Unitary Dilation of Freely Independent Contractions

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Sz.-Nagy Dilation Theorem

Theorem (Sz.-Nagy)

Given a contraction $T \in B(H)$, there is a Hilbert space K containing H and a unitary $U \in B(K)$ such that

$$T^n = P_H U^n|_H$$

for $n \in \mathbb{Z}$.

Sz.-Nagy Dilation Theorem

Proof (Schäffer).

Let $K = \ell_2(\mathbb{Z}) \otimes H$ and

$$U = \begin{bmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & I & 0 & & & & \\ & & D_{T^*} & T & & & \\ & & -T^* & D_T & 0 & & \\ & & & & I & 0 & \\ & & & & & \ddots & \ddots \end{bmatrix}$$

where for $||A|| \leq 1, D_A = (I - A^*A)^{\frac{1}{2}}$. □

Consequence: von Neumann's Inequality

Theorem (von Neumann)

Given a contraction $T \in B(H)$ and a polynomial p, we have that

$$||p(T)|| \leq ||p(z)||_{\mathbb{D}} = ||p(z)||_{\mathbb{T}}.$$

Proof.

$||p(T)|| = ||P_H p(U)|_H|| \leq ||p(U)|| = ||p(z)||_{\sigma(U)} \leq ||p(z)||_{\mathbb{T}}$. □

Ando's Dilation Theorem

Theorem (Ando)

Given two commuting contractions $S, T \in B(H)$, there is a Hilbert space K containing H and commuting unitaries $U, V \in B(K)$ such that

$$S^m T^n = P_H U^m V^n|_H$$

for $n, m \in \mathbb{N}$.

The above theorem does not hold for three or more commuting contractions. Counterexamples are due to Parrott and Varopoulos.

Sz.-Nagy-Foias Dilation Theorem

Definition

Two operators $S, T \in B(H)$ are said to *doubly commute* if $ST = TS$ and $S^*T = TS^*$.

Theorem (Sz.-Nagy-Foias)

Given n doubly commuting contractions $T_1, \ldots, T_n \in B(H)$, there is a Hilbert space K containing H and doubly commuting unitaries $U_1, \ldots, U_n \in B(K)$ so that

$$T_1^{k_1} \cdots T_n^{k_n} = P_H U_1^{k_1} \cdots U_n^{k_n}|_H$$

for $k_1, \ldots, k_n \in \mathbb{Z}$.

Non-Commutative Probability Space

Definition

A *non-commutative probability space* is given by a pair $(\mathfrak{A}, \varphi)$ where $\mathfrak{A}$ is a unital C^*-algebra and $\varphi \in S(\mathfrak{A})$ is a state on $\mathfrak{A}$.

Doubly Commuting $\leftrightsquigarrow$ Tensor Independence

Definition

Let $(\mathfrak{A}, \varphi)$ be a non-commutative probability space. The elements $T_1, \ldots, T_n \in \mathfrak{A}$ are *tensor independent* or *classically independent* (with respect to φ) if

1. The generated C^*-algebras $C^*(1, T_1), \ldots, C^*(1, T_n)$ pair-wise commute.
2. Given $a_j \in C^*(1, T_j), 1 \leq j \leq n$, we have that

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n).$$

Key observation: $T_1, \ldots T_n$ doubly commute if and only if $C^*(1, T_1), \ldots, C^*(1, T_n)$ pair-wise commute.

Unitary Dilation of Tensor Independent Contractions

Theorem (A.-Ramsey)

Let $T_1, \ldots, T_n$ be tensor independent contractions in the non-commutative probability space $(B(H), \varphi)$. There is a Hilbert space K containing H and unitaries $U_1, \ldots, U_n \in B(K)$ that are tensor independent with respect to $\psi = \varphi \circ Ad(P_H)$ such that

$$T_1^{k_1} \cdots T_n^{k_n} = P_H U_1^{k_1} \cdots U_n^{k_n}|_H$$

for $k_1, \ldots, k_n \in \mathbb{Z}$.

Free Independence

There are other notions of independence in the theory of non-commutative probability. Free independence was introduced by Voiculescu in the 1980's and has been heavily studied ever since.

Definition

Let $(\mathfrak{A}, \varphi)$ be a non-commutative probability space. The elements $T_1, \ldots, T_n \in \mathfrak{A}$ are *freely independent* (with respect to φ) if for $1 \leq j \leq m$, whenever $a_j \in C^*(1, T_{i_j})$ with $i_j \neq i_{j+1}$ and $\varphi(a_j) = 0$ then we have

$$\varphi(a_1 \cdots a_m) = 0.$$

OR: "alternating products of centered elements are centered."

Free Independence

Example

Let Γ be a discrete group with infinite cardinality. We say that n subgroups $\Gamma_1, \ldots, \Gamma_n \leq \Gamma$ are free from one another if for $1 \leq j \leq m$, whenever $g_j \in \Gamma_{i_j}$ with $i_j \neq i_{j+1}$ and $g_j \neq e$ then we have

$$g_1 \cdots g_m \neq e.$$

Consider $C_r^*(\Gamma) \subset B(\ell_2(\Gamma))$ with state φ given by $\varphi(\cdot) := \langle \cdot\, \delta_e | \delta_e \rangle$; observe that $g \neq e \Leftrightarrow \varphi(\lambda(g)) = 0$. So $\Gamma_1, \ldots, \Gamma_n \leq \Gamma$ are free from one another if for $1 \leq j \leq m$, whenever $g_j \in \Gamma_{i_j}$ with $i_j \neq i_{j+1}$ and $\varphi(\lambda(g_j)) = 0$ then we have

$$\varphi(\lambda(g_1) \cdots \lambda(g_m)) = 0.$$

Unitary Dilation of Freely Independent Contractions

Theorem (A.-Ramsey)

Let $T_1, \ldots, T_n$ be freely independent contractions in the non-commutative probability space $(B(H), \varphi)$. There is a Hilbert space K containing H and unitaries $U_1, \ldots, U_n \in B(K)$ that are freely independent with respect to $\psi = \varphi \circ Ad(P_H)$ such that

$$T_{i_1}^{k_1} \cdots T_{i_m}^{k_m} = P_H U_{i_1}^{k_1} \cdots U_{i_m}^{k_m}|_H$$

for $k_1, \ldots, k_m \in \mathbb{N}$.

Proof Sketch

- Let $V_i \in B(K_i)$ be the Schäffer dilation for each T_i.
- Let $\theta_i : C^*(V_i) \to C^*(1, T_i)$ be the ucp map given by

$$p(V_i) \mapsto p(T_i).$$

- By Boca, there is a ucp map (coherent with φ)

$$*_{i=1}^{n}\theta_i : \check{*}_{i=1}^{n} C^*(V_i) \to C^*(1, T_1, \ldots, T_n).$$

- By Stinespring, let K be a Hilbert space containing H and $\pi : \check{*}_{i=1}^{n} C^*(V_i) \to B(K)$ be a $*$-homomorphism be such that

$$*_{i=1}^{n}\theta_i(a) = P_H \pi(a)|_H.$$

- Put $U_i = \pi(V_i)$ for desired unitaries.

More Observations

- Because $C^*(U_i)$ is commutative for each $1 \leq i \leq n$, $\varphi \circ \mathrm{Ad}(P_H)|_{C^*(U_i)}$ is a trace. It is a well-known fact that free independence of the U_i's implies that $\varphi \circ \mathrm{Ad}(P_H)$ is a trace on $C^*(U_1, \ldots, U_n)$.

- If φ is additionally faithful, then

$$(C^*(U_1, \ldots, U_n), \varphi \circ \mathrm{Ad}(P_H) \cong *_{i=1}^{n}(C^*(V_i), \varphi \circ \theta_i).$$

Outlook

It would be nice to obtain a more concrete dilation (think Schäffer).

The Sz.-Nagy Unitary Dilation Theorem can be used for a slick proof of von Neumann's inequality. Can we use free dilation to establish an inequality involving freely independent contractions?

Can other forms of independence in n.c. probability be dilated?

Thanks!

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