

Compare and Discuss to Deepen Algebra Learning

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my.vanderbilt.edu/cems



From Resources Tab:

Select Materials

All of our materials can be downloaded

Materials

Worked Example Pairs (WEPs)

At the core of our materials are the Worked Example Pairs (WEPs). Each WEP shows the mathematical work and dialogue of two hypothetical students as they attempt to solve one or more algebra problems. Our curriculum contains four different types of WEPs, with the types varying in what is being compared and the instructional goal of the comparison.

- Which is better? WEPs show the same problem solved using two different correct strategies, with the goal of understanding which strategy is more efficient or easier than the other.
- When is correct? WEPs show the same problem solved with a correct and incorrect strategy, with the goal of understanding and avoiding common errors.
- How do they differ? WEPs show the same problem solved with two different correct strategies, but with the goal of illustrating the conceptual rationale in one strategy that is less apparent in the other strategy.
- How do they differ? WEPs show two different problems solved in related ways, with an interest in illustrating what the relationship between problems and answers of the two problems reveals about an underlying mathematical concept.

Student Worksheets

In conjunction with the WEPs, we have also created worksheets to facilitate class discussion about multiple solution methods.

Each GEMS worksheet features a question or prompt at the top of the page that is related to the WEP being taught. The worksheets are designed to follow a think-pair-share approach where students first think independently about the worksheet prompt, then turn and share ideas with a partner, and finally engage in whole class discussion.

At the end of the lesson, the class discusses the "Big Ideas," or main takeaway point that our team has developed for the WEP, and the last prompt on the worksheet asks for students to write the Big Idea using their own words.

*Click on the links below to view our CEMS materials.

- Topic 1: Linear Equations
- Topic 2: Functions & Graphing Linear Equations
- Topic 3: Solving Systems of Linear Equations
- Topic 4: Polynomials & Factoring

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Audience Polls 1 - 3

- Go to: my.vanderbilt.edu/cems/
 - Select NCTM 2019 from menu
 - Complete Polls 1, 2 & 3

The first screenshot shows the main navigation menu with options like Contact Us, NCTM 2019, Overview, People, Reports, and Resources. The 'Resources' tab is highlighted with a red circle. The second screenshot shows a poll titled 'Description of Study' with several questions and answer options.

The first screenshot shows a poll titled 'NCTM 2019' with the question '1. What is your primary role?' and five options: High school math teacher, Middle school math teacher, Teacher other than middle/high school math, Resource/Special Education teacher, and School leadership/administration. The second screenshot shows 'Poll #1' with the question 'What is your primary role?' and the same five options. The third screenshot shows 'Poll #2' with the question 'What is your primary role?' and the same five options.

Poll Results

- <http://www.easypolls.net/poll.html?p=5d8ce5c1e4b00d4af40b6824>
- <http://www.easypolls.net/poll.html?p=5d8ce744e4b00d4af40b6826>
- <http://www.easypolls.net/poll.html?p=5d8ce8c7e4b00d4af40b6828>

We Often Learn Through Comparison

Google - Pixel 3 64GB - Jet Black (Verizon)	Samsung - Galaxy Note10+ 256GB - Aura Black (Verizon)	Apple - iPhone XS Max 256GB - Gold (Verizon)	Motorola - moto z2 - Flash Gray (Verizon)
Model: G010N63JU8 MSRP: \$699 \$20.83/mo. for 24 mos., 0% APR  (982) See Details	Model: SM-N975UZKATXZ MSRP: \$1,099 \$20.84/mo. for 24 mos., 0% APR  (1,691) See Details	Model: A1901T19B04 MSRP: \$1,099 \$20.84/mo. for 24 mos., 0% APR  (129) See Details	Model: MZD171804 MSRP: \$499 \$20.84/mo. for 24 mos., 0% APR  (129) See Details
Save	Save	Save	Save
Specifications	Color: Gold	Color: Gold	Color: Gold
Key Specs	Compare	Compare	Compare
Voice Assistant Built-in			
Google Assistant			
Carrier	Verizon	Verizon	Verizon
Wireless Technology			
4G LTE	20, 30, 40, 4G LTE	4G LTE	20, 30, 40, 4G LTE
Operating System	Android 9.0 Pie	Android 9.0 Pie	Apple iOS 12
Android 9.0 Pie	Android 9.0 Pie	Apple iOS 12	Android 9.0 Pie
Usage Time(s)			
Standby (Always On Display Off), up to 384 hrs; Mixed usage (Always On Display Off), up to 25 hrs	Up to 35 hrs	Talk: up to 1500 min; Playback (wireless video), up to 10 hrs; Playback (wireless audio), up to 60 hrs; Active online usage, up to 13 hrs	—
Internal Memory	64 gigabytes	256 gigabytes	256 gigabytes
64 gigabytes	256 gigabytes	256 gigabytes	128 gigabytes
Screen Size	6.5 inches	6.8 inches	6.5 inches
6.5 inches	6.8 inches	6.5 inches	6.4 inches
Water Resistant	Yes	Yes	Yes
Yes	Yes	Yes	No

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Learn words



“Comparison is one of the most integral components of human thought”

(Goldstone, Day & Son, 2010, p. 103)

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Comparison is a “Best Practice” in Mathematics Instruction

- Share and compare solution strategies core to reform pedagogy in many countries (Australian Education Ministers, 2006; Brophy, 1999; Kultusministerkonferenz, 2004; NCTM, 2014; Singapore Ministry of Education, 2006; Treffers, 1991)
- Expert teachers use this approach (Lampert, 1990; Richland, Zur & Holyoak, 2007; Shimizu, 1999)

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The Essence of Our Approach

Student learning of math, and attitudes toward math, can be improved through the use of:

Comparison



Discussion



Research

- Over 15 years of research supporting the development and testing of this approach
- Several large grants totaling over \$5 million from the US Department of Education and the National Science Foundation
- 30+ publications in academic and teacher journals

Benefits of Compare & Discuss

- Comparing and discussing multiple strategies improves students'
 - Problem-solving accuracy
 - Flexibility: Knowing multiple strategies and when to use them
 - Understanding of key concepts and strategies



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EDUCATOR'S PRACTICE GUIDE WHAT WORKS CLEARINGHOUSE

Improving Mathematical Problem Solving in Grades 4 Through 8



Recommended Teaching Strategy

Recommendation 4.
Expose students to multiple problem-solving strategies.

1. Provide instruction in multiple strategies.
2. Provide opportunities for students to compare multiple strategies in worked examples.
3. Ask students to generate and share multiple strategies for solving a problem.

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Institute of Education Sciences

ies.ed.gov/ncee/wwc/PracticeGuides

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EDUCATOR'S PRACTICE GUIDE
A set of recommendations to address challenges in classrooms and schools

WHAT WORKS CLEARINGHOUSE™

Teaching Strategies for Improving Algebra Knowledge in Middle and High School Students



Recommended Teaching Strategy

Recommendation 3.
Teach students to intentionally choose from alternative algebraic strategies when solving problems.

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Need to Help Teachers Use Compare & Discuss More Frequently and Effectively

- Comparing strategies rarely done in textbook lessons on **Algebra**
- Only 1% of examples in a US Algebra I textbook included multiple strategies for solving *the same problem*, and comparison was not supported.

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Use of Compare and Discuss in Typical Algebra Classrooms is Infrequent

Instructional Practice	% of Algebra Lessons
Exposed students to multiple strategies	20
Multiple strategies were compared for at least a 1.5-minute continuous block	1
Engaged in partner/small group work for at least a 1-minute continuous block	29
Had a whole-class discussion for at least a 1.5-minute continuous block	6

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Compare & Discuss: Worked Example Pairs (WEPs)

- Side-by-side comparison of solved problems
- Shows hypothetical students' work and dialogue explaining process
- Includes discussion questions and prompts

Which is better?

Riley and Gloria were asked to graph the equation $3x - 2y = 6$.

Riley's "x- and y-intercept" way

Find the x-intercept by plugging in 0 for y.
 $3x - 2(0) = 6$
 $3x = 6$
 $x = 2$
x-intercept: $(2, 0)$

Then find the y-intercept by plugging in 0 for x.
 $3(0) - 2y = 6$
 $-2y = 6$
 $y = -3$
y-intercept: $(0, -3)$

I plotted the intercepts and connected them.

How did Riley graph the line? Why did Gloria solve the equation for y as a first step?
Which method is better?

Gloria's "slope-intercept" way

$3x - 2y = 6$

$2y = 3x + 6$

$y = \frac{3}{2}x + 3$

I substituted the y-intercept for y to put the equation in slope-intercept form.
I plotted the y-intercept of $(0, 3)$ and then one more point.
I connected the points to get the line.

Topic 2.6

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Our Supplemental Compare & Discuss Curriculum for Algebra I

- Accessible online at
 - my.vanderbilt.edu/cems
 - Resources tab
- Materials for each lesson:
 - Teacher Guide for planning
 - Worked-example pair
 - Graphic organizer for student discussion
 - Big Idea
 - (See handout for sample materials)
- 7-9 lessons per topic

Compare & Discuss Problems

Topic 1: Linear Equations



Video Guiding Questions

- In what ways is this lesson leveraging the power of comparison?
 - How do the **materials** support **comparison**?
 - How does the **teacher** facilitate **comparison**?
- See Handout for Topic 3.5:
 - Worked Example Pair
 - Discussion Connections graphic organizer
 - Summary of Big Idea
- See Handout 2 for space to record your ideas. Will use Think-Pair-Share routine because teachers use this routine with our materials.

Video (Using Topic 3.5 Which is Better?)

Opening to 2:42; 5:36-9:50

Which is better? Topic 3.5

Tim and Emma were asked to solve the linear system

$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$

Tim's "substitution" way

$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$

$$x = 3y + 10$$

$$3(3y + 10) + 2y = 8$$

$$9y + 30 + 2y = 8$$

$$11y + 30 = 8$$

$$11y = -22$$

$$y = -2$$

$$I \text{ plugged } y \text{ into the second equation to find } x.$$

$$x - 3(-2) = 10$$

$$x + 6 = 10$$

$$x = 4$$

$$The solution is (4, -2)$$

Emma's "elimination" way

$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$

$$3x + 2y = 8$$

$$-3(x - 3y = 10)$$

$$3x + 2y = 8$$

$$-3x + 9y = -30$$

$$11y = -22$$

$$y = -2$$

$$I \text{ multiplied the bottom equation by } -1.$$

$$I \text{ then used elimination and solved for } y.$$

$$x - 3(-2) = 10$$

$$x + 6 = 10$$

$$x = 4$$

$$The solution is (4, -2)$$



Why did Tim choose to plug $y = -2$ into the second equation to find x instead of the first equation?

Which method is better? What are some advantages of Tim's "substitution" way? Of Emma's "elimination" way?

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Discussion: Think-Pair-Share

- THINK: 1 minute to finish jotting down your thoughts to the prompts on handout
- PAIR: 5 minutes to pair with another participant to discuss your responses to the questions. (Groups of 3 if needed)
- SHARE: 3-4 groups share out given our limited time

PROMPTS reminder:

- In what ways is this lesson leveraging the power of comparison?
 - How do the **materials** support **comparison**?
 - How does the **teacher** facilitate **comparison**?

Your Observations

- SHARE: In what ways is this lesson leveraging the power of comparison?
 - How do the **materials** support **comparison**?
 - 1. x
 - How does the **teacher** facilitate **comparison**?
 - 1. x

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How do the materials support comparison?

Topic 3.5

1. Present two different strategies for solving the same problem.

2. Presented as students' solutions to encourage critical reflection.

3. Make both examples visible and clear; present side-by-side.

Tim and Emma were asked to solve the linear system
$$\begin{cases} 3x + 2y = 8 \\ x - 3y = 10 \end{cases}$$

Tim's "substitution" way

$$\begin{aligned} 3x + 2y &= 8 \\ x - 3y &= 10 \\ \downarrow & \\ x &= 3y + 10 \\ \downarrow & \\ 3(3y + 10) + 2y &= 8 \\ 9y + 30 + 2y &= 8 \\ 11y + 30 &= 8 \\ 11y &= -22 \\ y &= -2 \\ \downarrow & \\ x - 3(-2) &= 10 \\ x + 6 &= 10 \\ x &= 4 \\ \downarrow & \\ \text{The solution is } (4, -2) & \end{aligned}$$

Emma's "elimination" way

$$\begin{aligned} 3x + 2y &= 8 \\ x - 3y &= 10 \\ \downarrow & \\ 3x + 2y &= 8 \\ -3(x - 3y = 10) & \\ \downarrow & \\ -3x + 2y &= 8 \\ -3x + 9y &= -30 \\ 11y &= -22 \\ y &= -2 \\ \downarrow & \\ x + 6 &= 10 \\ x &= 4 \\ \downarrow & \\ \text{The solution is } (4, -2) & \end{aligned}$$

I multiplied the bottom equation by -3.

I then used elimination and solved for y.

I plugged y into the second equation to find x.

Why did Tim choose to plug $y = -2$ into the second equation instead of the first equation?

Which method is better? What are some advantages of Tim's "substitution" way? Of Emma's "elimination" way?

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4. Prompts for students to:
A. Understand each strategy.
B. Compare strategies to identify pros and cons.

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How does the teacher facilitate comparison?

1. Prepare to compare: Take time for students to understand each strategy

2. Make Comparisons
1. Ask students to explain similarities and differences.

1. Mark or list them.
2. Push students to reflect on a key point about the comparison.

Prepare to Compare

- What is the problem asking?
- What is happening in the first method?
- What is happening in the second method?

Make Comparisons

- What are the similarities and differences between the two methods?
 - Which method is better?

Big Idea

- ▣ Multiple techniques make comparing strategies more effective, including side-by-side presentation of the strategies and prompting students to identify similarities and differences and pros and cons of the strategies.

Big Idea. Write what you think is the big idea of this video example and discussion, in your own words.

Share. After reviewing together, summarize the ideas we agreed on on your handout.

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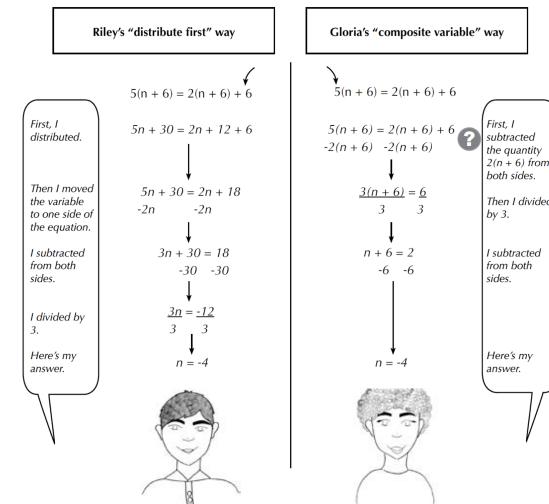
Video Guiding Questions

- In what ways is this lesson leveraging the power of comparison? What new ideas do you notice?
 - How does the **teacher** facilitate **comparison**?
- See Topic 1.7 worked example pair
- Record ideas on handout 2

Which is better?

Topic 1.7

Riley and Gloria were asked to solve $5(n + 6) = 2(n + 6) + 6$.



How did Riley and Gloria solve the equation?

Which method is better? What are some important differences between Riley's "distribute first" method and Gloria's "composite variable" method?

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Video 2 (Using Topic 1.7 Which is Better?)

Discussion: Think-Pair-Share

- **THINK:** 1 minute to finish jotting down your thoughts
- **PAIR:** 3 minutes to pair with another participant to discuss your responses to the question.
- **SHARE:** 2-3 groups share out given our limited time

PROMPTS:

- In what ways is this lesson leveraging the power of comparison?
What new ideas do you notice?
- How does the **teacher** facilitate **comparison**?

How does the teacher facilitate comparison?

2. Make Comparisons

1. Ask students to explain similarities and differences.

- 1. Use color coding to mark, write notes.
- 2. Doesn't stop at 1-2 similarities or differences.

2. Push students to reflect on a key point about the comparison.

Make Comparisons

- What are the similarities and differences between the two methods?
- Which method is better?

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Types of Comparisons

See your handout

■ Type 1: Which is better?

- Compare two correct strategies and reflect on when one strategy is better than another
- Examples Topic 3.5 and Topic 1.7 (video)

■ Type 2: Why does it work?

- Compare two correct strategies to better understand why the teacher-taught strategy works
- Example Topic 2.1

■ Type 3: Which is correct?

- Compare a correct and incorrect strategy to understand why common mistakes are incorrect and to increase use of correct strategies.
- Example Topic 4.5

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Why Does It Work?

Compare two correct strategies to better understand why the teacher-taught strategy works

Riley's "make a table" way

I made a table.
I saw that 2 in the domain is paired with both a 5 and a 4 in the range.
This means the relation is not a function.

x (domain)	y (range)
-3	6
2	5
2	4
3	1
5	1

Not a function

Gloria's "graph and vertical line test" way

I graphed the ordered pairs.
I found a vertical line that intersected two of the points.
This means the relation is not a function.

How did Riley determine if the relation was a function? How did Gloria determine if the relation was a function?

Why do both methods work? Why does the vertical line test tell us the same thing as the table of values?

Topic 2.1

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Which is correct?

Compare a correct and incorrect strategy to understand why common mistakes are incorrect and to increase use of correct strategies.

Layla's "set equal to 0" way

First, I set the equation equal to zero by adding 12 to both sides. Then, I factored.
I solved the equations to get my answers.

$$\begin{aligned} a^2 + 5a - 6 &= -12 \\ a^2 + 5a + 6 &= 0 \\ (a + 2)(a + 3) &= 0 \\ a + 2 = 0 \text{ or } a + 3 &= 0 \\ a = -2 \text{ or } a = -3 & \end{aligned}$$

How could you check to see if Layla's solutions are correct?

Which method is correct, Layla's "set equal to 0" method or Riley's "factor first" method?

Topic 4.5

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Riley's "factor first" way

First, I factored.
Since 6 times -2 is -12, I set the first part equal to 6 and the second part equal to -2. Then I solved the equations to get my answers.

$$\begin{aligned} a^2 + 5a - 6 &= -12 \\ (a + 6)(a - 1) &= -12 \\ a + 6 = 0 \text{ or } a - 1 &= -2 \\ a = 0 \text{ or } a = -3 & \end{aligned}$$

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How do the materials support discussion?

Graphic organizer to support a) Think – pair – share routine

b) Students summarizing the big idea in own words

Discuss Connections

Is there a situation where substitution would be better than elimination, or vice versa?

Think	Pair
-------	------

Share. After reviewing the worksheet as a class, summarize the answer(s) your class agrees on. Was this different from your original response?

Big Idea. When your teacher tells you to do so, write w example, in your own words.

Prepare to Discuss (Think, Pair)

Is there a situation where substitution would be better than elimination, or vice versa?

Discuss Connections (Share)

If one of the equations has a variable with a coefficient of 1, that equation is easy to rearrange, so substitution might be better. If the same variable in both equations has the same or opposite coefficients, then elimination might be better.

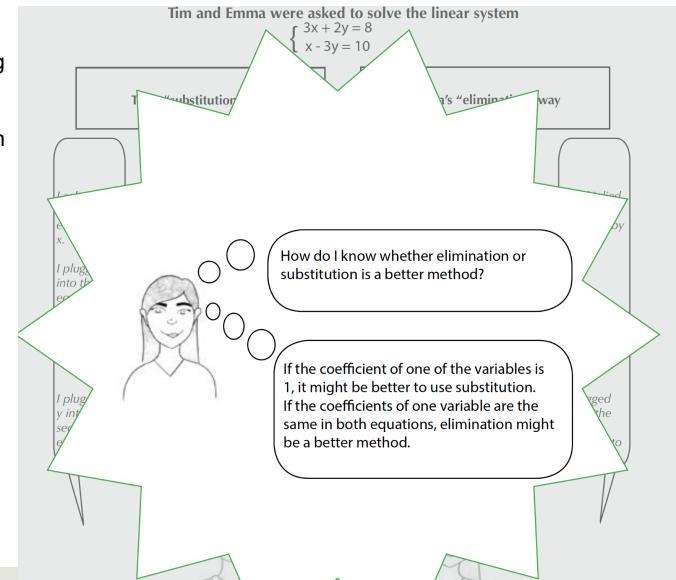
Identify the Big Idea

Can you always use either substitution or elimination? Which is better? When solving a system of linear equations, substitution and elimination are both correct methods that will give you the same answer. Substitution might be easier if the variable has a coefficient of 1.

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How do the materials support discussion?

Slide summarizing a Big Idea that should emerge from discussion.



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Video Guiding Questions

- In what ways is this lesson leveraging the power of discussion?
 - How does the **teacher** facilitate **discussion**?

- See Handouts for Topic 3.5:
 - Worked Example Pair
 - **Discussion Connections graphic organizer**
 - **Summary of Big Idea**

Video of Discussion (Using Topic 3.5 Which is Better?)

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Discussion: Think-Pair-Share

- THINK: 1 minute to finish jotting down your thoughts
- PAIR: 5 minutes to pair with another participant to discuss your responses to the questions
- SHARE: 3-4 groups share out given our limited time

PROMPTS:

- In what ways is this lesson leveraging the power of discussion?
 - How does the **teacher** facilitate **discussion**?
- X

Leveraging the Power of Discussion

How can teachers support **discussion**?

1. During the discussion:
 - Asking open-ended questions (e.g., "Why do you think that's true?")
 - Re-voicing and summarizing contributions
 - Hearing from many voices
 - Holding participants accountable for listening to others: "Do you agree or disagree with Morgan? Why?",

Big Idea

THINK: On a notecard:

- BIG IDEA: Write what you think is the big idea of this presentation, in your own words.
- USE IT: Write 1-3 things you learned from today that you plan to use in your own instruction.

PAIR: Share with your partner

SHARE: Share with me by dropping off your notecard on front table

Complete Polls 4 & 5 online

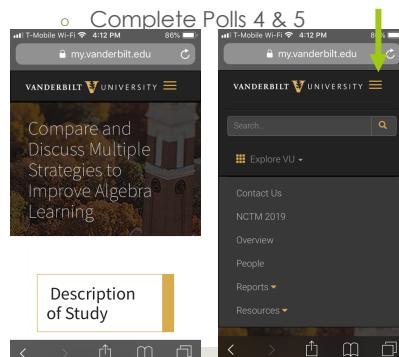
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Audience Exit Polls 4 & 5

- Go to: my.vanderbilt.edu/cems/

◦ Select NCTM 2019 from menu

◦ Complete Polls 4 & 5



NCTM 2019

Click on the links below to access the polls used in Dr. Bethany Little-Johnson's NCTM 2019 presentation.

Poll #1

Poll #2

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Poll Results

- <http://www.easypolls.net/poll.html?p=5d8d0dc8e4b00d4af40b68ab>
- <http://www.easypolls.net/poll.html?p=5d8d0dece4b00d4af40b68ac>
- Side note: We are hoping to work with Integrated Math I teachers in Metro Nashville Public Schools next year! Let me know if you are an MNPS teacher who might be interested in participating if the project works out.

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Acknowledgements

- ❑ Slide and Materials available at:
 - ❑ my.vanderbilt.edu/cems
- ❑ E-mail: b.rittle-johnson@vanderbilt.edu
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 - ❑ Opinions expressed are those of the authors only!



Select References

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