Compare and Discuss Multiple Strategies!

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Abstract

Comparing helps students learn mathematics, particularly when students have the opportunity to compare and discuss multiple ways for solving problems. This paper introduces an instructional

routine aimed at supporting use of comparison and discussions called Compare and Discuss Multiple Strategies or CDMS.

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Comparison is a powerful and important way that we learn. For example, comparing helps us choose among different brands and models of products, as well as helping us learn new ideas by comparing them to what we already know. Comparing also helps students learn mathematics, particularly when they have the opportunity to compare and discuss multiple ways for solving problems (Star, Kenyon, Joiner, and Rittle-Johnson 2010). Standards documents from many countries around the world (including the US: CCSS 2021) encourage teachers to prompt students to share, compare, and discuss multiple strategies for solving mathematics problems.

To support teachers in the use of comparison in their instruction, we developed an instructional routine called *Compare and Discuss Multiple Strategies* [CDMS]. Similar to other instructional routines, CDMS is a structured, specific, repeatable mini-lesson that teachers can use to enable all students to engage in mathematics learning.

CDMS is designed to allow teachers to organize aspects of their instruction around (a) comparison of multiple strategies, and (b) participation in mathematical discussion. With respect to the former, comparison is particularly powerful in mathematics learning, as it gives students the opportunity to consider and reason about multiple ways for solving problems. For the latter, discussion can provide students with opportunities to develop a deeper understanding of concepts and make sense of new ideas, by listening to peers' contribution, understanding multiple perspectives, and making connections to one's own understanding.

Comparison and discussion of multiple strategies is considered a best practice in mathematics education. But implementing comparison and discussion can be challenging for teachers, for several reasons. First, orchestrating a productive discussion requires teachers to listen to and spontaneously respond in-the-moment to students' contributions, knitting together what students offer in ways that move the lesson forward towards its objective. CDMS assists with this challenge by providing teachers with embedded questions that can help propel a discussion toward its desired goals. Second, the success of comparison and discussion often rest on the tasks that students are engaged in, as some tasks seem more amenable to productive comparison than others. Furthermore, learning from the comparison of multiple strategies depends on the particular methods that students produce, and sometimes a key strategy or idea does not naturally emerge from student work on a given day. To address this challenge, the CDMS routine relies upon solved problems (Barlow, Gerstenschlager, and Harmon 2016) – worked examples that have been designed to be particularly conducive to comparison and discussion. The structured CDMS routine can make it easier for teachers to incorporate comparison and discussion into their instruction, and it can allow all students to fully and consistently engage in these important mathematical thinking habits.

What does CDMS look like?

Figure 1 captures the flow of the CDMS instructional routine, which is designed to take about 20 minutes to complete. (But as noted below, there are many ways that this routine can be adapted for shorter or longer time periods.) The gist of CDMS is that students first compare and contrast two solved problems that have been selected and formatted to enable easy comparison, and then students engage in a discussion around a big idea or objective that the comparison is designed to stimulate conversation around.

Looking at CDMS in a bit more detail, the routine begins with students' comparing and contrasting the two strategies provided in the solved problem, in the COMPARE phase of the routine. This COMPARE phase (which is designed to take about 8 minutes) begins with students examining each of the two strategies one at a time, in order to understand the reasoning behind each strategy. We refer to this part of the routine as *Prepare to Compare* sub-phase. Teachers provide students with questions to engage in to support the process of preparing for comparison, including "What is happening in the first strategy?" and "What is happening in the second strategy?" Before comparing, it is important that students have the opportunity to fully understand each of the two strategies one at a time. Next, the routine prompts students to engage in the actual comparison of the two solved problems in the *Make Comparisons* sub-phase, by considering any similarities and differences that students notice about the two strategies and reflecting on teacher-initiated questions such as, "Why do both strategies work?" and "Which strategy is correct?" as well as other student-generated questions that may come up.

After engaging in this comparison, CDMS moves next to the DISCUSS phase of the routine. This phase leverages the think-pair-share routine: In order to prepare to engage in the discussion (in the *Prepare to Discuss* sub-phase), students first are asked to think independently and then to talk with a partner about one or more key questions that emerged from the comparison. To support this independent and partner work and to prepare for discussion, students are provided with a handout where they can record their (and their partner's) observations. Students are then encouraged to share and discuss their ideas with the whole class in the *Discuss Connections* sub-phase. Here the teacher facilitates a whole-class conversation where students share their thoughts and reasoning that have emerged from the independent and partner work. Here the teacher may re-voice or summarize student contributions to keep the conversation going and to involve as many students as possible. Additionally, the teacher asks follow-up questions to help students understand the important mathematical ideas illustrated by the solved problem and to hold students accountable for listening. For example, the teacher might ask students to elaborate on their explanations by asking, "Why do you think that's true?" or ask peers to make connections from what they have heard to their own thinking by asking, "Do you agree or disagree? Why?" Finally, the teacher guides students to identify the main take-aways (in *Identify the Big Idea* sub-phase) of the activity in the students' own words. These three sub-phases of the DISCUSS phase of CDMS is designed to take about 12 minutes.

<table>
<tr><td rowspan="2">Compare
(8 minutes)</td><td>Prepare to Compare
➢ What is the problem asking?
➢ What is happening in the first strategy?
➢ What is happening in the second strategy?</td></tr>
<tr><td>Make Comparisons
➢ What are the similarities and differences between the two strategies?
o Which strategy is correct?
o Why do both strategies work?
o How do the problems differ?
o Which strategy is better?</td></tr>
</table>

Discuss (12 minutes)	**Prepare to Discuss (think, pair)** ➢ How does this comparison help you understand this problem? ➢ How might you apply these strategies to a similar problem?
	Discuss Connections (share) ➢ What ideas would you like to share with the class?
	Identify the Big Idea ➢ Can you summarize the Big Idea in your own words?

Figure 1: The CDMS instructional routine is organized into different phases, each with a selection of possible guiding questions.

Worked Example Pairs for CDMS

As described above, the use of CDMS is facilitated by resources – essentially curricular supplements - around which students engage in comparison and discussion. At the core of these supplements are pairs of solved problems, which we call worked example pairs (WEPs), which are presented side-by-side to facilitate comparison and discussion.

Each WEP has three pages that are tightly linked to the structure of CDMS (see Figures 2 and 3 for examples). The first page of each WEP shows the two hypothetical students' strategies and dialogue used for approaching the given problem(s). This first page (see pane (a) of Figures 2 and 3) is intended to be used for the COMPARE phase of CDMS, where teachers engage students in understanding the reasoning behind each strategy one at a time and then comparing the two strategies. Note that WEP-specific embedded discussion questions are provided for use in each of these sub-phases of CDMS, as a supplement to the more generic questions provided for the CDMS routine in general (see Figure 1).

The second page each of WEP (see pane (b) of Figures 2 and 3) is used as the DISCUSS phase begins and provides a space for students to record their independent, partner, and whole class thinking during the think-pair-share routine. At the top of this page is a key question or problem that is to be discussed as part of the WEP; this question closely follows from the question that is explored in the first page of the WEP. Note that this page provides an additional question for students to discuss that builds upon the questions that arose during the COMPARE phase. Finally, CDMS culminates with the *Identify the Big Idea* subphase (pane (c) of Figures 2 and 3), where teacher and students identify and discuss the main take-aways from the WEP.

Worked Example Pair – Type 1: Why does it work? There are several different types of WEPs, distinguishable by their different learning goals. For example, Figure 2 is an example of the "*Why does it work?*" WEP type, as its objective is for students to engage in thinking about important mathematical ideas related to slope, y-intercept, parallel lines, and solutions to systems of linear equations. In this WEP, two hypothetical students (Emma and Layla) are given the first equation in a system of linear equations ($4x + 2y = 8$) and asked to generate a second equation such that the system has no solution. Emma explores this problem graphically, by graphing the given line, constructing a parallel line, and then determining the equation of the parallel line. Layla relies more on symbolic representations, determining the equation of a parallel line from the given line's slope and y-intercept. Emma's and Layla's strategies are presented side-by-side, using similar terminology, consistent with research findings on effective ways to promote students' engagement with multiple strategies.

After students finish discussing Emma's and Layla's ways in the COMPARE phase (using the first page of the WEP), the second page of the WEP asks, as follow-up questions, for students (a) to find another equation (different from the one that Emma and Layla produced) such that the system has no solution, and (b) to indicate whether they used Emma's or Layla's way in doing so.

In a discussion of this solved problem, there are two key learning goals. A first goal is that students deepen their understanding of the features of lines (slope and *y*-intercept), as represented symbolically and graphically, that are useful in describing and determining the equations of parallel lines. A second learning goal is that students think about graphical and symbolic representations of inconsistent linear systems (e.g., those with no solution). Each WEP provides student dialogue as well as embedded WEP-specific questions that can stimulate discussion and that are linked to the phases of CDMS.

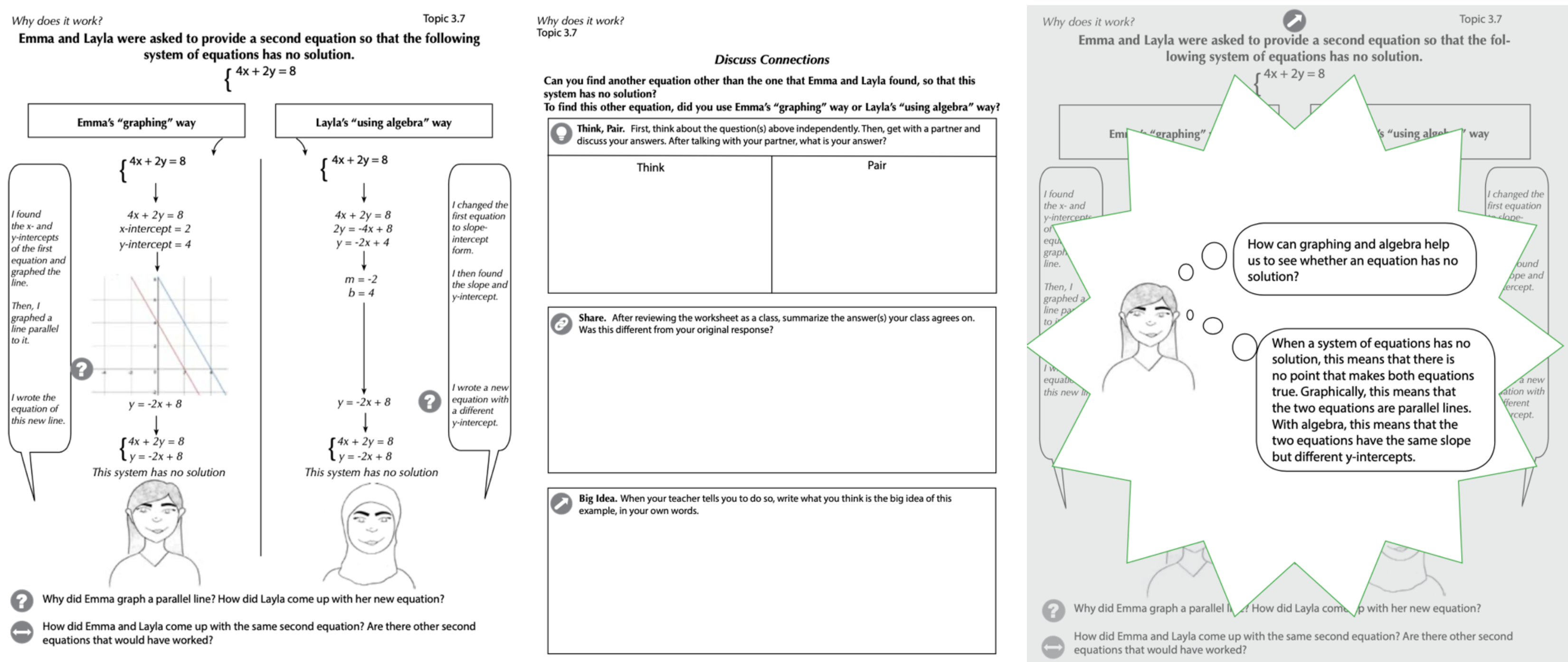

Figure 2: A "*Why does it work*?" WEP that is used for implementing the CDMS routine. Pane (a) is used for the COMPARE phase of CDMS. Pane (b) is used for the *Prepare to Discuss* and *Discuss Connections* sub-phases. Pane (c) is used for the *Identify the Big Idea* sub-phase of CDMS. Figure taken from Star et al. (2019).

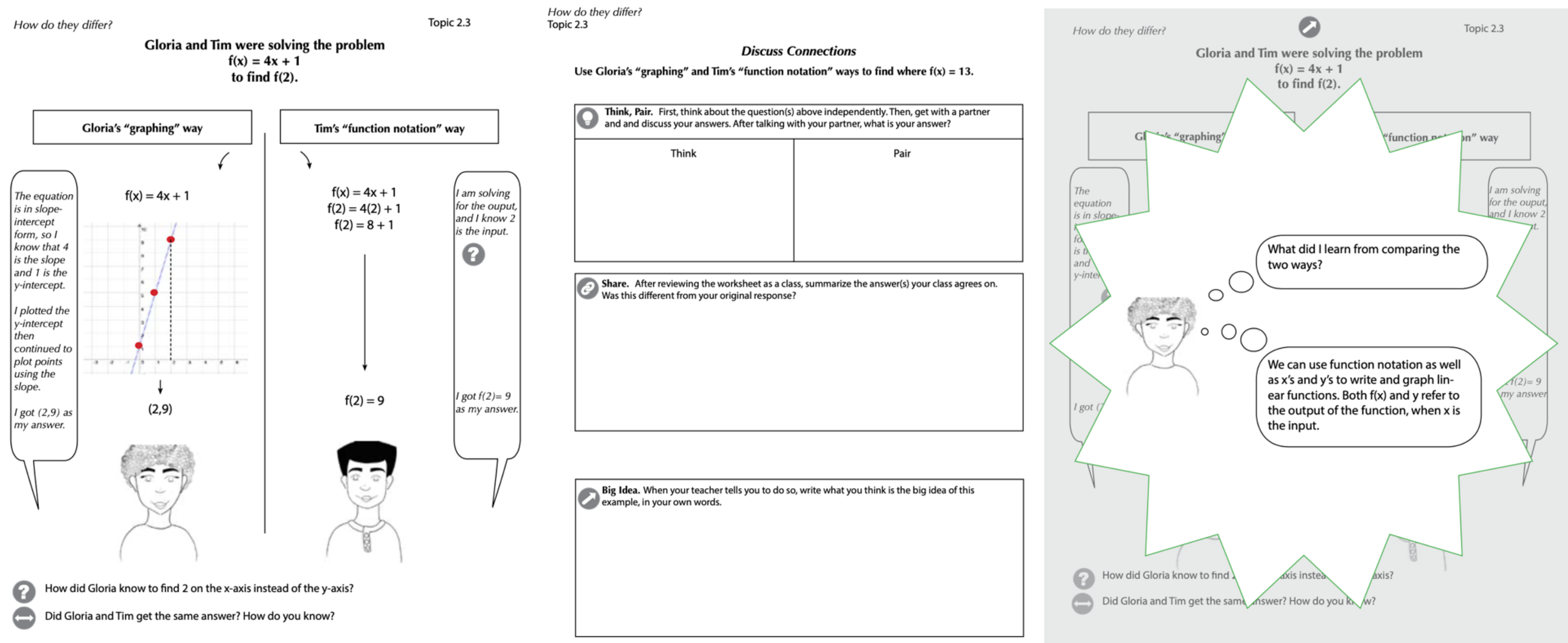

Figure 3: A "*How do they differ?*" WEP that is used for implementing the CDMS routine. Pane (a) is used for the COMPARE phase of CDMS. Pane (b) is used for the *Prepare to Discuss* and *Discuss Connections* sub-phases. Pane (c) is used for the *Identify the Big Idea* sub-phase of CDMS. Figure taken from Star et al. (2019).

Worked Example Pair – Type 2: How do they differ? As another example of a different type of WEP (with a different learning goal) that can be used with CDMS, Figure 3 shows a "*How do they differ?*" example. Here Gloria and Tim are asked to evaluate the value of the function $f(x) = 4x + 1$, at $x = 2$. Gloria's strategy involves graphing, while Tim's involves the use of function notation. Embedded questions provided with the WEP can help students to understand what each student did and also to compare their strategies. After completing the COMPARE phase using the first page of the WEP, students turn to the second page of the WEP and participate in a think-pair-share discussion around the (new) problem of using Gloria's and Tim's ways to find where $f(x) = 13$. This WEP engages students in comparing graphical and symbolic representations related to a linear function, with the goal of having students consider what it means to evaluate a function at a given value and how graphical and symbolic means of evaluation are linked.

Worked Example Pair – Types 3 (Which is correct?) and 4 (Which is better?) In addition to the "*Why does it work?* and the "*How do they differ?*" WEP types, there are two additional types of WEPs that we have created for use with the CDMS routine. Each of these additional types focuses on a comparison of two strategies for solving the same problem, sometimes with the goal of understanding sources of common errors for solving the problem (in *Which is correct?* WEPs) and sometimes with the goal of engaging students in a discussion of the relative advantages and disadvantages of different ways of solving the same problem (in *Which is better?* WEPs). All four types of WEPs are structured in the same way as the examples illustrated in Figures 2 and 3 to align with the CDMS routine, with a page used for the COMPARE phase of CDMS and two pages (one page for the think-pair-share in the *Prepare to Discuss* and *Discuss Connections* sub-phases and one page for the *Identify the Big Idea* sub-phase) for the DISCUSS phase. Examples of all four types of WEPs can be explored and downloaded at www.compareanddiscuss.com.

Teachers' experiences using CDMS

We have used CDMS and the WEPs described above with many algebra teachers over the past 10 years. We have found that algebra teachers – especially those who have limited experience in orchestrating mathematical discussions around comparison of multiple strategies – are much more likely to engage their students in sustained comparison and discussion when they use the CDMS routine.

When we asked teachers who have used this approach to describe what they and their students find most beneficial about CDMS, teachers reported the following:

- CDMS provides a structure that students easily become comfortable with and are able to productively engage with;
- Use of CDMS seems to be related to an increase in students' confidence in mathematics;
- Students enjoyed taking on the perspective of the different hypothetical students in the WEPs and discussing and defending those students' ideas
- Some WEPs – particularly the "*Which is better?*" type - can inspire passionate debates among many students and make for very engaging and stimulating lessons
- CDMS helps push students to provide greater depth in the reasoning behind their and their peers' answers
- CDMS makes the classroom a generally more reflective environment, where both teacher and students need to explain, refine, and revoice others' reasoning frequently

- CDMS is easily adaptable and can be used for longer or shorter portions of lessons as well as for introducing new material as well as for review

With respect to this last point, we found that as teachers become more comfortable and experienced with CDMS, they made a number of productive adaptations to the routine in order to make it work better for their particular lesson objectives and instructional preferences. Some of these adaptations included:

- Changing the duration of CDMS, including lengthening it (so that the routine lasted for a full lesson) or shortening it (so that the whole routine could be completed in about 10 minutes)
- Using CDMS at various points within mathematics lessons, including as a warmup to begin a class period, a cooldown activity to conclude a lesson, and/or as the main instructional focus of a lesson
- Constructing their own WEPs, using the examples we have provided as a guide. This adaptation included, for some teachers, encouraging students to construct their own WEPs, as well as incorporating real students' names into the WEPs in place of the hypothetical students' names
- Spontaneously incorporating students' own generated (in-the-moment) strategies into the comparison, replacing either one or both of the hypothetical students' work provided with each WEP
- Engaging students in a comparison of more than two strategies

Conclusion

In summary, through its use of solved problems with embedded discussion prompts and its focus on comparison of multiple strategies, CDMS – and the associated WEP resources that facilitate its use - is a highly structured but quite adaptable instructional routine that can bring numerous benefits for the teaching and learning of algebra (Durkin, Rittle-Johnson, Star and Loehr, 2021).

The WEPs – pairs of solved problems that are discussed as part of CDMS – can take many forms, including comparing a more with a less efficient way to solve a problem, comparing a correct method to one with an error, and comparing a commonly used method to a less commonly used but viable method. And beyond using premade WEPs, teachers can implement CDMS by using the three-page WEP template (shown in Figures 2 and 3) as a resource for creating and comparing methods produced by their own students.

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