Convex Sets Associated to C^*-Algebras

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Overview

Goal: Define and investigate invariants for a unital separable tracial C*-algebra $\mathfrak{A}$.

These invariants will be convex sets that in some sense generalize the trace space of $\mathfrak{A}$.

They are built out of equivalence classes of $*$-homomorphisms from $\mathfrak{A}$ into certain II_1-factors.

This is an adaptation of a 2011 contruction of Nate Brown involving ultrapowers. Our construction uses no ultrapowers, and these invariants are always separable–Brown's are either nonseparable or trivial.

$\mathfrak{A}$ will denote a unital separable tracial C*-algebra, and R will denote the separable hyperfinite II_1-factor throughout.

Classical Situation: BDF Theory and Ext($\mathfrak{A}$)

BDF theory and (more generally) the theory of Ext($\mathfrak{A}$) are classical examples of placing a nice structure on equivalence classes of $*$-homomorphisms.

Ext($\mathfrak{A}$) is given by the set of unital $*$-monomorphisms $\pi : \mathfrak{A} \to B(H)/K(H)$ modulo $B(H)$-unitary equivalence.

A semigroup structure on Ext($\mathfrak{A}$) is described by the following picture.

$$[\pi] + [\rho] = \left[\begin{pmatrix} \pi & 0 \\ 0 & \rho \end{pmatrix} \right]$$

Preliminaries

Definition

For a separable, unital C*-algebra $\mathfrak{A}$, and a separable II_1-factor N, we define $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N)$ to be the space of unital $*$-homomorphisms $\pi : \mathfrak{A} \to N$ modulo the equivalence relation of weak approximate unitary equivalence (w.a.u.e.).

That is, $[\pi] = [\rho]$ if there is a sequence $\{u_n\}$ of unitaries in N such that for every $a \in \mathfrak{A}$ we have

$$\lim_n ||\pi(a) - u_n \rho(a) u_n^*||_2 = 0$$

where $||x||_2^2 = \tau_N(x^* x)$ for τ_N the unique tracial state on N.

Convex Structure

If the target algebra is McDuff ($M \cong M \otimes R$) then $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ yields a convex structure, rather than a semigroup structure as in $\mathrm{Ext}(\mathfrak{A})$. Picture:

$$t[\pi] + (1-t)[\rho] \rightsquigarrow \left(\begin{array}{c|c} p\pi p & 0 \\ \hline 0 & p^\perp \rho p^\perp \end{array}\right), \tau(p) = t$$

Definition

Let M be a separable McDuff II_1-factor. For $t \in [0,1]$ and $[\pi], [\rho] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$, we define

$$t[\pi] + (1-t)[\rho] := [\sigma_M(\pi \otimes p + \rho \otimes p^\perp)]$$

where $\sigma_M : M \otimes R \to M$ is an isomorphism satisfying $\sigma_M \circ (\mathrm{id}_M \otimes 1_R) \sim \mathrm{id}_M$ and p is a projection in R with $\tau_R(p) = t$.

Non-McDuff? No Problem!

Theorem

Let N be a separable II_1-factor. Given $\pi, \rho : \mathfrak{A} \to N$, consider $\pi \otimes 1_R, \rho \otimes 1_R : \mathfrak{A} \to N \otimes R$. If $\pi \otimes 1_R \sim \rho \otimes 1_R$ then $\pi \sim \rho$.

Consequence: For N non-McDuff,

$$\mathbb{H}\mathrm{om}_w(\mathfrak{A}, N) \subseteq \mathbb{H}\mathrm{om}_w(\mathfrak{A}, N \otimes R)$$

via

$$[\pi] \mapsto [\pi \otimes 1_R]$$

Extreme Points

By a characterization due to Brown and Capraro-Fritz, we may consider $\mathbb{H}\text{om}_w(\mathfrak{A}, M)$ as a closed, bounded (sometimes compact but not always), separable, convex subset of a Banach space.

So what about extreme points?

Extreme Points

Let $\pi : \mathfrak{A} \to M$ be given.

Theorem (A.)

If $[\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ is extreme then $W^*(\pi(\mathfrak{A}))$ is a factor, but the converse fails in general.

$\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M) \subseteq \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M^{\mathcal{U}})$ via constant sequence embedding $M \subset M^{\mathcal{U}}$.

Theorem (Brown, A.)

If $\pi^{\mathcal{U}}(\mathfrak{A})' \cap M^{\mathcal{U}}$ is a factor then $[\pi]$ is extreme in $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$.

Extreme Points in Amenable Cases

Theorem (A.)

If either $\mathfrak{A}$ or M is amenable, then given $\pi : \mathfrak{A} \to M$, TFAE:

- $[\pi]$ *is extreme;*
- $W^*(\pi(\mathfrak{A}))$ *is a factor;*
- $\pi^{\mathcal{U}}(\mathfrak{A})' \cap M^{\mathcal{U}}$ *is a factor.*

Quotients $\to$ Faces

Theorem (A.)

If $J \triangleleft \mathfrak{A}$ is a closed two-sided ideal of $\mathfrak{A}$, then $\mathbb{H}\mathrm{om}_w(\mathfrak{A}/J, M)$ is naturally embedded as a(n exposed) face of $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$.

With the observation that any separable unital C*-algebra is a quotient of $C^*(\mathbb{F}_\infty)$, we get the following corollary.

Corollary (A.)

For any separable unital $\mathfrak{A}$, $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ may be embedded as a face of $\mathbb{H}\mathrm{om}_w(C^*(\mathbb{F}_\infty), M)$.

Connection to Trace Space

Given $[\pi] \in \mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ we get a tracial state on $\mathfrak{A}$ given by

$$\tau_M \circ \pi.$$

The map $[\pi] \mapsto \tau_M \circ \pi$ is well-defined, continuous, and affine.

Natural question: For a fixed M, how much data does $\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ share with $T(\mathfrak{A})$?

Nuclear Case

Theorem (A., Ding-Hadwin)

If $\mathfrak{A}$ is nuclear then for any McDuff M, $\mathbb{H}om_w(\mathfrak{A}, M)$ is affinely homeomorphic to $T(\mathfrak{A})$ via $[\pi] \leftrightarrow \tau_M \circ \pi$.

English Version: All traces of a separable unital nuclear algebra "lift" through any fixed McDuff factor; and the traces "remember" their homomorphisms up to w.a.u.e.

Alternative Characterization of Hyperfiniteness

Let N be a separable, finite, tracial, $R^{\mathcal{U}}$-embeddable von Neumann algebra.

Theorem (Jung)

N is hyperfinite if and only if any two embeddings $\pi, \rho : N \to R^{\mathcal{U}}$ are unitarily conjugate.

(embedding = unital, injective, normal, trace-preserving $*$-homomorphism)

Corollary (A.)

N is hyperfinite if and only if for any separable McDuff II_1-factor M, any two embeddings $\pi, \rho : N \to M$ are weakly approximately unitarily equivalent.

Table of Examples

Property	$\mathfrak{A}$	M
$\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ compact	nuclear	any McDuff
$\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ not compact	$C^*(\Gamma)$ for Γ a non-amenable, residually finite, discrete group	R
$\mathbb{H}\mathrm{om}_w(\mathfrak{A}, M)$ has non extreme point with factorial closure	Dense in $\otimes_{\mathbb{Z}} L(\mathbb{F}_2)$	$\otimes_{\mathbb{Z}} L(\mathbb{F}_2)$
$[\pi] \mapsto \tau_M \circ \pi$ not injective	$C_r^*(\mathbb{F}_2)$	$L(\mathbb{F}_2) \otimes R$
$[\pi] \mapsto \tau_M \circ \pi$ not surjective	$C^*(\mathbb{F}_\infty)$	any McDuff

Minimal Faces in $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$

Theorem (A.)

Let $\pi : N \to R^{\mathcal{U}}$ be an embedding such that the center of $\pi(N)' \cap R^{\mathcal{U}}$ has dimension $n < \infty$. Then the minimal face in $\mathbb{H}om(N, R^{\mathcal{U}})$ containing $[\pi]$ is an n-vertex simplex.

Conjecture (A.)

If $[\pi]$ is a nontrivial average of $n(< \infty)$ extreme points in $\mathbb{H}om(N, R^{\mathcal{U}})$ then the center of $\pi(N)' \cap R^{\mathcal{U}}$ has dimension n. (And thus the minimal face containing $[\pi]$ is an n-vertex simplex)

So, for example, the hull of four extreme points in $\mathbb{H}\mathrm{om}(N, R^{\mathcal{U}})$ cannot be a square–it has to be a tetrahedron.

Thanks!

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