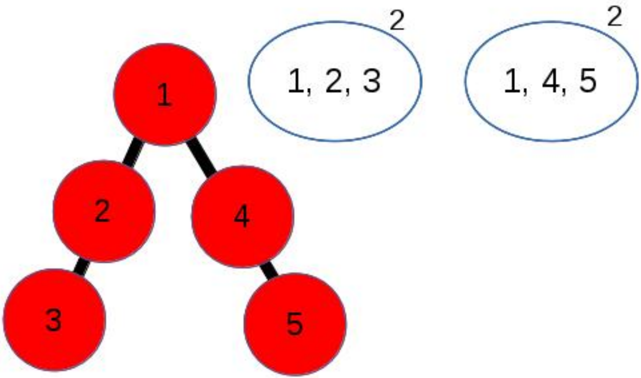
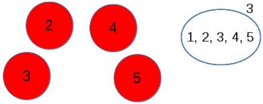
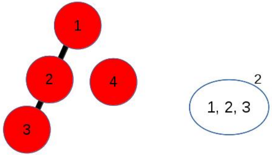


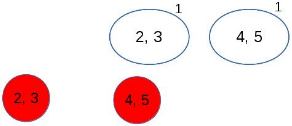
Not Laminar

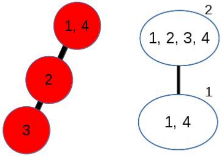


An Excluded Minor







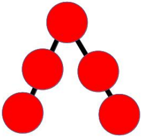


The Excluded Minors

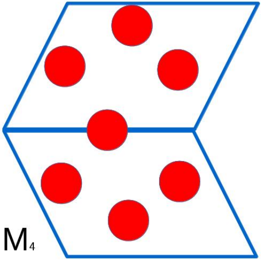
Theorem

The excluded minors of laminar matroids are:

M_3



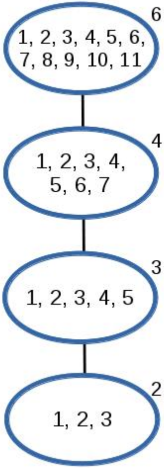
M_4



. . .

Nested Matroids

These are laminar matroids which have a representation where the family $\mathscr{A}$ looks like a path.



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Theorem (O., Prendergast, and Row)

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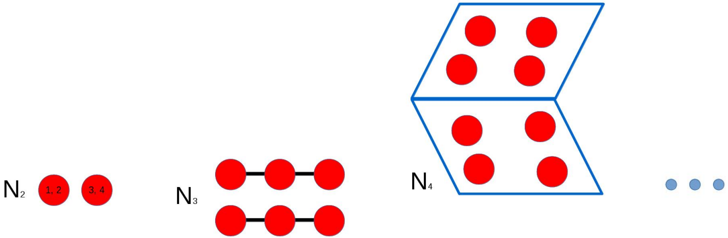
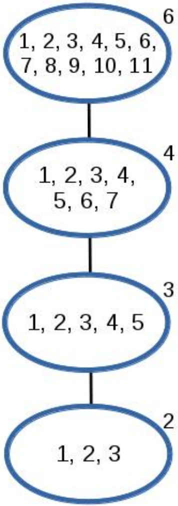


Another Look at Circuits



A circuit is a minimal dependent set.

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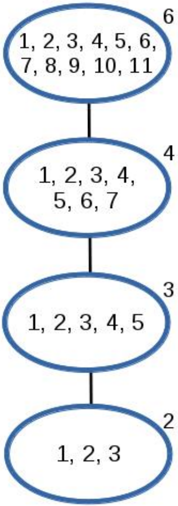

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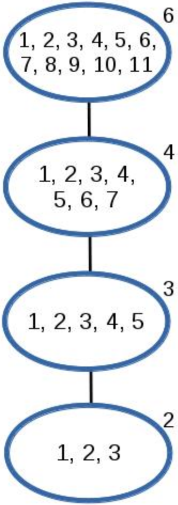

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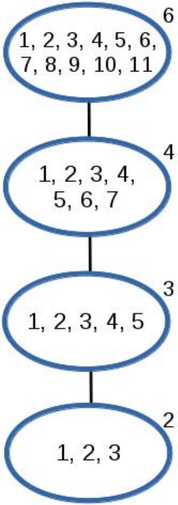

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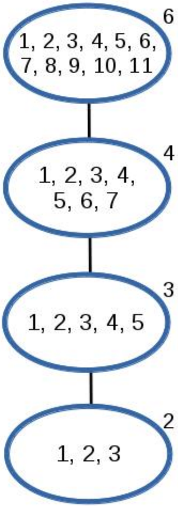

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etc.

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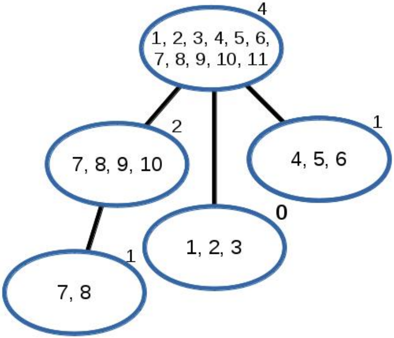
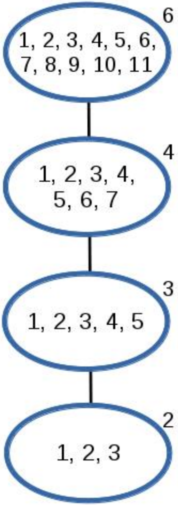


A Circuit is a minimally dependent set.

Theorem

For a laminar matroid $M(E, \mathscr{A}, c)$, a set C is a circuit if it is a minimal set such that $C \subseteq A$ and $|C| = c(A) + 1$ for some $A \in \mathscr{A}$.

Hamiltonian Flats



If $X \subseteq E$, we define $cl(X)$, the closure of X, to be

$$X \cup \{e : \text{there is a circuit } C \text{ with } e \in C \subseteq e \cup X\}.$$

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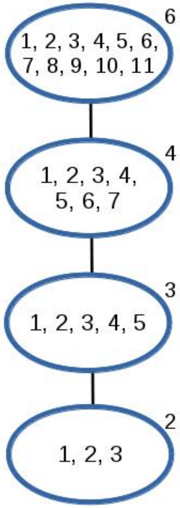

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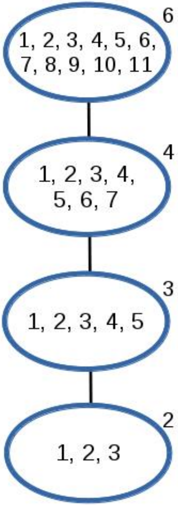

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Our Hamiltonian flats are:

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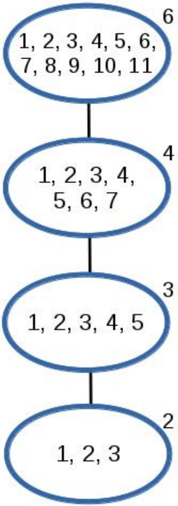

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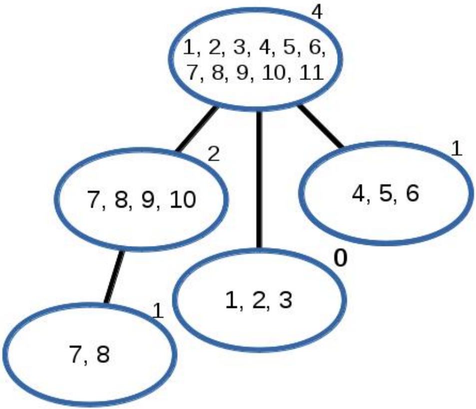
Hamiltonian Flats



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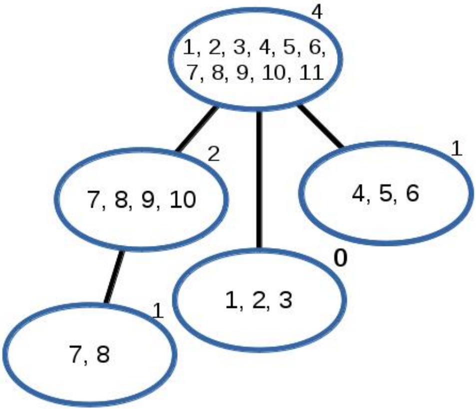

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Hamiltonian Flats

Theorem

A matroid is nested if and only if its Hamiltonian flats form a chain under inclusion.

Hamiltonian Flats

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Theorem

A matroid M is laminar if and only if, for every independent set X of size 1, the Hamiltonian flats of M containing X form a chain under inclusion.

A Generalization

Let $\mathscr{M}_k$ be the class of all matroids such that for every independent set X of size k, the Hamiltonian flats of M containing X form a chain under inclusion.

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- $\mathscr{M}_2$ is minor-closed, and its excluded minors are known.
- $\mathscr{M}_3$ is minor-closed.
- $\mathscr{M}_k$ is not minor-closed for any $k \geq 4$.

Thank You

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