A property on reinforcing edge-disjoint spanning hypertrees in uniform hypergraphs

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- "Dynamic" process

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Reinfore edge-disjoint spanning trees

The following theorem was conjectured by Payan in 1986 and proved by Lai, Lai and Payan in 1996.

- Theorem
  If G is a simple graph with $|E(G)| = k(|V(G)| - 1)$ and $\varepsilon(G) > 0$, then there exist $e \in E(G)$ and $e' \in E(G^c)$ such that $\varepsilon(G - e + e') < \varepsilon(G)$.

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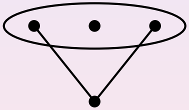

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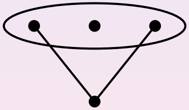

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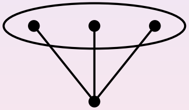

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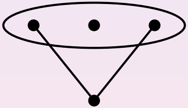

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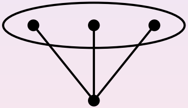

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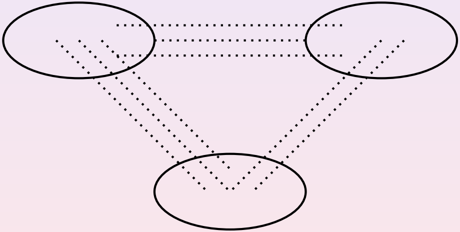
- If in addition, $|E(H)| = |V(H)| - 1$, then H is a hypertree.

Definition

- A hypergraph H is k-partition-connected if $e(\pi) \geq k(|\pi| - 1)$ for every partition π of $V(H)$, where $e(\pi)$ denotes the number of edges intersecting at least 2 parts of π.



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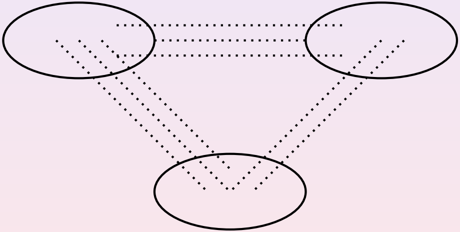

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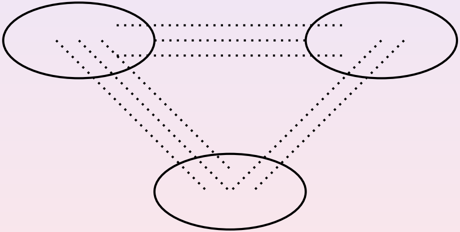

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- $c(H)$ denotes the number of maximal partition-connected components in H.

Packing spanning hypertrees

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  For a graph, k-partition-connectedness is equivalent to k edge-disjoint spanning trees.
- Theorem (Frank, Király and Kriesell, 2003)
  A hypergraph H is k-partition-connected if and only if H has k edge-disjoint spanning hypertrees.

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- Lemma For any uniform hypergraph H with $|E(H)| = k(|V(H)| - 1)$, we have $\varepsilon(H) = \varepsilon'(H)$.

Sketch of the proof

- Find a sub-hypergraph S of H that contains k edge-disjoint spanning hypertrees, i.e., there is a partition $(Y_1, Y_2, \cdots, Y_k)$ of $E(S)$ such that each part induces a partition-connected sub-hypergraph.

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- The corresponding partition of $E(F)$ from π is denoted by π'. Show $\varepsilon(F) < \varepsilon(H)$.

Thanks