Generalized MV-algebras

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Abstract

We generalize the notion of an MV-algebra in the context of residuated lattices to include non-commutative and unbounded structures. We investigate a number of their properties and prove that they can be obtained from lattice-ordered groups via a truncation construction that generalizes the Chang-Mundici $\Gamma$ functor. This correspondence extends to a categorical equivalence that generalizes the ones established by D. Mundici and A. Dvurečenskij. The decidability of the equational theory of the variety of generalized MV-algebras follows from our analysis.

Key words: residuated lattice, MV-algebra, lattice-ordered group, nucleus, categorical equivalence

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1 Introduction

A residuated lattice is an algebra $L = \langle L, \wedge, \vee, \cdot, \backslash, /, e \rangle$ such that $\langle L, \wedge, \vee \rangle$ is a lattice; $\langle L, \cdot, e \rangle$ is a monoid; and for all $x, y, z \in L$,

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z.$$
Residuated lattices form a finitely based equational class of algebras (see, for example, [4]), denoted by $\mathcal{RL}$.

It is important to remark that the elimination of the requirement that a residuated lattice have a bottom element has led to the development of a surprisingly rich theory that includes the study of various important varieties of cancellative residuated lattices, such as the variety of lattice-ordered groups. See, for example, [18], [4], [2], [20], [9], [12], [13], [14].

A lattice-ordered group ($\ell$-group) is an algebra $G = \langle G, \land, \lor, \cdot, -1, e \rangle$ such that $\langle G, \land, \lor \rangle$ is a lattice, $\langle G, \cdot, -1, e \rangle$ is a group, and multiplication is order preserving (or, equivalently, it distributes over the lattice operations). The variety of $\ell$-groups is term equivalent to the subvariety, $\mathcal{LG}$, of residuated lattices defined by the equations $(e/x)x \approx e \approx x(x/e)$; the term equivalence is given by $x^{-1} = e/x$ and $x\setminus y = x^{-1}y$, $y/x = yx^{-1}$. See [1] for an accessible introduction to the theory of $\ell$-groups.

A residuated bounded-lattice is an algebra $L = \langle L, \land, \lor, \cdot, \setminus, /, e, 0 \rangle$ such that $\langle L, \land, \lor, \cdot, \setminus, /, e \rangle$ is a residuated lattice and $L$ satisfies the equation $x \lor 0 \approx x$. Note that $\top = 0\setminus 0 = 0/0$ is the greatest element of such an algebra. A residuated (bounded-) lattice is called commutative if it satisfies the equation $xy \approx yx$ and integral if it satisfies $x \land e \approx x$.

Commutative, integral residuated bounded-lattices have been studied extensively in both algebraic and logical form, and include important classes of algebras, such as the variety of MV-algebras, which provides the algebraic setting for Łukasiewicz's infinite-valued propositional logic. Several term equivalent formulations of MV-algebras have been proposed (see, for example, [8]). Within the context of commutative, residuated bounded-lattices, MV-algebras are axiomatized by the identity $(x \rightarrow y) \rightarrow y \approx x \lor y$, which is a relativized version of the law $\neg\neg x \approx x$ of double negation; in commutative residuated lattices we write $x \rightarrow y$ for the common value of $x\setminus y$ and $y/x$, and $\neg x$ for $x \rightarrow 0$. The term equivalence with the standard signature is given by $x \odot y = x \cdot y$, $\neg x = x \rightarrow 0$, $x \oplus y = -(\neg x \cdot \neg y)$ and $x \rightarrow y = \neg x \oplus y$. The appropriate non-commutative generalization of an MV-algebra is a residuated bounded-lattice that satisfies the identities $x/(y\setminus x) \approx x \lor y \approx (x/y)\setminus x$. These algebras have recently been considered in [10], [15] and [16] under the name pseudo-MV-algebras.

C. C. Chang proved in [7] that if $G = \langle G, \land, \lor, \cdot, -1, e \rangle$ is a totally ordered Abelian group and $u < e$, then the residuated-bounded lattice $\Gamma(G, u) = \langle [u, e], \land, \lor, \cdot, \setminus, /, e, u \rangle$ – where $x \circ y = xy \lor u$, $x\setminus y = x^{-1}y \land e$ and $x/y = xy^{-1} \land e$ – is an MV-algebra. Conversely, if $L$ is a totally-ordered MV-algebra, then there exists a totally ordered Abelian group with a strong order unit $u < e$ such that $L \cong \Gamma(G, u)$. This result was subsequently generalized for
arbitrary Abelian ℓ-groups by D. Mundici [24] and recently for arbitrary ℓ-
groups by A. Dvurečenskij [10]. It should be noted that all three authors
have expressed their results in terms of the positive, rather than the negative,
cone. Mundici and Dvurečenskij have also shown that the object assignment
Γ can be extended to an equivalence between the category of MV-algebras
(respectively, pseudo-MV-algebras), and the category with objects Abelian
(respectively, arbitrary) ℓ-groups with a strong order unit, and morphisms
ℓ-group homomorphisms that preserve the unit.

We generalize the concept of an MV-algebra in the setting of residuated lat-
tices - by dropping integrality \((x \land e \approx x)\), commutativity \((xy \approx yx)\) and
the existence of bounds - to a class that includes ℓ-groups, their negative
cones, generalized Boolean algebras and the 0-free reducts of MV-algebras.
The aim of this paper is to extend the aforementioned results of Mundici and
Dvurečenskij.

A generalized MV-algebra (GMV-algebra) is a residuated lattice that satisfies
the identities \(x/(((x \lor y) \setminus x) \approx x \lor y \approx (x/(x \lor y)) \setminus x)\). It is shown in Section 2,
see Lemma 2.9, that every GMV-algebra has a distributive lattice reduct.

The negative cone of a residuated lattice \(L = \langle L, \land, \lor, \cdot, \setminus, /, e \rangle\) is the algebra
\(L^- = \langle L^-, \land, \lor, \cdot, \setminus_{L^-}, /_{L^-}, e \rangle\), where \(L^- = \{x \in L \mid x \leq e\}\), \(x_{L^-} y = x \setminus y \land e\) and
\(x_{L^-} / = x / y \land e\). It is easy to verify that \(L^-\) is a residuated lattice. It will
be shown that if \(L\) is a GMV-algebra, then \(L^-\) is a GMV-algebra, as well. As
noted before a residuated lattice is called integral, if \(e\) is the greatest element
of its lattice reduct. The negative cone of every residuated lattice is, obviously,
integral.

By a nucleus on a residuated lattice \(L\) we understand a closure operator \(\gamma\) on
\(L\) satisfying \(\gamma(a) \gamma(b) \leq \gamma(ab)\), for all \(a, b\) in \(L\).

We note that if \(L = \langle L, \land, \lor, \cdot, \setminus, /, e \rangle\) is a residuated lattice and \(\gamma\) is a nucleus
on \(L\), then the image \(L_\gamma\) of \(\gamma\) can be endowed with a residuated lattice structure
as follows (see Lemma 3.3):

\[
L_\gamma = \langle L_\gamma, \land, \lor, \circ_\gamma, \setminus, /, \gamma(e) \rangle
\]

\[
\gamma(a) \lor \gamma(b) = \gamma(a \lor b)
\]

\[
\gamma(a) \circ_\gamma \gamma(b) = \gamma(ab)
\]

As an illustration, let \(u\) be a negative element of an ℓ-group \(G\), and let \(\gamma_u : G^- \rightarrow G^-\) be defined by \(\gamma_u(x) = x \lor u\), for all \(x \in G^-\). Then, \(\gamma_u\) is a nucleus
on \(G^-\) and \(G^-_{\gamma_u}\) is equal to the 0-free reduct of \(\Gamma(G, u)\).

We say that a residuated lattice \(A\) is the direct sum of two of its subalgebras
B, C, in symbols $A = B \oplus C$, if the map $f : B \times C \to A$ defined by $f(x, y) = xy$ is an isomorphism.

The primary purpose of this paper is to establish the following six results.

**Theorem A** (See Theorem 5.6.) A residuated lattice $M$ is a GMV-algebra if and only if there are residuated lattices $G, L$, such that $G$ is an $\ell$-group, $L$ is the negative cone of an $\ell$-group, $\gamma$ is a nucleus on $L$ and $M = G \oplus L \gamma$.

**Theorem B** (See Theorem 3.12.) A residuated lattice $M$ is an integral GMV-algebra if and only if there exists an $\ell$-group $H$ and a nucleus $\gamma$ on $H^-$, such that $M \cong H^- \gamma$.

Let $\text{IGMV}$ be the category with objects integral GMV-algebras and morphisms residuated lattice homomorphisms. Also, let $\text{LG}^-_\gamma$ be the category with objects algebras $(L, \gamma)$, such that $L$ is the negative cone of an $\ell$-group and $\gamma$ is a nucleus on it such that its image generates $L$ as a monoid. Let the morphisms of this category be homomorphisms between these algebras. The generalization of Mundici’s and Dvurečenskij’s results is provided by the following theorem.

**Theorem C** (See Theorem 4.12.) The categories $\text{IGMV}$ and $\text{LG}^-_\gamma$ are equivalent.

**Theorem D** (See Theorem 6.6.) A residuated lattice $L$ is a GMV-algebra if and only if $L \cong G_\beta$, for some $\ell$-group $G$ and some core $\beta$ on $G$. (For the concept of a core, see page 32 and Lemma 6.8.)

Let $\text{GMV}$ be the category with objects GMV-algebras and morphisms residuated lattice homomorphisms. Also, let $\text{LG}^*$ be the category with objects algebras $(G, \beta)$ such that $G$ is an $\ell$-group and $\beta$ is core on $G$ whose image generates $G$; let the morphisms of this category be homomorphisms between these algebras.

**Theorem E** (See Theorem 6.9.) The categories $\text{GMV}$ and $\text{LG}^*$ are equivalent.

Let $\text{G}M\text{V}$ be the variety of GMV-algebras and let $\text{IG}M\text{V}$ be the variety of integral GMV-algebras.

**Theorem F** (See Theorem 7.3.) The varieties $\text{G}M\text{V}$ and $\text{IG}M\text{V}$ have decidable equational theories.
2 Definitions and basic properties

We refer the reader to [4] and [20] for basic results in the theory of residuated lattices. Here, we only review background material needed in the remainder of the paper.

The operations \ and / may be viewed as generalized division operations, with \ being read as “\ over y” and /x as “y under x”. In either case, x is considered the numerator and y is the denominator. We refer to \ as the left division operation and / as the right division operation. Other commonly used terms for these operations are left residuation and right residuation, respectively.

As usual, we write \ for \y\ and adopt the convention that, in the absence of parenthesis, \ is performed first, followed by \ and \. For example, \w\ represents the expression \w/. We tend to favor \ in calculations, but any statement about residuated structures has an opposite “mirror image” obtained by reading terms backwards (i.e., replacing \ by \ and interchanging \ and \). Examples of opposite equations can be seen in properties (i)-(vi) of Lemma 2.1 below.

The existence of the division operations has the following basic consequences.

**Lemma 2.1** [4] Residuated lattices satisfy the following identities:

1. \( x(y \lor z) \approx xy \lor xz \) and \( (y \lor z)x \approx yx \lor zx \)
2. \( \frac{x}{y} \lor z \approx \left( \frac{x}{y} \lor x \right) \) and \( z/y \lor x \approx \left( z/y \lor z \right) \)
3. \( x(y \lor z) \approx xy \lor (xz) \) and \( (y \lor z)x \approx (yx) \lor (zx) \)
4. \( (x/y)y \leq x \) and \( y(y/x) \leq x \)
5. \( x(y/z) \leq (xy)/z \) and \( (x\lor y)x \leq y(x) \)
6. \( x(y/z) \approx x/\left( yz \right) \) and \( z\left( y \lor x \right) \approx (yz)x \)
7. \( x(y/z) \approx \left( xy \right)/z \)
8. \( x/e \approx x \approx e\lor x \)
9. \( x/e \approx x/y/e \) and \( e \leq x \lor x \)
10. \( x(y/z)^2 \approx x/y \) and \( (x\lor y)^2 \approx x\lor y \).

A residuated lattice is called commutative (respectively, cancellative), if its monoid reduct is commutative (respectively, cancellative). It is shown in [2] that the class CanRL of all cancellative residuated lattices is a variety with defining equations \( xy/y \approx x \approx y\lor x \). As mentioned before, a residuated lattice is called integral if it satisfies the identity \( e \lor x \approx x \). The variety of all integral residuated lattices will be denoted by IRL. We will also have the occasion to refer to the subvariety of RL generated by all totally ordered residuated lattices. We denote this variety by RL and refer to its members as representable residuated lattices. It follows from Jónsson’s Lemma on congruence-distributive varieties (see [21]) that all subdirectly irreducible...
algebras in $\mathcal{RL}^C$ are totally ordered and whence every representable residuated lattice is a subdirect product of totally ordered residuated lattices. The following result provides a concise equational basis for $\mathcal{RL}^C$.

**Theorem 2.2** ([4], [20], see also [18]) A residuated lattice is representable, i.e., it is a member of the variety $\mathcal{RL}^C$, if and only if it satisfies the identity $(z\langle x/(x \lor y) \rangle z \land e) \lor (w(y/(x \lor y))/w \land e) \approx e$.

**Definition 2.3**

(i) A *generalized BL-algebra* (GBL-algebra) is a residuated lattice that satisfies the identities

\[(x \land y)/y \approx x \land y \approx y(x \land y).\]

(ii) A *generalized MV-algebra* (GMV-algebra) is a residuated lattice that satisfies the identities

\[x/(x \lor y) \approx x \lor y \approx (x/(x \lor y)) \setminus x.\]

We denote the variety of all GBL-algebras by $\mathcal{GBL}$ and that of GMV-algebras by $\mathcal{GMV}$. GBL-algebras generalize BL-algebras, the algebraic counterparts of basic logic (see [17]). In particular, representable, commutative, bounded (integral) GBL-algebras are (term equivalent to) the 0-free reducts of BL-algebras.

**Lemma 2.4** [2] The preceding sets of identities have, respectively, the following quasi-identity formulations:

\[x \leq y \Rightarrow (x/y)y \approx x \approx y(y \setminus x)\]

and

\[x \leq y \Rightarrow x/(y \setminus x) \approx y \approx (x/y) \setminus x.\]

Moreover, the first set of identities is also equivalent to the property of divisibility in the setting of residuated lattices:

\[x \leq y \Rightarrow (\exists z, w)(zy \approx x \approx yw).\]

**Lemma 2.5** [2] Every GMV-algebra is a GBL-algebra.

**PROOF.** Let $x, y$ be elements of $L$ such that $x \leq y$. Set $z = (x/y)y$ and note that, by Lemma 2.1, $z \leq x$ and $z/y \leq z/x$.

Using Lemma 2.1(vii), (vi) and the defining quasi-equation for GMV-algebras,
we have the following:

\[
\begin{align*}
z \leq x & \Rightarrow (z/x)\backslash z = x \\
& \Rightarrow ((z/x)\backslash z)/y = x/y \\
& \Rightarrow (z/x)\backslash(z/y) = x/y \\
& \Rightarrow (z/y)/(z/x)\backslash(z/y) = (z/y)/(x/y) \\
& \Rightarrow z/x = z/(x/y)y \\
& \Rightarrow (z/x)\backslash z = (z/(x/y)y)\backslash z \\
& \Rightarrow x = (x/y)y
\end{align*}
\]

Thus, \( x \leq y \) implies \( x = (x/y)y \). Likewise, \( x \leq y \) implies \( y(y\backslash x) = x \).  

Lattice-ordered groups and their negative cones are examples of cancellative GMV-algebras. Non-cancellative examples include generalized Boolean algebras.

**Definition 2.6** An element \( a \) in a residuated lattice \( L \) is called **invertible**, if \( a(a\backslash e) = e = (e/a)a \); \( a \) is called **integral**, if \( e/a = a\backslash e = e \). We denote the set of invertible elements of \( L \) by \( G(L) \) and the set of integral elements by \( I(L) \).

Note that \( a \) is invertible if and only if there exists an element \( a^{-1} \) such that \( aa^{-1} = e = a^{-1}a \). In this case \( a^{-1} = e/a = a\backslash e \). It is easy to see that multiplication by an invertible element is an order automorphism.

**Lemma 2.7** Let \( L \) be a GBL-algebra.

(i) Every positive element of \( L \) is invertible.
(ii) \( L \) satisfies the identities \( x/x \approx x\backslash x \approx e \).
(iii) \( L \) satisfies the identity \( e/x \approx x\backslash e \).

**Proof.** For the first property, let \( a \) be a positive element; by the defining identity for GBL-algebras, we get \( a(a\backslash e) = e = (e/a)a \); that is, \( a \) is invertible. By (i) and Lemma 2.1(ix), \( x/x \) and \( x\backslash x \) are invertible for every \( x \). Hence, by Lemma 2.1(x), \( x/x = e = x\backslash x \). Finally, by (ii) and Lemma 2.1(v), \( x(e/x) \leq x/x = e \), hence \( e/x \leq x\backslash e \). Likewise, \( x\backslash e \leq e/x \).  

**Lemma 2.8** A residuated lattice is a GBL-algebra if and only if it satisfies the identities \( x(x\backslash y \wedge e) \approx x \wedge y \approx (y/x \wedge e)x \).
PROOF. Assume that $L$ is a GBL-algebra and $x, y \in L$. By Lemma 2.7(ii) and Lemma 2.1(ii), we get

$$x \land y = x(x \backslash (x \land y)) = x(x \backslash x \land y) = x(e \backslash x \land y).$$

Likewise, we get the opposite identity.

Conversely assume that the identities in the statement of the lemma hold. We first show that every positive element $a$ is invertible. Indeed, $e = a(a \backslash e \land e) \leq a(a \backslash e) \leq e$. So, $a(a \backslash e) = e$ and likewise $(e/a)a = e$. Arguing as in the proof of (ii) of Lemma 2.7, we show that $x/x = x/x$, for every $x \in L$. Using Lemma 2.1(ii), we get

$$x(x \backslash (x \land y)) = x(x \backslash x \land y) = x(e \backslash x \land y) = x \land y.$$ 

Likewise, we obtain the opposite equation. $\square$

Lemma 2.9 Every GBL-algebra has a distributive lattice reduct.

PROOF. Let $L$ be a GBL-algebra and $x, y, z \in L$. Invoking Lemmas 2.1 and 2.8, we have

$$x \land (y \lor z) = [x/(y \lor z) \land e]y \lor z$$

$$= [x/(y \lor z) \land e]y \lor [x/(y \lor z) \land e]z$$

$$\leq (x/y \land e)y \lor (x/z \land e)z$$

$$= (x \land y) \lor (x \land z),$$

for all $x, y, z$. Thus, the lattice reduct of $L$ is distributive. $\square$

Lemma 2.10 If $x, y$ are elements of a GBL-algebra and $x \lor y = e$, $(x, y$ are orthogonal), then $xy = x \land y$.

PROOF. We have, $x = x/e = x/(x \lor y) = x/x \lor x/y = e \lor x/y = y \land x/y = (y \land x)/y$. So, $xy = ((x \land y) \lor y)y = x \land y$. $\square$

The variety of integral GBL-algebras is denoted by $\mathcal{IGBL}$ and that of integral GMV-algebras by $\mathcal{IGMV}$. Obviously, $\mathcal{IGBL} = \mathcal{IRL} \cap \mathcal{GBL}$ and $\mathcal{IGMV} = \mathcal{IRL} \cap \mathcal{GMV}$.

Lemma 2.11
(i) The variety \( IGBL \) is axiomatized, relative to \( RL \), by the equations \( (x/y)y \approx x \land y \approx y(x\backslash x) \).

(ii) The variety \( IGMV \) is axiomatized, relative to \( RL \), by the equations \( x/(y\backslash x) \approx x \lor y \approx (x/y)\backslash x \).

**PROOF.** In view of the alternative axiomatizations of \( GBL \) and \( GMV \) given in Lemma 2.8, the proposed equations hold in the corresponding varieties. For the reverse direction we verify that the proposed identities imply integrality. This is obvious for the first set of identities for \( y = e \). For the second set, observe that for every \( x \),

\[
e \leq e \lor e/x = e/((e/x)\backslash e) = e/(e \lor x);
\]

so \( e \lor x \leq e \), i.e., \( x \leq e \).

Negative cones of \( \ell \)-groups are examples of integral GMV-algebras, hence also of integral GBL-algebras. Moreover, these are cancellative residuated lattices, that is, members of \( \text{Can}RL \). It is shown in [2] that the class \( LG^- \) of negative cones of \( \ell \)-groups is a variety and \( LG^- = IGMV \cap \text{Can}RL = IGBL \cap \text{Can}RL \). This result provides an equational basis for \( LG^- \).

**Theorem 2.12** [2] The class, \( LG^- \), of negative cones of \( \ell \)-groups is a variety and the equations \( xy/y \approx x \approx y\backslash yx \), \( x(x\backslash y) \approx x \land y \approx (y/x)x \) form an equational basis for it, relative to \( RL \).

The variety of Brouwerian algebras is term equivalent to the subvariety \( Br \) of residuated lattices axiomatized by the identity \( xy \approx x \land y \). It is clear that \( Br \subseteq IGBL \). The variety \( GBA \) of generalized Boolean Algebras is generated, in the setting of residuated lattices, by the two-element residuated lattice \( 2 \) and \( GBA = IGMV \cap Br \) (see [13]).

**Lemma 2.13**

(i) Every integral GBL-algebra satisfies the identity \( (y/x)\backslash (x/y) \approx x/y \) and its opposite.

(ii) Every integral GMV-algebra satisfies the identity \( x/y \lor y/x \approx e \) and its opposite.

(iii) Every integral GMV-algebra satisfies the identities \( x/(y\land z) \approx x/y \lor x/z \), \( (x \lor y)/z \approx x/z \lor y/z \) and the opposite ones.

(iv) Every commutative integral GMV-algebra is representable. Consequently, the subdirectly irreducible, commutative, integral GMV-algebras are totally ordered.

**PROOF.**
(i) For every integral GBL-algebra, \( y/x \leq e \), so \( (y/x) \backslash (x/y) \geq x/y \). To show the reverse inequality, we need to check that

\[
((y/x) \backslash (x/y)) y \leq x.
\]

By Lemma 2.1(vii), it suffices to show that

\[
(((y/x) \backslash x) / y) y \leq x.
\]

Using Lemma 2.11(i), we see that the last equation is equivalent to

\[
(y/((y/x) \backslash x))(y/x) \leq x,
\]

which in turn is equivalent to

\[
y/((y/x) \backslash x) \leq x/((y/x) \backslash x).
\]

To show that this holds note that

\[
y/((y/x) \backslash x) \leq y/x,
\]

since \( y/x \leq e \), and that

\[
y/x \leq x/((y/x) \backslash x),
\]

since \( u \leq v/(u \wedge v) \) is valid in every residuated lattice, by Lemma 2.1(iv).

(ii) Using Lemma 2.11(ii), we have \( x/y \vee y/x = (x/y)/(y/x)((y/x) \backslash (x/y)) \), which simplifies to \( (x/y)/(y/x) \), by invoking (i) and the fact that integral GMV-algebras are integral GBL-algebras. Finally, the last term is equal to \( e \) in integral residuated lattices.

(iii) Since every GMV-algebra has a distributive lattice reduct by Lemma 2.9, the equations in (iii) follow from (ii) and Proposition 6.10(ii) of [4].

(iv) This follows from (ii) and [18].

It will be shown in Section 5, refer to Corollary 5.5, that the assumption of integrality in condition (iv) is not needed.

3 A concrete realization of integral generalized MV-algebras

A closure operator on a poset \( P \) is a map \( \gamma : P \to P \) with the usual properties of preserving the order, being extensive \( (x \leq \gamma(x)) \), and being idempotent.
Such a map is completely determined by its image

\[(3.1) \quad C = \text{im} \gamma\]

by virtue of the formula

\[(3.2) \quad \gamma(x) = \min\{c \in C : x \leq c\}.\]

A \textit{closure retract} is any subset \(C \subseteq P\) such that the minima \((3.2)\) exist for all \(x \in P\). Conditions \((3.1)\) and \((3.2)\) establish a bijective correspondence between all closure operators \(\gamma\) and all closure retracts \(C\) of \(P\). In what follows, we will use \(P_\gamma\) to denote the closure retract on \(P\) corresponding to the closure operator \(\gamma\).

A \textit{nucleus} on a residuated lattice \(L\) is a closure operator \(\gamma\) on the lattice reduct of \(L\) such that \(\gamma(a) \gamma(b) \leq \gamma(ab)\), for all \(a, b \in L\). It is clear that a closure operator \(\gamma\) on \(L\) is a nucleus if and only if \(\gamma(\gamma(a) \gamma(b)) = \gamma(ab)\), for all \(a, b \in L\). Note that the monotonicity condition in the definition of a nucleus can be replaced by the inequality \(\gamma(x) \gamma(x \setminus y) \leq \gamma(y)\); so, the property that \(\gamma\) is a nucleus on a residuated lattice can be expressed equationally in the expansion of the language of residuated lattices by a unary operation. A closure retract \(C\) of a residuated lattice \(L\) is called a \textit{subact} of \(L\) if \(x/ y, y \setminus x \in C\), for all \(x \in C\) and \(y \in L\).

As an example, note that if \(u\) is an element of an integral residuated lattice \(L\), then \(\gamma_u : L \rightarrow L - \text{defined by} \gamma_u(x) = x \lor u\), for all \(x \in L -\) is a nucleus on \(L\). Its image \(L_{\gamma_u}\) is the principal filter \(\uparrow u = \{x \in L \mid u \leq x\}\).

The next result describes the relationship between nuclei and subacts (see [25], page 30; and [26], Corollary 3.7, for an earlier result in the setting of Brouwerian meet-semilattices).

**Lemma 3.1** Let \(\gamma\) be a closure operator on a residuated lattice \(L\), and let \(L_\gamma\) be the closure retract associated with \(\gamma\). The following statements are equivalent.

(i) \(\gamma\) is a nucleus.
(ii) \(\gamma(a)/ b, b \setminus \gamma(a) \in L_\gamma\), for all \(a, b \in L\).
(iii) \(L_\gamma\) is a subact of \(L\).

**Proof.**

(i) \(\Rightarrow\) (ii) Let \(a, b \in L\). We have,
\[\gamma(\gamma(a)/b)b \leq \gamma((\gamma(a)/b)b) \quad (\gamma \text{ is extensive})\]
\[\leq \gamma((\gamma(a)/b)b) \quad (i)\]
\[\leq \gamma(\gamma(a)) \quad (\gamma \text{ is monotone, Lemma 2.1(iv)})\]
\[= \gamma(a). \quad (\gamma \text{ is idempotent})\]

So, \(\gamma(\gamma(a)/b) \leq \gamma(a)/b\), by the defining property of residuated lattices. Since the reverse inequality follows from the fact that \(\gamma\) is extensive, we have \(\gamma(a)/b = \gamma(\gamma(a)/b) \in L_\gamma\). Likewise, we obtain the result for the other division operation.

(ii) \(\Rightarrow\) (i) Let \(a, b \in L\). Since \(\gamma\) is extensive, \(ab \leq \gamma(ab)\), so \(a \leq \gamma(ab)/b\). By the monotonicity of \(\gamma\) and the hypothesis, \(\gamma(a) \leq \gamma(ab)/b\). Using the defining property of residuated lattices, we get \(b \leq \gamma(a)\gamma(ab)\). Invoking, once more, the monotonicity of \(\gamma\) and the hypothesis, we obtain \(\gamma(b) \leq \gamma(a)\gamma(ab)\), namely \(\gamma(a)\gamma(b) \leq \gamma(ab)\).

(ii) \(\Leftrightarrow\) (iii) This is trivial by the definition of a subact. \(\square\)

Actually, it can be shown that an arbitrary map \(\gamma\) on a residuated lattice \(L\) is a nucleus if and only if \(\gamma(a)/b = \gamma(a)\gamma(b)\) and \(b\gamma(a) = \gamma(b)\gamma(a)\), for all \(a, b \in L\) (see [25], page 30).

**Corollary 3.2** Conditions (3.1) and (3.2) establish a bijective correspondence between nuclei on and subacts of a residuated lattice.

**PROOF.** Use Lemma 3.1. \(\square\)

The next result shows that every subact of a residuated lattice is a residuated lattice in its own right.

**Lemma 3.3** Let \(L = \langle L, \wedge, \vee, \cdot, \backslash, /, e \rangle\) be a residuated lattice, \(\gamma\) be a nucleus on \(L\) and \(L_\gamma\) be the subact associated with \(\gamma\). Then the algebraic system \(L_\gamma = \langle L_\gamma, \wedge, \vee, \cdot, \backslash, /, \gamma(e) \rangle\) – where \(x \cdot \gamma y = \gamma(x \cdot y)\) and \(x \vee \gamma y = \gamma(x \vee y)\) – is a residuated lattice.

**PROOF.** Obviously, \(\gamma(e)\) is the multiplicative identity of \(L_\gamma\). Further, \(L_\gamma\), being the image of a closure operator on \(L\), is a lattice with joins and meets defined by \(x \vee \gamma y = \gamma(x \vee y)\) and \(x \backslash \gamma y = x \backslash y\), for all \(x, y \in L_\gamma\). One can easily check that multiplication is associative. Finally, to check that \(\cdot \gamma\) is residuated, recall that \(L_\gamma\) is closed under the division operations by Lemma 3.1. Consider
We have \( x \circ \gamma y \leq z \iff \gamma(xy) \leq z \iff xy \leq z \) (since \( z = \gamma(z) \) and \( xy \leq \gamma(xy) \)) \( \iff y \leq x \setminus z \). Likewise, we have \( x \circ \gamma y \leq z \iff x \leq z/y \).

\[ \square \]

**Theorem 3.4** If \( L = \langle L, \land, \lor, \cdot, \setminus, /, e \rangle \) is a GMV-algebra, \( \gamma \) a nucleus on it and \( L_\gamma \) the associated subact, then

(i) \( \lor_\gamma = \lor \);

(ii) \( \gamma \) preserves binary joins;

(iii) \( \gamma(e) = e \);

(iv) \( L_\gamma = \langle L_\gamma, \land, \lor, \circ, \setminus, /, e \rangle \) is a GMV-algebra; and

(v) \( L_\gamma \) is a filter in \( L \).

**Proof.**

(i) Since \( L \) is a GMV-algebra, if \( x \in L_\gamma \), then \( x \lor y = x/(x \lor y) \setminus x \) \( \in L_\gamma \), by Lemma 3.1(ii). Thus, \( \lor_\gamma \) is the restriction of \( \lor \) on \( L_\gamma \).

(ii) It is well-known, and easy to prove, that if \( \gamma \) is a closure operator on a poset \( P \) and \( X \) is a subset of \( P \) such that \( P \lor X \) exists, then \( P \lor \gamma(X) \) exists and \( P \lor \gamma(X) = \gamma(P \lor X) \). Thus, (i) and (ii) are equivalent.

(iii) Since \( \gamma \) is extensive, \( e \leq \gamma(e) \). Hence, \( \gamma(e) \) is invertible, by Lemma 2.7(i). Since \( \gamma \) is a nucleus, \( \gamma(e) \gamma(e) \leq \gamma(e) \), so \( \gamma(e) \leq e \). Thus, \( \gamma(e) = e \).

(iv) By Lemma 3.3, \( L_\gamma \) is a residuated lattice. It is a GMV-algebra because the join and division operations of \( L_\gamma \) are the restrictions of the corresponding operations in \( L \), and \( L \) is a GMV-algebra.

(v) If \( x \in L_\gamma \), \( y \in L \) and \( x \leq y \), then by Lemma 3.1, \( y = x \lor y = x/(x \lor y) \setminus x \) is an element of \( L_\gamma \). Since \( L_\gamma \) is also a sublattice, it is a lattice-filter.

\( \square \)

**Corollary 3.5** If \( L \) is an integral GMV-algebra and \( \gamma \) is a nucleus on \( L \), then \( L_\gamma \) is an integral GMV-algebra.

**Lemma 3.6** Let \( \gamma \) be a nucleus on the negative cone \( L \) of an \( \ell \)-group. If \( z \in L \) and \( u = \gamma(z) \), then \( \gamma \) agrees with the nucleus \( \gamma_u \) on the principal filter \( \uparrow z \).

**Proof.** Let \( x \geq z \). We will show that \( \gamma(x) = u \lor x \). Note that \( u \lor x = \gamma(z) \lor x \leq \gamma(x) \), since \( \gamma \) is monotone and extensive. On the other hand, \( x \leq u \lor x \), so \( \gamma(x) \leq \gamma(u \lor x) = u \lor x \), because \( L_\gamma \) is a filter, by Theorem 3.4(v).

\( \square \)

**Corollary 3.7** Every nucleus on a GMV-algebra is a lattice homomorphism.
PROOF. In view of Theorem 3.4(ii), we need only show that \( \gamma \) preserves binary meets. Let \( x, y \) be elements of a GMV-algebra and set \( z = x \wedge y \) and \( u = \gamma(z) \). Recall that a GMV-algebra has a distributive lattice reduct; refer to Lemma 2.5 and Lemma 2.9. Whence by Lemma 3.6, \( \gamma(x \wedge y) = \gamma_u(x \wedge y) = u \vee (x \wedge y) = (u \vee x) \wedge (u \vee y) = \gamma_u(x) \wedge \gamma_u(y) = \gamma(x) \wedge \gamma(y) \). \( \square \)

By Corollary 3.5, the image of a nucleus on the negative cone of an \( \ell \)-group is an integral GMV-algebra. In the remainder of this section we are concerned with the proof of the converse, namely that every integral GMV-algebra is the image of a nucleus on the negative cone of an \( \ell \)-group. Our proof is based on Theorem 3.11, which is due to B. Bosbach, see [5] and [6].

**Definition 3.8** A cone algebra is an algebra \( C = \langle C, \backslash, /, e \rangle \) that satisfies:

\[
\begin{align*}
(x \backslash y)(x \backslash z) &\approx (y \backslash x)(y \backslash z) & (z/y)(x/y) &\approx (z/x)/(y/x) \\
e \backslash y &\approx y & y/e &\approx y \\
x \backslash (y/z) &\approx (x \backslash y)/z & x/(y \backslash x) &\approx (y/x) \backslash y \\
x \backslash x &\approx e & x/x &\approx e
\end{align*}
\]

**Lemma 3.9** ([5], [6]) If \( C = \langle C, \backslash, /, e \rangle \) is a cone algebra, then

(i) for all \( a, b \in C \), \( a \backslash b = e \iff b/a = e \);

(ii) the relation \( \leq \) on \( C \), defined by \( a \leq b \iff a \backslash b = e \), is a semilattice order with \( a \vee b = a/(b \backslash a) \); in particular \( a \leq e \), for all \( a \);

(iii) if \( a \leq b \), then \( c \backslash a \leq c \backslash b \) and \( a/c \leq b/c \).

It is easy to see that if \( L = \langle L, \wedge, \vee, \cdot, \backslash, /, e \rangle \) is an integral GMV-algebra – for example, \( L \in \mathcal{L} \mathcal{G}^- \) – then \( \langle L, \backslash, /, e \rangle \) is a cone algebra, called the cone algebra of \( L \).

It will be shown that every cone algebra is a subalgebra of the cone algebra of a residuated lattice in \( \mathcal{L} \mathcal{G}^- \). In the following construction, the algebra in \( \mathcal{L} \mathcal{G}^- \) is defined as the union of an ascending chain \( \langle C_n \mid n \in \mathbb{N} \rangle \) of cone algebras, each of which is a subalgebra of its successor. In the process of constructing the algebras \( C_n \), we also define in \( C_{n+1} \) binary products of elements of \( C_n \). Each such product is identified with the congruence class of the corresponding ordered pair. The definition below of the division operations becomes less opaque if we note that the negative cones of any \( \ell \)-group satisfies the law \( ab \backslash cd = (b \backslash (a \backslash c)) \cdot ((c \backslash ((a \backslash c) \backslash b))\backslash (c \backslash a) \backslash d)) \) and its opposite.

Let \( C \) be a cone algebra. Define the operations \( \backslash \) and \( / \) and the relations \( \Theta \) and \( \Theta' \) on \( C \times C \), by

\[
(a, b) \backslash (c, d) = (b \backslash (a \backslash c), ((a \backslash c) \backslash b)\backslash ((c \backslash a) \backslash d))
\]
\[(d, c)/(b, a) = ((d/(a/c))/(b/(c/a))", 
\[(a, b) \Theta (c, d) \iff (a, b)/(c, d) = (e, e) \text{ and } (c, d)/(a, b) = (e, e)\]
\[(a, b) \Theta' (c, d) \iff (a, b)/(c, d) = (e, e) \text{ and } (c, d)/(a, b) = (e, e)\]

Lemma 3.10 [6] Let \(C = \langle C, \setminus, /, e \rangle\) be a cone algebra. Then:

(i) \(\Theta = \Theta'\).

(ii) \(\Theta\) is a congruence relation of \(C \times C\).

(iii) \(s(C) = \langle C \times C, \setminus, /, e \rangle/\Theta\) is a cone algebra.

(iv) For each \(x \in C\), let \([[(x, e)]_\Theta\) denote the \(\Theta\)-congruence class of \((x, e)\). Then the map \(x \mapsto [[(x, e)]_\Theta\) is an embedding of \(C\) into \(s(C)\).

Let \(C_n = C\), \(C_{n+1} = s(C_n)\), for every natural number \(n\), and \(\mathcal{C} = \bigcup C_n\), the directed union of the \(C_n\)’s.

We can now establish the main result of [6].

Theorem 3.11 [6] Every cone algebra \(C\) is a subalgebra of the cone algebra of some \(\hat{C} \in LG^-\). Moreover, every element of \(\hat{C}\) can be written as a product of elements of \(C\).

**Proof.** We will show that the algebra \(\mathcal{C}\) defined above is the cone algebra, i.e., the \(\{\setminus, /, e\}\)-reduct, of a \(\hat{C} \in LG^-\).

For two elements of \(\mathcal{C}\), we define their product, \(ab\), to be the element \([[(a, b)]_\Theta\). This is well defined, because of the embedding of \(C_n\) into \(C_{n+1}\), for every \(n\).

Let \(\mathcal{C} = \langle \mathcal{C}, \wedge, \vee, \cdot, \setminus, /, e \rangle\), where \(\setminus = \setminus_{\mathcal{C}}, \ / = /_{\mathcal{C}}, \ x \vee y = x/(y\setminus x)\) and \(x \wedge y = (x/y)y\). We will show that \(\mathcal{C} \in LG^-\).

By the definition of the operations in \(\hat{C}\) and Lemma 3.9(ii), \(\hat{C}\) is a join semilattice. Note that \(ab\setminus cd = (b\setminus((a\setminus c)) \cdot (((a\setminus c)\setminus b)\setminus((c\setminus a)\setminus d))\). In particular, \(ab\setminus c = b\setminus(a\setminus c)\) and \(a\setminus ab = b\). The opposite equations hold, as well. Finally, note that \(e/a = e\).

**Multiplication is order preserving:**

Let \(a \leq c\); then \(e = a\setminus c\), by the definition of \(\leq\). To show that \(ab \leq cb\), we note that
\[ab\setminus cb = b\setminus[(c\setminus a)\setminus b] = [(c\setminus a)b]\setminus b.\]

On the other hand,
\[b/[(c\setminus a)b] = (b/b)/(c\setminus d) = e/(c\setminus d) = e.\]

This successively yields, \((c\setminus a)b \leq b, [(c\setminus a)b]\setminus b = e, ab\setminus cb = e\) and \(ab \leq cb\).

Likewise \(a \leq c\) implies \(ba \leq bc\).
Multiplication is residuated:
Note that \( a(a \setminus c) \leq c \), since \( [a(a \setminus c)] \setminus c = (a \setminus c)(a \setminus c) = e \). If \( ab \leq c \), then \( a \setminus ab \leq a \setminus c \), so \( b \leq a \setminus c \). Conversely, if \( b \leq a \setminus c \), then \( ab \leq a(a \setminus c) \leq c \). The other equivalence is obtained similarly.

Multiplication is associative:
We have the following sequence of equivalences:

\[
(ab)c \leq d \iff ab \leq d/c \iff b \leq (a \setminus d)/c \iff bc \leq (a \setminus d) \iff a(bc) \leq d.
\]

\( \wedge \) is the meet operation:
We have \( a(a \setminus b) \leq b \) and \( a(a \setminus b) \leq ae = a \). Additionally, if \( c \leq a \) and \( c \leq b \), then \( e = c \setminus a = c \setminus b \). We have, \( c \setminus a(a \setminus b) = (c \setminus a)(c \setminus b) = e \), so \( c \leq a(a \setminus b) \). Interchanging the roles of \( a \) and \( b \) we get that \( c \leq a, b \iff c \leq b(b \setminus a) \). The opposite properties are obtained similarly.

Thus, \( \hat{C} \) is a residuated lattice. Since it satisfies the identities \( x \setminus xy \approx y \approx yx/x \) and \( x/(y \setminus x) \approx x \lor y \approx (x/y) \setminus x \), it is in \( \mathcal{LG}^- \), by Theorem 2.12. Finally, by construction, every element of \( \hat{C} \) is the product of elements of \( C \).

The algebra \( \hat{C} \) is called the product extension of \( C \).

We can now prove the main result of this section.

**Theorem 3.12** The residuated lattice \( M \) is an integral GMV-algebra if and only if \( M \cong L_\gamma \), for some \( L \in \mathcal{LG}^- \) and some nucleus \( \gamma \) on \( L \).

**Proof.** One direction follows from Corollary 3.5. For the opposite implication, let \( M = \langle M, \wedge, \lor, \setminus, /, e \rangle \) be an integral GMV-algebra. Using Lemmas 2.5, 2.11(ii), 2.7(ii), 2.1(viii), 2.1(vii) and 2.11(i), we see that \( \langle M, \setminus, /, e \rangle \) is a cone algebra. So, by Theorem 3.11, it is a subreduct of a residuated lattice \( L = \hat{M} \in \mathcal{LG}^- \) such that \( M \) generates \( L \) as a monoid.

Since the division operations of \( M \) are the restrictions of the division operations of \( L \), we use the symbols \( \setminus \) and \( / \) for the latter, as well. Moreover, the same holds for the join and the constant \( e \), because in integral GMV-algebras they are term definable by the division operations: \( x \lor y \approx x/(y \setminus x) \) and \( e \approx x/x \). We use “·” to denote the multiplication of \( L \).
Since $M$ generates $L$ as a monoid, for every $x \in L$ there exists a sequence $(x_1, \ldots, x_n)$ of elements of $M$ such that $x = x_1 \cdots x_n$.

Claim 1: If $z \in M, x \in L$ and $(x_1, \ldots, x_n)$ is a sequence of elements of $M$ such that $x = x_1 \cdots x_n$, then $z \vee x = z \vee (x_1 \circ \cdots \circ x_n)$.

Indeed, $z \vee x = z/(x \setminus z)$ (axiom of IGMV-algebras)
\[= z/((x_1 \cdots x_n) \setminus z)\]
\[= z/[x_n \setminus (\ldots (x_2 \setminus (x_1 \setminus z)) \ldots)]\] (Lemma 2.1(vi))
\[= z/((x_1 \circ \cdots \circ x_n) \setminus z)\] (Lemma 2.1(vi))
\[= z \vee (x_1 \circ \cdots \circ x_n)\] (axiom of IGMV-algebras)

Claim 2: Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ be sequences of elements of $M$ such that $x_1 \cdots x_n = y_1 \cdots y_m$. Then, $x_1 \circ \cdots \circ x_n = y_1 \circ \cdots \circ y_m$.

Indeed, $x_1 \circ \cdots \circ x_n \vee y_1 \circ \cdots \circ y_m = x_1 \circ \cdots \circ x_n \vee x_1 \circ \cdots \circ x_n$, by the preceding claim. It follows that $y_1 \circ \cdots \circ y_m \leq x_1 \circ \cdots \circ x_n$, and likewise, $x_1 \circ \cdots \circ x_n \leq y_1 \circ \cdots \circ y_m$. Hence, $x_1 \circ \cdots \circ x_n = y_1 \circ \cdots \circ y_m$.

We now define a map $\gamma$ on $L$ as follows: if $x \in L$ and $(x_1, \ldots, x_n)$ is a sequence of elements of $M$ such that $x = x_1 \cdots x_n$, we let $\gamma(x) = x_1 \circ \cdots \circ x_n$. By Claim 2, $\gamma$ is well-defined. We will show that it is a nucleus on $L$, $L \gamma = M$ and $L, \sim \cong M$.

Note that $\gamma(x) \in M$, for all $x \in L$, so by setting $z = \gamma(x)$ in the statement of Claim 1, we get $\gamma(x) \vee x = \gamma(x)$. So, $x \leq \gamma(x)$, for all $x \in L$. If $x \leq y$, then

\[
\gamma(x) \leq \gamma(y) \vee \gamma(x) = \gamma(y) \vee x \quad \text{(Claim 1 for } z = \gamma(y))
\]
\[\leq \gamma(y) \vee y \quad (x \leq y)
\]
\[\leq \gamma(y) \quad \text{(since } \gamma \text{ is extensive)}
\]

This shows that $\gamma$ is monotone. The following computation shows that $\gamma$ is idempotent, and hence a closure operator.
\[
\gamma(\gamma(x)) = \gamma(x_1 \circ \cdots \circ x_n) = x_1 \circ \cdots \circ x_n = \gamma(x).
\]

Finally, if $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_m$, are two representations of $x$ and $y$ in terms of elements of $M$, then
\[ \gamma(x)\gamma(y) \leq \gamma(\gamma(x)\gamma(y)) \quad \text{(since \(\gamma\) is extensive)} \]
\[ = \gamma(x) \circ \gamma(y) \quad \text{(definition of \(\gamma\))} \]
\[ = (x_1 \circ \cdots \circ x_n) \circ (y_1 \circ \cdots \circ y_m) \quad \text{(definition of \(\gamma\))} \]
\[ = \gamma(xy) \quad \text{(definition of \(\gamma\))} \]

Thus, \(\gamma\) is a nucleus.

It is clear that \(L_\gamma = M\), by the definition of \(\gamma(x)\). Further, we have already observed that the division operations, join and \(e\) agree on \(L_\gamma\) and \(M\). Also, for \(x, y \in M\), \(x \circ_\gamma y = \gamma(xy) = x \circ y\). Finally, the meet operation on the two structures is the same, since integral GMV-algebras satisfy the identity \(x \land y \approx (x/y)y\). Thus, the two structures \(M\) and \(L_\gamma\) are identical. \(\square\)

As an example, we note that the collection of all co-finite subsets of \(\omega\) is the universe of a generalized Boolean algebra \(A\), hence an integral GMV-algebra. It is easy to see that \(A \cong ((\mathbb{Z}^-)^\omega)_\gamma\), where \(\mathbb{Z}^-\) is the \(\ell\)-group of the integers under addition and the natural order, and \(\gamma(((x_n)_{n \in \omega})) = (x_n \lor (-1))_{n \in \omega}\).

## 4 A categorical equivalence for integral GMV-algebras

In this section we extend the representation of integral GMV-algebras, discussed in the previous section, to a categorical equivalence.

Let \(\mathcal{IGMV}\) be the category with objects integral GMV-algebras and morphisms residuated lattice homomorphisms. Also, let \(\mathcal{LG}^-\) be the category with objects algebras \(\langle L, \land, \lor, \cdot, \\setminus, /, e, \gamma \rangle\), where \(L = \langle L, \land, \lor, \cdot, \\setminus, /, e \rangle \in \mathcal{LG}^-\) and \(\gamma\) is a nucleus on \(L\) such that its image generates \(L\) as a monoid. (In what follows, we will use the notation \(\langle L, \gamma \rangle\) for the objects of \(\mathcal{LG}^-\).) Let the morphisms of this category be homomorphisms between these algebras. The main result of this section, Theorem 4.12, asserts that the two categories defined above are equivalent.

Definition 4.2 and Lemma 4.3 below have been influenced by results in [24]. Lemmas 4.5 and 4.7 are non-commutative, unbounded versions of results in the same paper.

**Lemma 4.1** Let \(a, b, c\) be elements of a residuated lattice \(L \in \mathcal{LG}^-\). Then, \(ab = c\) iff \((a = c/b \text{ and } c \leq b)\) iff \((b = a \setminus c \text{ and } c \leq a)\).

**PROOF.** We prove only the first equivalence. If \(ab = c\), then \(ab/b = c/b\),
so, by Theorem 2.12, \( a = c/b \). Moreover, \( c = ab \leq eb \leq b \), by integrality. Conversely, if \( a = c/b \) and \( c \leq b \), then \( ab = (c/b)b \). So, \( ab = c \wedge b \), because \( L \in \mathcal{IGBL} \), by Theorem 2.12. Since \( c \leq b \), we get \( ab = c \). \( \square \)

Recall that if \( \gamma \) is a nucleus on some \( L \in \mathcal{LG}^- \), the monoid multiplication \( \circ \gamma \) of \( L_\gamma \) is defined by \( x \circ \gamma y = \gamma(xy) \), for all elements \( x, y \in L \) (see Lemma 3.3).

**Definition 4.2** Let \( \gamma \) be a nucleus on \( L \in \mathcal{LG}^- \) and let \( x \) be an element of \( L \). A sequence \((x_1, \ldots, x_n)\) of elements of \( L_\gamma \) is called a *decomposition* of \( x \) with respect to \( \gamma \) if \( x = x_1 \cdots x_n \). A decomposition is called *canonical* if, in addition, \( x_i \circ \gamma x_{i+1} = x_i \), for all \( i \in \{1, \ldots, n-1\} \).

**Lemma 4.3** Let \( L \in \mathcal{LG}^- \) and let \( \gamma \) be a nucleus on \( L \) such that \( L_\gamma \) generates \( L \) as a monoid. Then every element of \( L \) has a canonical decomposition with respect to \( \gamma \). Moreover, if \((x_1, \ldots, x_n)\) and \((x'_1, \ldots, x'_m)\) are canonical decompositions of the same element with respect to \( \gamma \) and \( m \geq n \), then \( x_i = x'_i \), for all \( i \in \{1, \ldots, n\} \) and \( x'_i = \epsilon \) for all \( i \in \{n+1, \ldots, m\} \).

**PROOF.** We first construct a canonical decomposition of an arbitrary element \( x \) of \( L \). Let \( x_1 = \gamma(x) \) and \( x_{i+1} = \gamma((x_1 \cdots x_i) \backslash x) \), for all \( i \geq 1 \). We will prove that there exists a natural number \( n \) such that \( x = x_1 \cdots x_n \) and \( x_i \circ \gamma x_{i+1} = x_i \) for all \( i \in \{1, \ldots, n-1\} \).

We show, by induction, that \( x \leq x_1 \cdots x_k \), for every integer \( k \geq 1 \). For \( k = 1 \) we have \( x \leq \gamma(x) = x_1 \). If \( x \leq x_1 \cdots x_k \), then Lemma 2.11(i) yields

\[
\begin{align*}
x &= x_1 \cdots x_k \wedge x \\
&= x_1 \cdots x_k \cdot ((x_1 \cdots x_k) \backslash x) \\
&\leq x_1 \cdots x_k \cdot \gamma((x_1 \cdots x_k) \backslash x) \\
&= x_1 \cdots x_k \cdot x_{k+1}.
\end{align*}
\]

Next, let \( z \) be any element of \( L \) such that \( z \leq x \) and set \( u = \gamma(z) \). By Lemma 3.6, the maps \( \gamma \) and \( \gamma_u \) agree on \( \uparrow x \). The arguments of \( \gamma \) in the definition of the elements \( x_i \), as well as in the equality \( \gamma(x_i \cdot x_{i+1}) = x_i \), are in \( \uparrow x \), so we can replace \( \gamma \) by \( \gamma_u \). In particular, a decomposition of an element \( x \) is canonical with respect to \( \gamma \) if and only if it is canonical with respect to some/every \( \gamma_u \) such that \( u = \gamma(z) \) and \( z \leq x \).

Applying Lemma 4.1, for \( a = x_i, b = x_i \backslash ((x_1 \cdots x_{i-1}) \backslash x) \) and \( c = (x_1 \cdots x_{i-1}) \backslash x \), we obtain for all \( i \geq 1 \),

\[
x_i[x_i \backslash ((x_1 \cdots x_{i-1}) \backslash x)] = (x_1 \cdots x_{i-1}) \backslash x,
\]

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where \( x_1 \cdots x_{i-1} = e \) for \( i = 1 \). It follows that, for all \( i \geq 1 \),

\[
x_i \circ_{\gamma} x_{i+1} = \gamma(x_i x_{i+1}) = \gamma_u(x_i x_{i+1})
\]

\[
= u \lor \, (x_i x_{i+1}) = u \lor \, (x_i \gamma_u((x_1 \cdots x_i)\backslash x))
\]

\[
= u \lor \, x_i (u \lor \, [(x_1 \cdots x_i)\backslash x])
\]

\[
= u \lor \, x_i u \lor \, x_i [(x_1 \cdots x_i)\backslash x]
\]

\[
= u \lor \, x_i u \lor \, x_i [(x_1 \cdots x_{i-1})\backslash x]
\]

\[
= u \lor \, [(x_1 \cdots x_{i-1})\backslash x]
\]

\[
= \gamma_u((x_1 \cdots x_{i-1})\backslash x)
\]

\[
= \gamma((x_1 \cdots x_{i-1})\backslash x) = x_i.
\]

We next show that \((x_1 \cdots x_k)\backslash x = u^k \backslash x\), for all \( k \geq 1 \), using induction on \( k \).

For \( k = 1 \), we have

\[
x_1 \backslash x = \gamma(x) \backslash x = \gamma_u(x) \backslash x = (x \lor u) \backslash x = x \lor \, u \lor \, x \land u \lor \, x = e \land u \lor \, x = u \lor \, x.
\]

Assume that the statement is true for \( k \). To show that it is true for \( k + 1 \), note that, using properties (iii) and (vi) of Lemma 2.1, we get

\[
(x_1 \cdots x_{k+1}) \backslash x = x_{k+1} \backslash [(x_1 \cdots x_k) \backslash x]
\]

\[
= \gamma((x_1 \cdots x_k) \backslash x) \backslash [(x_1 \cdots x_k) \backslash x]
\]

\[
= [u \lor \, ((x_1 \cdots x_k) \backslash x)] \backslash [(x_1 \cdots x_k) \backslash x]
\]

\[
= u \lor \, [(x_1 \cdots x_k) \backslash x] \land
\]

\[
[(x_1 \cdots x_k) \backslash x] \land (x_1 \cdots x_k) \backslash x
\]

\[
= u \lor \, [(x_1 \cdots x_k) \backslash x] \land e
\]

\[
= u \lor \, (u^k \backslash x) = u^{k+1} \backslash x.
\]

We have shown that \((x_1 \cdots x_k) \backslash x = u^k \backslash x\), for all \( k \geq 1 \).

Since \( L \) is the monoid generated by \( L_\gamma \), there exist a natural number \( n \) and elements \( a_1, \ldots, a_n \in L_\gamma \) such that \( x = a_1 \cdots a_n \). Thus, \( u \leq \gamma(x) = \gamma(a_1 \cdots a_n) = a_1 \circ_{\gamma} \cdots \circ_{\gamma} a_n \leq a_i \), for all \( i \). It follows that \( u^n \leq a_1 \cdots a_n = x \). Consequently, \( e \leq u^n \backslash x = (x_1 \cdots x_n) \backslash x \), that is, \( x_1 \cdots x_n \leq x \). Since the reverse inequality was established above, we have \( x = x_1 \cdots x_n \).

To establish uniqueness, let \((x_1, \ldots, x_n)\) and \((x'_1, \ldots, x'_m)\) be canonical decompositions of an element \( x \) with respect to \( \gamma \) and \( m \geq n \). Then, \( x_i \circ_{\gamma} x_{i+1} = x_i \).
$x'_i \circ \gamma x'_{i+1} = x'_i$, for all appropriate values of $i$, and $x_1 \cdots x_n = x'_1 \cdots x'_m$. So, 
$\gamma(x_1 \cdots x_n) = \gamma(x'_1 \cdots x'_m)$, i.e., 
\[ x_1 \circ \gamma \cdots \circ \gamma x_n = x'_1 \circ \gamma \cdots \circ \gamma x'_m. \]

Hence $x_1 = x'_1$, by the defining property of canonical decompositions. Consequently, $x_1 x_2 \cdots x_n = x'_1 x'_2 \cdots x'_m$, so $x_2 \cdots x_n = x'_2 \cdots x'_m$, by cancellativity. Proceeding inductively, we get $x_i = x'_i$, for all $i \in \{1, \ldots, n\}$. Another application of cancellativity yields $e = x'_{n+1} \cdots x'_m$, hence $x'_i = e$ for all $i \in \{n+1, \ldots, m\}$, by integrality.

It follows from the preceding lemma that each element has a canonical decomposition unique up to the addition of extra terms, equal to $e$, at the end of the sequence. Thus, when we consider canonical decompositions of a finite set of elements, we may assume that all have the same length.

**Corollary 4.4** Let $L \in LG^{-}$ and let $\gamma$ be a nucleus on $L$. If $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are canonical decompositions of the elements $x$ and $y$, respectively, with respect to $\gamma$ and $x \leq y$, then $x_i \leq y_i$, for all $i \leq n$.

**Proof.** In view of the preceding lemma, we may assume that $x_i$ and $y_i$ are given by the formulas at the beginning of its proof. Let $z$ be an element of $L$ such that $z \leq x \wedge y$ and let $u = \gamma(z)$. From the proof of the previous theorem we have that $(x_1 \cdots x_k) \wedge x = u^k \wedge x$, and $(y_1 \cdots y_k) \wedge y = u^k \wedge y$, for all $k \in \{1, \ldots, n\}$. Thus, for all $i \in \{1, \ldots, n\}$,

\[
x_i = \gamma((x_1 \cdots x_{i-1}) \wedge x) \\
\quad = \gamma(u^{i-1} \wedge x) \\
\quad \leq \gamma(u^{i-1} \wedge y) \\
\quad = \gamma((y_1 \cdots y_{i-1}) \wedge y) \\
\quad = y_i,
\]

where $x_1 \cdots x_{i-1} = y_1 \cdots y_{i-1} = e$, if $i = 1$. \qed

**Lemma 4.5** Let $L \in LG^{-}$ and let $\gamma$ be a nucleus on $L$ such that $L_{\gamma}$ generates $L$ as a monoid. Also, let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be canonical decompositions of the elements $x$ and $y$, respectively. Then,

\[
x \wedge y = \prod_{i=1}^{n}(x_i \wedge y_i) \quad \text{and} \quad x \vee y = \prod_{i=1}^{n}(x_i \vee y_i).
\]
**PROOF.** Let \((z_1, \ldots, z_n)\) be a canonical decomposition of \(z = x \wedge y\). Without loss of generality we assume that the length of the decomposition of \(z\) is \(n\). We can do that by extending the decompositions of \(x\) and \(y\) or of \(z\) with extra terms each equal to \(e\). Obviously, 

\[
\prod_{i=1}^{n} (x_i \wedge y_i) \leq \prod_{i=1}^{n} x_i \wedge \prod_{i=1}^{n} y_i = x \wedge y = z
\]

Moreover, \(z \leq x, y\), so \(z_i \leq x_i, y_i\), for all \(i\), by Corollary 4.4; hence \(z_i \leq x_i \wedge y_i\). Consequently,

\[
z = \prod_{i=1}^{n} z_i \leq \prod_{i=1}^{n} (x_i \wedge y_i).
\]

Thus,

\[
z = \prod_{i=1}^{n} (x_i \wedge y_i).
\]

The proof for joins is analogous. \(\square\)

The following refinement lemma can be found in [11]. Its importance in the proof of the categorical equivalence was suggested to us by the considerations in [10]. For completeness, we give the proof in the language of negative cones of \(\ell\)-groups.

**Lemma 4.6** ([11], Theorem 1, p. 68) Let \(L \in \mathcal{L}G^-\) and let \(a_1, \ldots, a_n, b_1, \ldots, b_m\) be elements of \(L\). The following statements are equivalent.

1. The equality \(a_1 \cdots a_n = b_1 \cdots b_m\) holds.
2. There exist elements \(c_{ij}\) of \(L\), where \(1 \leq i \leq n\) and \(1 \leq j \leq m\), such that for all \(i, j\),

\[
\begin{align*}
    a_j &= \prod_{i=1}^{m} c_{ij}, \\
    b_i &= \prod_{j=1}^{n} c_{ij} \quad \text{and} \quad \prod_{l=j+1}^{m} c_{il} \lor \prod_{k=i+1}^{n} c_{kj} = e.
\end{align*}
\]

**Notation:** We denote the fact that condition (2) holds by the following configuration:

\[
\begin{bmatrix}
    a_1 & \cdots & a_n \\
    b_1 & c_{11} & \cdots & c_{1n} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_m & c_{m1} & \cdots & c_{mn}
\end{bmatrix}
\]

Thus, with respect to this description, condition (2) states that for all \(i\) and \(j\), \(a_j\) is the product of the elements of the \(j\)-th column, \(b_i\) is the product of
the elements of the \( i \)-th row and that the product of the elements to the right of \( c_{ij} \) is orthogonal to the product of elements below it.

**PROOF.** First, we show that (2) implies (1). Recall that if \( x \lor y = e \), then \( xy = yx \), by Lemma 2.10. For \( m = n = 2 \), we have \( a_1a_2 = c_{11}c_{21}c_{12}c_{22} = b_1b_2 \). We proceed by induction on the pair \( (m, n) \). Let \( m \geq 2 \), \( n > 2 \) and assume that the lemma is true for all pairs \( (m, k) \), where \( k < n \). We will show it is true for the pair \( (m, n) \).

Suppose that condition (2) holds. It is easy to see that

\[
\begin{bmatrix}
  a_1 & c_1 & c_2 & \cdots & c_m \\
  c_2 & & & & \\
  \vdots & & & & \\
  c_m & & & & \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  a_1 & c_1 & c_2 & \cdots & c_m \\
  c_2 & & & & \\
  \vdots & & & & \\
  c_m & & & & \\
\end{bmatrix}
\]

where \( c = c_1 \cdots c_m \). So, \( a_1a_2 \cdots a_n = a_1(c_1 \cdots c_m) = a_1c = b_1b_2 \cdots b_m \). Note that the lemma holds for the pair \( (m, n) \) if and only if it holds for the pair \( (n, m) \), a fact that completes the induction proof.

For the converse we use induction, as well. We first prove it for \( m = n = 2 \). Assume that \( a_1a_2 = b_1b_2 = c \) and set

\[
c_{11} = a_1 \lor b_1, \quad c_{12} = a_2/c_{22} \\
c_{21} = c_{11}\setminus a_1, \quad c_{22} = a_2 \lor b_2
\]

Using Lemma 2.13(iii), Lemma 4.1 and Lemma 2.1 we get

\[
c_{12} = a_2/c_{22} = a_2/(a_2 \lor b_2) \\
= (a_1 \setminus c)/(a_1 \setminus c \lor b_1 \setminus c) = (a_1 \setminus c)/((a_1 \land b_1) \setminus c) \\
= a_1 \setminus [c/((a_1 \land b_1) \setminus c)] = a_1 \setminus [(a_1 \land b_1) \lor c] \\
= a_1 \setminus (a_1 \land b_1) = a_1 \setminus a_1 \land a_1 \lor b_1 \\
= e \land a_1 \setminus b_1 = a_1 \setminus b_1 \lor b_1 \\
= (a_1 \lor b_1) \setminus b_1 = c_{11} \setminus b_1.
\]

Similarly, we show that \( c_{21} = b_2/c_{22} \). So, we have

\[
c_{11}c_{21} = c_{11}(c_{11} \setminus a_1) = c_{11} \land a_1 = (a_1 \lor b_1) \land a_1 = a_1. \\
c_{12}c_{22} = (a_2/c_{22})c_{22} = a_2 \land c_{22} = a_2.
\]
\[ c_{11}c_{12} = c_{11}(c_{11} \setminus b_1) = c_{11} \land b_1 = b_1. \]
\[ c_{21}c_{22} = (b_2/c_{22})c_{22} = b_2 \land c_{22} = b_2. \]

Finally, \( c_{21} = c_{11} \setminus a_1 \lor c_{11} \setminus b_1 = c_{11} \setminus (a_1 \lor b_1) = c_{11} \setminus c_{11} = e. \)

For the general case, we proceed by induction on the pair \((m, n)\). Let \(m \geq 2\), \(n > 2\) and assume that the lemma is true for all pairs \((m, k)\), where \(k < n\). We will show it is true for the pair \((m, n)\). Assume that \(a_1a_2 \cdots a_n = b_1b_2 \cdots b_n\) and set \(a = a_2a_3 \cdots a_n\). So, \(a_1a = b_1b_2 \cdots b_n\). By the induction hypothesis, we get

\[
\begin{bmatrix}
 a_1 & a \\
 b_1 & c_{11} & c_{12} \\
 \vdots & \vdots & \vdots \\
 b_m & c_{m1} & c_{m2}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
 a_2 & \cdots & a_n \\
 c_{12} & d_{12} & \cdots & d_{1n} \\
 \vdots & \vdots & \vdots \\
 c_{m2} & d_{m2} & \cdots & d_{mn}
\end{bmatrix}
\]

for some \(c_{ij}, d_{kl}\), with appropriate indices. So, we have,

\[
\begin{bmatrix}
 a_1 & a_2 & \cdots & a_n \\
 b_1 & c_{11} & d_{12} & \cdots & d_{1n} \\
 \vdots & \vdots & \vdots \\
 b_m & c_{m1} & d_{m2} & \cdots & d_{mn}
\end{bmatrix}
\]

\[ \square \]

**Lemma 4.7** Let \( L \in \mathcal{LG}^{-}\), \( \gamma \) be a nucleus on it and \( a, a_1, \ldots, a_n \in L_\gamma \). Then \( a = a_1 \cdot a_2 \cdots a_n \) if and only if \( a = a_1 \circ_\gamma a_2 \circ_\gamma \cdots \circ_\gamma a_n \) and \( a_k = (a_k \circ_\gamma a_{k+1} \circ_\gamma \cdots \circ_\gamma a_n)/(a_{k+1} \circ_\gamma a_{k+2} \circ_\gamma \cdots \circ_\gamma a_n) \), for all \( 1 \leq k < n \).

**PROOF.** We use induction on \( n \). For \( n = 2 \), if \( a = a_1a_2 \), then \( \gamma(a) = \gamma(a_1a_2) \), so \( a = a_1 \circ_\gamma a_2 \). Moreover, by Lemma 4.1, \( a_1 = a/a_2 \), so \( a_1 = (a_1 \circ_\gamma a_2)/a_2 \). Conversely, if \( a = a_1 \circ_\gamma a_2 \) and \((a_1 \circ_\gamma a_2)/a_2 = a_1\), then \( a = \gamma(a_1a_2) \leq \gamma(a_2) = a_2 \). Since \( a_1 = a/a_2 \), we get \( a = a_1a_2 \), by Lemma 4.1.

Assume now that the statement is true for all numbers less than \( n \). Note that if \( a_1a_2 \cdots a_n \in L_\gamma \), then \( a_2 \cdots a_n \in L_\gamma \), since \( a_1a_2 \cdots a_n \leq a_2 \cdots a_n \) and \( L_\gamma \) is a filter, by Theorems 3.4 and 2.12.
\[ a = a_1(a_2 \cdots a_n) \]

\[ \iff a = a_1 b, \quad b = a_2 \cdots a_n \text{ and } b \in L_\gamma \]

\[ \iff a = a_1 \circ \gamma b, \quad a_1 = a/b, \quad b = a_2 \circ \gamma \cdots \circ \gamma a_n \text{ and } \]

\[ a_k = (a_k \circ \gamma a_{k+1} \circ \gamma \cdots \circ \gamma a_n)/(a_{k+1} \circ \gamma a_{k+2} \circ \gamma \cdots \circ \gamma a_n), \]

for all \( 2 \leq k < n \)

\[ \iff a = a_1 \circ \gamma a_2 \circ \gamma \cdots \circ \gamma a_n \text{ and } \]

\[ a_k = (a_k \circ \gamma a_{k+1} \circ \gamma \cdots \circ \gamma a_n)/(a_{k+1} \circ \gamma a_{k+2} \circ \gamma \cdots \circ \gamma a_n), \]

for all \( 1 \leq k < n \).

\[ \square \]

**Lemma 4.8** Assume that \( K, L \in LG^-, \gamma_1, \gamma_2 \) are nuclei on \( K, L \) and \( K_{\gamma_1}, L_{\gamma_2} \) generate \( K \) and \( L \) as monoids, respectively. Let \( f : K_{\gamma_1} \rightarrow L_{\gamma_2} \) be a residuated lattice homomorphism and let \( a_1, \ldots, a_n, b_1, \ldots, b_m \) be elements of \( K_{\gamma_1} \), such that \( a_1a_2 \cdots a_n = b_1b_2 \cdots b_m \), where multiplication is in \( K \). Then, \( f(a_1)f(a_2) \cdots f(a_n) = f(b_1)f(b_2) \cdots f(b_m) \), where multiplication takes place in \( L \).

**PROOF.** First note that, for all \( c_1, c_2, \ldots, c_n \in K_{\gamma_1} \), if \( c_1c_2 \cdots c_n \in K_{\gamma_1} \), then

\[ f(c_1c_2 \cdots c_n) = f(c_1)f(c_2) \cdots f(c_n). \]

Indeed, by Lemma 4.7, the statement \( c = c_1c_2 \cdots c_n \), for an element \( c \in K \), is equivalent to a system of IGMV-algebra equations in \( K_{\gamma_1} \). Since \( f \) is a homomorphism, the same equations hold for the images of the elements under \( f \). Applying Lemma 4.7 again, we get \( f(c) = f(c_1)f(c_2) \cdots f(c_n) \).

Next, the equality \( a_1a_2 \cdots a_n = b_1b_2 \cdots b_m \) implies, by Lemma 4.6, that there exist \( c_{ij} \in K_{\gamma_1} \), such that for all \( i, j \),

\[ a_j = \prod_{i=1}^{m} c_{ij}, \quad b_i = \prod_{j=1}^{n} c_{ij} \quad \text{and} \quad \prod_{l=j+1}^{m} c_{il} \lor \prod_{k=i+1}^{n} c_{kj} = e. \]

Note that all of the products above are in \( K_{\gamma_1} \). Using the observation above and the fact that \( f \) preserves joins (recall that the join operation in \( K_{\gamma_1} \) is the restriction of the join operation in \( K \), by Theorems 3.4 and 2.12), we get that, for all \( i, j \),

\[ f(a_j) = \prod_{i=1}^{m} f(c_{ij}), \quad f(b_i) = \prod_{j=1}^{n} f(c_{ij}) \quad \text{and} \quad \prod_{l=j+1}^{m} f(c_{il}) \lor \prod_{k=i+1}^{n} f(c_{kj}) = e. \]
Finally, we obtain

\[ f(a_1)f(a_2) \cdots f(a_n) = f(b_1)f(b_2) \cdots f(b_m) \]

by applying Lemma 4.6 once more.

The following result is an immediate consequence of Theorem 1.4.5 of [3].

**Lemma 4.9** Any multiplicative meet-homomorphism between two members of \( \mathcal{LG}^- \) is a residuated lattice homomorphism.

**Lemma 4.10** Let \( K, L \in \mathcal{LG}^- \), and let \( \gamma_1, \gamma_2 \) be nuclei on \( K, L \), respectively, such that \( K_{\gamma_1}, L_{\gamma_2} \) generate \( K, L \) as monoids. If \( f : K_{\gamma_1} \to L_{\gamma_2} \) is a residuated lattice homomorphism, then there exists a unique homomorphism \( \bar{f} : K \to L \), such that \( f \circ \gamma_1 = \gamma_2 \circ \bar{f} \).

**PROOF.** By assumption every element of \( K \) is a product of elements of \( K_{\gamma_1} \). By Lemma 4.8, the map \( \bar{f} : K \to L \), defined by \( \bar{f}(x_1x_2 \cdots x_n) = f(x_1)f(x_2) \cdots f(x_n) \), for \( x_1, x_2, \ldots, x_n \in K_{\gamma_1} \), is well defined and obviously preserves multiplication.

If \( x \in K \), then there exist \( x_1, \ldots, x_n \in K_{\gamma_1} \) such that \( x = x_1 \cdots x_n \). Hence,

\[ \bar{f}(\gamma_1(x)) = f(\gamma_1(x)) = f(\gamma_1(x_1 \cdots x_n)) = f(x_1 \circ \gamma_1 \cdots \circ \gamma_1 x_n) = f(x_1) \circ \gamma_1 \cdots \circ \gamma_1 f(x_n) = \gamma_2(f(x_1) \cdots f(x_n)) = \gamma_2(\bar{f}(x)). \]

Thus, \( f \circ \gamma_1 = \gamma_2 \circ \bar{f} \).

Moreover, if \( (x_1, \ldots, x_n) \) is a canonical decomposition for \( x \) with respect to \( \gamma_1 \), then \( x = x_1 \cdots x_n \) and \( x_i \circ \gamma_1 x_{i+1} = x_i \). So, \( \bar{f}(x) = f(x_1) \cdots f(x_n) \) and \( f(x_i) \circ \gamma_2 f(x_{i+1}) = f(x_i) \), i.e., \( (f(x_1), \ldots, f(x_n)) \) is a canonical decomposition for \( \bar{f}(x) \) with respect to \( \gamma_2 \).

We can now show that \( \bar{f} \) preserves meets. Let \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_n) \) be canonical decompositions for \( x, y \). Then, by Lemma 4.5,

\[ \bar{f}(x \land y) = \bar{f}\left(\prod_{i=1}^{n}(x_i \land y_i)\right) \]
\[ = \prod_{i=1}^{n} f(x_i \land y_i) \]
\[ = \prod_{i=1}^{n} (f(x_i) \land f(y_i)) \]
\[ = \bar{f}(x) \land \bar{f}(y), \]

where the last equality is given by Lemma 4.5, since \( f \) preserves canonical decompositions. Thus \( \bar{f} \) preserves multiplication and meet, and hence it is a residuated lattice homomorphism, by Lemma 4.9. \( \square \)
Corollary 4.11 Under the hypothesis of the previous lemma, if $f$ is an injection, a surjection or an isomorphism, then so is $\bar{f}$.

PROOF. Assume that $f$ is onto and let $y \in L$. There exist $y_1, \ldots, y_n \in L_{\gamma_2}$, such that $y = y_1 \cdots y_n$. Moreover, there exist $x_1, \ldots, x_n \in K_{\gamma_1}$, such that $f(x_i) = y_i$ for all $i$. Then, $\bar{f}(x_1 \cdots x_n) = f(x_1) \cdots f(x_n) = y_1 \cdots y_n = y$.

Assume that $f$ is injective. If $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ are canonical decompositions for $x, y$ and $\bar{f}(x) = \bar{f}(y)$, namely $f(x_1) \cdots f(x_n) = f(y_1) \cdots f(y_n)$ then, by the preservation of the canonicity of the decomposition under $\bar{f}$, established in the proof of the previous lemma, we get $f(x_i) = f(y_i)$ for all $i$.

By the injectivity of $f$ we get $x_i = y_i$, for all $i$, so $x = y$. □

Theorem 4.12 The categories $IGMV$ and $LG^-$ are equivalent.

PROOF. For an object $(K, \gamma)$ of $LG^-$, let $\Gamma((K, \gamma)) = K_{\gamma}$; for a homomorphism $f : (K, \gamma_1) \rightarrow (L, \gamma_2)$, let $\Gamma(f)$ be the restriction of $f$ to $K_{\gamma_1}$.

By Corollary 3.5, $\Gamma((K, \gamma))$ is an object in $IGMV$. Using the fact that $f$ commutes with the nuclei $\gamma_1$ and $\gamma_2$, it is easy to see that $\Gamma(f)$ is a morphism of $IGMV$. To check, for example, that it preserves multiplication, note that $\Gamma(f)(x \circ_{\gamma_1} y) = f(\gamma_1(xy)) = \gamma_2(f(xy)) = \gamma_2(f(x)f(y)) = f(x) \circ_{\gamma_2} f(y)$.

Moreover, it is obvious that $\Gamma(f \circ g) = \Gamma(f) \circ \Gamma(g)$ and that $\Gamma(id_{K_{\gamma_1}}) = id_{K_{\gamma_2}}$. Thus, $\Gamma$ is a functor between the two categories.

By Theorem 3.12, $\Gamma$ is onto the objects of $IGMV$ and by Lemma 4.10, $\Gamma$ is full. Finally, $\Gamma$ is faithful, because if two morphisms agree on a generating set, they agree on the whole negative cone of the $\ell$-group. Thus, $\Gamma$ is a categorical equivalence between the two categories, by Theorem 1, page 93, of [23]. □

In addition to $IGMV$ and $LG^-$, we also consider the following categories, the definitions of which we organize in Table 1.

We first explain the notation that is used. A bounded GMV-algebra is a residuated bounded-lattice whose 0-free reduct is a GMV-algebra; bounded GMV-algebras are called pseudo-MV-algebras in [15]. It is easy to see that every bounded GMV-algebra is integral. Bounded GMV-algebras form a variety, which we denote by $bIGMV$. We denote the class of integral GMV-algebras by $IGMV$, and the class of objects of the category $LG^-$ by $LG^-$. If $\mathcal{K}$ is a class of algebras, we denote by $H(\mathcal{K})$ the class of all homomorphisms between the algebras of $\mathcal{K}$; we denote the submonoid of a residuated lattice generated by a set $X$ by $(X)$. The category $K$ in the first column takes as values the
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<td>( Kb = )</td>
<td>( \text{Obj} )</td>
<td>( \mathcal{IgMV} )</td>
<td>( \langle L, \gamma \rangle \in \mathcal{L}_* )</td>
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<tr>
<td></td>
<td></td>
<td>( L = \langle \gamma(L) \rangle )</td>
<td>( (G, \gamma), G \in \mathcal{L}G )</td>
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<td></td>
<td></td>
<td>( G^- = \langle \gamma(G^-) \rangle )</td>
<td>( \gamma ) is a nucleus on ( G^- )</td>
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<tr>
<td>Mor</td>
<td>( f \in H(\mathcal{IgMV}) )</td>
<td>( f \in H(\mathcal{L}_*) )</td>
<td>( f \in H(\mathcal{L}G), f : G \rightarrow H )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( f : M \rightarrow N, ) ( f : K \rightarrow L, ) ( \uparrow f[M] = N ) ( \uparrow f[K] = L ) ( \uparrow f[G] = H )</td>
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<tr>
<td>( bKb = )</td>
<td>( \text{Obj} )</td>
<td>( b\mathcal{IgMV} )</td>
<td>( \langle L, \gamma_u \rangle \in \mathcal{L}_* )</td>
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<td></td>
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<td>( u \in L = \langle \gamma(L) \rangle )</td>
<td>( (G, \gamma_u), G \in \mathcal{L}G )</td>
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<td>( u \in G^- = \langle \gamma_u(G^-) \rangle )</td>
<td>( \gamma_u ) is a nucleus on ( G^- )</td>
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<td>Mor</td>
<td>( f \in H(\mathcal{IgMV}) )</td>
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<td></td>
<td>( f : M \rightarrow N, ) ( f : K \rightarrow L, ) ( \uparrow f[M] = N ) ( \uparrow f[K] = L ) ( \uparrow f[G] = H )</td>
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</tbody>
</table>

Table 1
Categorical equivalences.
categories in the top row. For example, the last entry of the table describes the category \( \mathbf{LG}^* \).

Note that the functor defining the equivalence of Theorem 4.12 specializes to pairs of domain and range as described in (the first two columns of) the last three rows of the table. Moreover, since the category of \( \ell \)-groups and the category of their negative cones are equivalent, by [2], the categories \( \mathbf{LG}^- \) and \( \mathbf{LG}^* \) are equivalent. Consequently, all three categories in the first row of the table are equivalent. The same arguments apply to the last two columns of the remaining three rows, so each of the four rows consists of a triple of equivalent categories. The categorical equivalence of the last row is the one established by A. Dvurečenskij in [10]. If we restrict further to the commutative case, we obtain D. Mundici’s result in [24].

5 Decomposition of GBL-algebras

The primary objective of this section is to establish Theorem A (see Theorem 5.6 below). Its proof is based on the decomposition result of Theorem 5.2. We refer the reader to [22] for a comprehensive discussion of products of residuated structures.

Lemma 5.1 GBL-algebras satisfy the identity \( x \approx (x \vee e)(x \wedge e) \).

\textbf{PROOF.} By Lemma 2.8, \((e/x \wedge e)x = x \wedge e\). Moreover, by Lemma 2.7(i), \(x \vee e\) is invertible and \((x \vee e)^{-1} = e/(x \vee e) = e/x \wedge e\). Thus, \((x \vee e)^{-1}x = x \wedge e\), or \(x = (x \vee e)(x \wedge e)\). \(\Box\)

The following theorem shows that if \(L\) is a GBL-algebra then the sets \(G(L)\) and \(I(L)\), given in definition 2.6, are subuniverses of \(L\). We denote the corresponding subalgebras by \(G(L)\) and \(I(L)\).

\textbf{Theorem 5.2} Every GBL-algebra \(L\) decomposes into the direct sum \(G(L) \oplus I(L)\).

\textbf{PROOF.} We begin with a series of claims.

Claim 1: \(G(L)\) is a subuniverse of \(L\).

Let \(x, y\) be invertible elements. It is clear that \(xy\) is invertible. Additionally, for all \(x, y \in G(L)\) and \(z \in L\), \(z \leq x^{-1}y \iff xz \leq y \iff z \leq x\setminus y\). It follows that \(x\setminus y = x^{-1}y\), hence \(x\setminus y\) is invertible. Likewise, \(y/x = yx^{-1}\) is invertible.
Moreover, \( x \lor y = (xy^{-1} \lor e)y \). So, \( x \lor y \) is invertible, since every positive element is invertible, by Lemma 2.7(i), and the product of two invertible elements is invertible. By Lemma 2.1(iii), \( x \land y = e/(x^{-1} \lor y^{-1}) \), which is invertible, since we have already shown that \( G(L) \) is closed under joins and the division operations.

**Claim 2:** \( I(L) \) is a subuniverse of \( L \).

Note that every integral element \( a \) is negative, since \( e = e/a \) implies \( e \leq e/a \) and \( a \leq e \). For \( x, y \in I(L) \), using Lemma 2.1 repeatedly, we get:

\[
e/xy = (e/y)/x = e/x = e, \text{ so } xy \in I(L).
e/(x \lor y) = e/x \land e/y = e, \text{ so } x \lor y \in I(L).
e \leq e/x \leq e/(x \land y) \leq e/xy = e, \text{ so } x \land y \in I(L).
e = e/(e/y) \leq e/(x/y) \leq e/(e/e) = e/x = e, \text{ so } x/y \in I(L).
\]

**Claim 3:** For every \( g \in (G(L))^− \) and every \( h \in I(L) \), \( g \lor h = e \).

Let \( g \in (G(L))^− \) and \( h \in I(L) \). We have \( e/(g \lor h) = e/g \land e/h = e/g \land e = e \), since \( e \leq e/g \). Moreover, \( g \leq g \lor h \), so \( e \leq g^{-1}(g \lor h) \). Thus, by the GBL-algebra identities and Lemma 2.1

\[
e = (e/[g^{-1}(g \lor h)]/[g^{-1}(g \lor h)])
= ([e/(g \lor h)]/[g^{-1}]g^{-1}(g \lor h)
= (e/g^{-1})g^{-1}(g \lor h)
= gg^{-1}(g \lor h)
= g \lor h.
\]

**Claim 4:** For every \( g \in (G(L))^− \) and every \( h \in I(L) \), \( gh = g \land h \).

In light of Lemma 5.1, \( g^{-1}h = (g^{-1}h \lor e)(g^{-1}h \land e) \). Multiplication by \( g \) yields \( h = (h \lor g)(g^{-1}h \land e) \). Using Claim 3, we have \( gh = g(g^{-1}h \land e) = h \land g \), since multiplication by an invertible element is an order automorphism.

**Claim 5:** For every \( g \in G(L) \) and every \( h \in I(L) \), \( gh = hg \).

The statement is true if \( g \leq e \), by Claim 4. If \( g \geq e \) then \( g^{-1} \leq e \), thus \( g^{-1}h = hg^{-1} \), hence \( hg = gh \). For arbitrary \( g \), note that both \( g \lor e \) and \( g \land e \) commute with \( h \). Using Lemma 5.1, we get \( gh = (g \lor e)(g \land e)h = (g \lor e)h(g \land e) = h(g \lor e)(g \land e) = hg \).
Claim 6: For every $x \in L$, there exist $g_x \in G(L)$ and $h_x \in I(L)$, such that $x = g_xh_x$.

By Lemma 5.1, $x = (x \lor e)(x \land e)$. Since $e \leq x \lor e$ and $e \leq e/(x \land e)$, by Lemma 2.7(i), these elements are invertible. Set $g_x = (x \lor e)(e/(x \land e))^{-1}$ and $h_x = (e/(x \land e))(x \land e)$. It is clear that $x = g_xh_x$, $g_x$ is invertible and $h_x$ is integral.

Claim 7: For every $g_1, g_2 \in G(L)$ and $h_1, h_2 \in I(L)$, $g_1h_1 \leq g_2h_2$ if and only if $g_1 \leq g_2$ and $h_1 \leq h_2$.

For the non-trivial direction we have

$$g_1h_1 \leq g_2h_2 \Rightarrow g_2^{-1}g_1h_1 \leq h_2 \Rightarrow g_2^{-1}g_1 \leq h_2/h_1 \leq e \Rightarrow g_1 \leq g_2.$$ 

Moreover,

$$g_2^{-1}g_1 \leq h_2/h_1 \Rightarrow e \leq g_1^{-1}g_2(h_2/h_1)$$

$$\Rightarrow e = [e/g_1^{-1}g_2(h_2/h_1)]g_1^{-1}g_2(h_2/h_1)$$

$$\Rightarrow e = [(e/(h_2/h_1))/g_1^{-1}g_2]g_1^{-1}g_2(h_2/h_1)$$

$$\Rightarrow e = g_1^{-1}g_2g_1^{-1}g_2(h_2/h_1)$$

$$\Rightarrow e = h_2/h_1$$

$$\Rightarrow h_1 \leq h_2.$$ 

By Claims 1 and 2, $G(L)$ and $I(L)$ are subalgebras of $L$. Define $f : G(L) \times I(L) \to L$ by $f(g, h) = gh$. We will show that $f$ is an isomorphism. It is onto by Claim 6 and an order isomorphism by Claim 7. So, it is a lattice isomorphism, as well. To verify that $f$ preserves the other operations note that $gg'h' = ghg'h'$, for all $g, g' \in G(L)$ and $h, h' \in I(L)$, by Claim 5. Moreover, for all $g, g', \bar{g} \in G(L)$ and $h, h', \bar{h} \in I(L)$, $\bar{g}h \leq gh/g'h'$ if and only if $\bar{g}g'h' \leq gh$. By Claim 5, this is equivalent to $\bar{g}g'h' \leq gh$, and, by Claim 7, to $\bar{g}g' \leq g$ and $\bar{h}h' \leq h$. This is in turn equivalent to $\bar{g} \leq g/g'$ and $\bar{h} \leq h/h'$, which is equivalent to $\bar{g}h \leq (g/g')(h/h')$ by Claim 7. Thus, for all $g, g' \in G(L)$ and $h, h' \in I(L)$, $gh/g'h' = (g/g')(h/h')$ and, likewise, $g'h'\bar{g}h = (g/g')(h/h')$.

\[\square\]

**Corollary 5.3** The varieties $\mathcal{GBL}$ and $\mathcal{GMV}$ decompose as follows:

$$\mathcal{GBL} = LG \times IGBL = LG \lor IGBL$$

$$\mathcal{GMV} = LG \times IGMV = LG \lor IGMV$$

Taking intersections with $\text{CanRL}$ and recalling Theorem 2.12, we get:

**Corollary 5.4** $\text{CanGMV} = \text{CanGBL} = LG \times LG^{-}$. 

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Here we have set $\text{CanGMV} = \text{CanRL} \cap \text{GMV}$ and $\text{CanGBL} = \text{CanRL} \cap \text{GBL}$. Moreover, in conjunction with Lemma 2.13(iv) and Theorem 2.2, Corollary 5.3 yields:

**Corollary 5.5** Every commutative GMV-algebra is representable.

By combining Theorem 5.2 and Theorem 3.12, we obtain the main result of this section.

**Theorem 5.6** A residuated lattice $M$ is a GMV-algebra if and only if there exist residuated lattices $G, L$, such that $G$ is an $\ell$-group, $L \in \mathcal{L}^-$, $\gamma$ is a nucleus on $L$ and $M = G \oplus L_\gamma$. Equivalently, $M$ is a GMV-algebra if and only if it has a direct product decomposition $M \cong G \times H_\gamma$, where $G, H$ are $\ell$-groups and $\gamma$ is a nucleus on $H^-$. 

6 A categorical equivalence for GMV-algebras

The goal of this section is to establish Theorems D and E (see Theorems 6.6 and 6.9 below).

If $G, H$ are $\ell$-groups and $\gamma$ is a nucleus on $H^-$, define $\delta(g, h) = (g, h \land e)$ and $\gamma'(g, h') = (g, \gamma(h'))$, for all $g \in G$, $h \in H$ and $h' \in H^-$. It follows from Theorem 5.6 that the underlying set of every GMV-algebra $M$ is of the form $\gamma'(\delta(G \times H))$, where $G, H$ are $\ell$-groups and $\gamma$ is a nucleus on $H^-$. 

Note that $\delta$ is an interior operator on $L = G \times H$, i.e. it is contracting ($\delta(x) \leq x$, for all $x \in L$), monotone (if $x \leq y$, then $\delta(x) \leq \delta(y)$, for all $x, y \in L$) and idempotent ($\delta(\delta(x)) = \delta(x)$, for all $x \in L$). Moreover, its image $L_\delta = \delta(L)$ is a submonoid and a lattice ideal of $L$. More explicitly, we have $\delta(\delta(x)\delta(y)) = \delta(x)\delta(y)$, $\delta(e) = e$, $\delta(x) \land y = \delta(\delta(x) \land y)$ and $\delta(x) \lor \delta(y) = \delta(\delta(x) \lor \delta(y))$, for all $x, y$ in $L$. We call an interior operator on a residuated lattice that satisfies the above properties a kernel operator; note that the last equality follows from the fact that $\delta$ is an interior operator and is not needed in the definition of a kernel. A core operator on a residuated lattice $L$ is the composition $\gamma \circ \delta$ of a kernel operator $\delta$ on $L$ and a nucleus $\gamma$ on the image $L_\delta$ of $\delta$; see Lemma 6.1.

### 6.1 The object level: Representations of GMV-algebras

The main result of this subsection is Theorem D (see Theorem 6.6 below). En route, we show that any core on a GMV-algebra has a unique representation as the composition of a nucleus and a kernel operator.
Lemma 6.1 If \( L \) is a residuated lattice and \( \delta \) a kernel on it, then the algebra \( L_\delta = \langle \delta(L), \wedge, \vee, \cdot, \setminus_\delta, /_\delta, e \rangle \), where \( x/_\delta y = \delta(x/y) \) and \( x\setminus_\delta y = \delta(x\setminus y) \), is a residuated lattice. Moreover, \( L_\delta \) is a lattice ideal of \( L \). If \( L \) is a GMV-algebra or a GBL-algebra, then so is \( L_\delta \).

PROOF. \( L_\delta \) is closed under join, since \( \delta \) is an interior operator, and under multiplication, by the first property of a kernel. Moreover, it contains \( e \) and it is obviously closed under \( \setminus_\delta \) and \( /_\delta \). By the third property of a kernel and the fact that it is closed under joins, \( L_\delta \) is an ideal of \( L \). So, \( L_\delta \) is a submonoid and a subsemilattice of \( L \). Moreover, \( L_\delta \) is residuated. For all \( x, y, z \in L_\delta \), \( x \leq z/_\delta y \) is equivalent to \( x \leq \delta(x/y) \), which in turn is equivalent to \( x \leq z/y \), since \( \delta \) is contracting and \( x = \delta(x) \).

If \( L \) is a GMV-algebra, then
\[
(x \lor y)\setminus x = x \setminus (x \lor y) \leq e.
\]
Since \( L_\delta \) is an ideal that contains \( e \), we have \( \delta((x \lor y)\setminus x) = (x \lor y)\setminus x \), for \( x, y \in L_\delta \). So,
\[
x/_\delta ((x \lor y)\setminus x) = \delta(x/((x \lor y)\setminus x)) = \delta(x \lor y) = x \lor y.
\]

Similarly, if \( L \) is a GBL-algebra, we have
\[
((x \land y)/_\delta y)y = \delta((x \land y)/y)y = (x \land y)/y = x \land y,
\]
since \( (x \land y)/y \leq e \). \qed

Note that the map \( \delta \) on a residuated lattice \( L \), defined by \( \delta(x) = x \land e \), is a kernel on \( L \) and \( L_\delta = L^- \).

For a class of algebras \( K \) we denote by \( n(K) \) and \( k(K) \) the class of all images of nuclei and kernels, respectively, of members of \( K \). We already know that \( n(LG^-) = IGMV \), from Theorem 3.12, and \( GMV \subseteq n(k(LG)) \). We will show that \( k(LG) = CanGMV \) and \( n(CanGMV) = GMV \). Moreover, we will give an alternative characterization of core operators. It follows from the lemma below that \( n(LG) = LG \) and \( k(IGMV) = IGMV \)

Lemma 6.2

(i) The identity map is the only nucleus on an \( \ell \)-group.
(ii) The identity is the only kernel on an integral GMV-algebra.
PROOF. i) Assume $\gamma$ is a nucleus on the $\ell$-group $G$. Since $G$ is a GMV-algebra, we have $e = \gamma(e) \in G_\gamma$, by Theorem 3.4; hence $G^+ \subseteq G_\gamma$. Moreover, by Lemma 3.1, for every $x \in G$, $e/x \in G_\gamma$, that is, $x^{-1} \in G_\gamma$. Thus, $G_\gamma = G$. Since a closure operator is uniquely defined by its image, $\gamma$ is the identity on $G$.

ii) Assume that $\delta$ is a kernel on an integral GMV-algebra $M$. By Lemma 6.1, $M_\delta$ is an ideal of $M$. Moreover, $e = \delta(e) \in M_\delta$. So, $M_\delta = M$ and $\delta$ is the identity map on $M$. \hfill $\square$

The following corollary describes the action of a kernel on a GMV-algebra and shows that $k(L_G) \subseteq \text{CanGMV}$. In what follows, we will use the term $\ell$-subgroup for a subalgebra of a residuated lattice that happens to be an $\ell$-group.

Corollary 6.3 If $\delta$ is a kernel on a GMV-algebra $M$, then there exist a GMV-subalgebra $N$ and an $\ell$-subgroup $H$ of $M$, such that $M = N \oplus H$ and $\delta(yh) = n(h \land e)$, for all $n \in N$ and $h \in H$. Thus, $M_\delta = N \oplus H^-$. If $M$ is an $\ell$-group, then so is $N$.

PROOF. By Theorem 5.6, there exist $\ell$-groups $G, L$, and a nucleus $\gamma$ on $L^{-}$, such that $M = G \oplus L_\gamma^-$. The restrictions of $\delta$ on $G$ and $L_\gamma^-$, also denoted by $\delta$, are kernels, because of the equational definition of a kernel.

First, note that $\delta(L_\gamma^-) \subseteq L_\gamma^-$ and $\delta(G) \subseteq G$. To verify this, observe that the image of $M$ under $\delta$ is an ideal of $M$, that contains the identity $e$, by Lemma 6.1; hence the negative cone of $M$ is fixed by $\delta$. In particular, $L_\gamma^-$ and $G^-$ are fixed by $\delta$. Consider an element $x$ in $G$. We will show that $\delta(x)$ is also in $G$. Let $\delta(x) = yk$, where $y \in G$ and $k \in L_\gamma^-$. Since $yk = \delta(x) \leq x = xe$, we have $y \leq x$. Both $yk$ and $e$ are fixed by $\delta$, so the same holds for their join $(y \lor e)(k \lor e) = y \lor e$, since the image of $\delta$ is a lattice ideal. Likewise, $y$ is fixed by $\delta$ since $y \leq y \lor e$. The element $\delta(x)$ is the maximum element below $x$ fixed by $\delta$; so $y \leq \delta(x)$, since $y \leq x$. On the other hand, $\delta(x) = yk \leq y$; hence $\delta(x) = y \in G$.

We will show that there exist $\ell$-subgroups $K, H$ of $G$, such that $G = K \oplus H$ and $\delta(kh) = k(h \land e)$, for all $k \in K$ and $h \in H$. Observe that $G_\delta$ is a GMV-algebra, by Lemma 6.1, so there are $\ell$-groups $K, H$ and a nucleus $\gamma$ on $H^-$, such that $G_\delta = K \oplus H^-\gamma$, by Theorem 5.6. Since $K \times H^-\gamma$ is isomorphic to $G_\delta$, the negative cones $K^- \times H^-\gamma$ and $G^-\delta$ are isomorphic. Moreover, we have $(G_\delta)^- = G^-$, because $G_\delta$ is an ideal of $G$ that contains $e$. The operations on $(G_\delta)^-$ and $G^-$ agree, since the lattice and monoid operations on both algebras are the restrictions to $(G_\delta)^- = G^-$ of the operations on $G$. Additionally, for
all \( z \in G \), \( z \wedge e \) is the greatest element fixed under \( \delta \) that is below \( z \); so, \( z \wedge e = \delta(z) = \delta(z) \wedge e \), and for all \( x, y \in G^- \), \( x \setminus (G^-)_y = x \setminus \delta y \wedge e = \delta(x \setminus y) \wedge e = x \setminus y \wedge e = x \setminus G^- y \) and likewise for right division. Consequently, \( K^- \times H^- \) is isomorphic to \( G^- \) via the map \((k, h) \mapsto kh\); i.e., \( G^- = K^- \oplus H^- \). Since \( H^- \) is a subalgebra of \( G^- \in L G^- \), we have \( H^- \in L G^- \). For simplicity of the presentation, and without loss of generality, we assume that \( H \) is such that \( \gamma \) is the identity on \( H^- \). So, \( G^- = K^- \oplus H^- = (K \oplus H)^- \) and \( G \) is isomorphic to \( K \oplus H \). We simplify notation by identifying isomorphic algebras, so \( G = K \oplus H \).

We have shown that \((K \oplus H)_\delta = K \oplus H^- \). Thus, \( \delta(K \oplus H) = \delta'(K \oplus H) \), where \( \delta'(gh) = g(h \wedge e) \) is an interior operator. Since an interior operator is defined by its image, we get \( \delta'(gh) = g(h \wedge e) \). So \( M = K \oplus H \oplus L^- \). Moreover, \( \delta \) is the identity on \( L^- \). If we set \( N = K \oplus L^- \), we get \( M = N \oplus H \) and \( \delta(nh) = n(h \wedge e) \), for all \( n \in N \) and \( h \in H \).

**Definition 6.4**

(i) If \( \delta \) is a map on a residuated lattice \( L \) and \( \gamma \) a map on \( \delta(L) \), define the map \( \beta_{(\gamma, \delta)} \) on \( L \) by \( \beta_{(\gamma, \delta)}(x) = \gamma(\delta(x)) \).

(ii) If \( \beta \) is a map on a residuated lattice \( L \), define the maps \( \delta_{\beta} \) on \( L \) and \( \gamma_{\beta} \) on \( \delta_{\beta}(L) \) by \( \delta_{\beta}(x) = \beta(x) \wedge x \) and \( \gamma_{\beta}(x) = \beta(x) \).

**Lemma 6.5** Let \( L \) be a GMV-algebra. If \( \delta \) is a kernel on \( L \) and \( \gamma \) a nucleus on \( L_{\delta} \), then \( \delta_{\beta_{(\gamma, \delta)}} = \delta \), \( \gamma_{\beta_{(\gamma, \delta)}} = \gamma \).

**Proof.** We have \( \delta_{\beta_{(\gamma, \delta)}}(x) = \beta_{(\gamma, \delta)}(x) \wedge x = \gamma(\delta(x)) \wedge x \). In view of Corollary 6.3, to show that \( \delta_{\beta_{(\gamma, \delta)}} = \delta \), it will suffice to verify that \( \gamma(\delta(x)) \wedge x = \delta(x) \), only for the cases \( \delta(x) = x \) and \( \delta(x) = x \wedge e \). In the first case, the equation holds, because \( \gamma \) is extensive. In the second case, the equation reduces to \( \gamma(x \wedge e) \wedge x = x \wedge e \). Since \( \gamma \) is extensive, we have \( x \wedge e = x \wedge e \wedge x \leq \gamma(x \wedge e) \wedge x \). Invoking the monotonicity of \( \gamma \) we get \( \gamma(x \wedge e) \wedge x = \gamma(e) \wedge x = e \wedge x \), by Theorem 3.4(iii).

For every \( x \) in the range of \( \delta_{\beta_{(\gamma, \delta)}} = \delta \), namely for \( x = \delta(x) \), we have \( \gamma_{\beta_{(\gamma, \delta)}}(x) = \beta_{(\gamma, \delta)}(x) = \gamma(\delta(x)) = \gamma(x) \).

Therefore cores on GMV-algebras decompose uniquely as compositions of kernels and nuclei. For a GMV-algebra \( L \) and a core \( \beta \) on it, define \( L_{\beta} = (L_{\delta_{\beta}})_{\gamma_{\beta}} \).

**Theorem 6.6** A residuated lattice \( L \) is a GMV-algebra if and only if \( L \cong G_{\beta} \), for some \( \ell \)-group \( G \) and some core \( \beta \) on \( G \).
Proof. By Lemma 6.1, if $G$ is an $\ell$-group and $\delta$ a kernel on it, then $G_\delta$ is a GMV-algebra. Moreover, by Theorem 3.4, $(G_\delta)_\gamma$ is a GMV-algebra, as well.

Conversely, let $L$ be a GMV-algebra. By Corollary 5.6, $L \cong K \times H^-_{\gamma}$, for some $\ell$-groups $K$ and $H$, and a nucleus $\gamma$ on $H^-$. Define a map $\delta$ on $K \times H$, by $\delta(k, h) = (k, h \wedge e)$. We will show that $\delta$ is a kernel. It is obviously an interior operator and $\delta(e, e) = (e, e)$. Note that

\[
\delta(k, h)\delta(k', h') = (k, h \wedge e)(k', h' \wedge e) \\
= (kk', (h \wedge e)(h' \wedge e)) \\
= (kk', hh' \wedge h \wedge h' \wedge e)
\]

and $\delta(kk', hh' \wedge h \wedge h' \wedge e) = (kk', hh' \wedge h \wedge h' \wedge e)$. Similarly

\[
\delta(k, h) \wedge (k', h') = (k, h \wedge e) \wedge (k', h) = (k \wedge k', h \wedge e \wedge h')
\]

and $\delta(k \wedge k', h \wedge e \wedge h') = (k \wedge k', h \wedge e \wedge h')$.

Note that the underlying set of $(K \times H)_\delta$ is $K \times H^-$. Define $\bar{\gamma}$ on $K \times H^-$, by $\bar{\gamma}(k, h) = (k, \gamma(h))$. We will show that $\bar{\gamma}$ is a nucleus on $(K \times H)_\delta$. It is obviously a closure operator. Moreover,

\[
\bar{\gamma}(k, h)\bar{\gamma}(k', h') = (k, \gamma(h))(k', \gamma(h')) \\
= (kk', \gamma(h)\gamma(h')) \\
\leq (kk', \gamma(hh')) \\
= \bar{\gamma}(kk', hh') \\
= \bar{\gamma}((k, h)(k', h')).
\]

We have $\bar{\gamma}((K \times H)_\delta) = \bar{\gamma}(K \times H^-) = K \times H^-_{\bar{\gamma}}$. So $K \times H^-_{\bar{\gamma}}$ and $((K \times H)_\delta)_\gamma$ have the same underlying set. Recalling the definitions of the image of a residuated lattice under a kernel and under a nucleus, we see that the lattice operations on the two algebras coincide. To show that the other operations coincide, note
that for all \((k, h), (k'h') \in K \times H^-\),

\[
(k, h) \circ ((K \times H) \delta) \gamma (k', h') = (k, h) \circ \gamma (k', h')
\]

\[
= \gamma ((k, h) \cdot (k', h'))
\]

\[
= \gamma (kk', hh')
\]

\[
= (kk', \gamma (hh'))
\]

\[
= (kk', h \circ \gamma h')
\]

\[
= (k, h) \circ_{K \times H^-} (k', h')
\]

\[
(k, h) \setminus ((K \times H) \delta) \gamma (k', h') = \delta ((k, h) \setminus_{K \times H} (k', h'))
\]

\[
= \delta ((k \setminus_{K} k', h \setminus_{H} h'))
\]

\[
= (k \setminus_{K} k', h \setminus_{H} h' \land e)
\]

\[
= (k \setminus_{K} k', h \setminus_{H} h')
\]

\[
= (k \setminus_{K} k', h \setminus_{H^-} h')
\]

\[
= (k, h) \setminus_{K \times H^-} (k', h')
\]

The proof for the other division operation is analogous. \(\square\)

It follows from the preceding theorem that \(k(n(LG)) = GMV\). We show below that \(k(LG) = CanGMV\) and \(n(CanGMV) = GMV\). Further, we provide an equational description for a core operator.

**Corollary 6.7**

(i) A residuated lattice \(L\) is a cancellative GMV-algebra if and only if \(L \cong G_\delta\), for some \(\ell\)-group \(G\) and some kernel \(\delta\) on \(G\).

(ii) A residuated lattice \(L\) is a GMV-algebra if and only if \(L \cong K_\gamma\), for some cancellative GMV-algebra \(K\) and some nucleus \(\gamma\) on \(K\).

**Proof.**

i) One direction follows from Corollary 6.3 and Corollary 5.4. For the other direction, assume that \(L\) is a cancellative GMV-algebra. By Corollary 5.4, \(L = K \times H^-\), for some \(\ell\)-groups \(K, H\). We have already remarked that the map \(\delta\) on \(K \times H\), defined by \(\delta(k, h) = (k, h \land e)\), is a kernel and that \((K \times H) \delta = K \times H^- = L\).

ii) One direction follows from Theorem 3.4. Conversely, if \(L\) is a GMV-algebra, then, by Theorem 5.6, there exist \(\ell\)-groups \(G, H\) and a nucleus \(\gamma\) on \(H^-\), such
that \( L = G \times H^- \). It is easy to check that the map \( \bar{\gamma} \) on \( G \times H^- \), defined by 
\( \bar{\gamma}(g, h) = (g, \gamma(h)) \), is a nucleus and that \( (G \times H^-)_\bar{\gamma} = G \times H^- = L \). Finally, 
\( K = G \times H^- \) is a cancellative GMV-algebra, by Corollary 5.4. \( \square \)

**Lemma 6.8** A map \( \beta \) on a GMV-algebra \( L \) is a core if and only if it is
monotone, idempotent and satisfies the following properties:

(i) \( \beta(x)\beta(y) \leq \beta(xy) \),
(ii) \( \beta(e) = e \),
(iii) \( (\beta(x) \wedge x)(\beta(y) \wedge y) \leq \beta((\beta(x) \wedge x)(\beta(y) \wedge y)) \),
(iv) \( \beta(x) \wedge x \wedge y \leq \beta(\beta(x) \wedge x \wedge y) \)
(v) \( \beta(\beta(x) \wedge x) = \beta(x) \).

**PROOF.** The result is a consequence of the following two claims and Lemma 6.5.

**Claim 1:** If \( \beta \) is a monotone, idempotent map on \( L \) that satisfies the properties above, then \( \delta_\beta \) is a kernel on \( L \), \( \gamma_\beta \) is a nucleus on \( L_{\delta_\beta} \) and \( \beta_{(\gamma_\beta, \delta_\beta)} = \beta \).

Since \( \gamma_\beta \) is the restriction of \( \beta \), we have \( \gamma_\beta(x)\gamma_\beta(y) \leq \gamma_\beta(xy) \), by the first
property. Moreover, \( \gamma_\beta \) is monotone and idempotent, being a restriction of \( \beta \).
It is also extensive on \( L_{\delta_\beta} \) because \( \delta_\beta(x) = \beta(x) \wedge x \leq \beta(\beta(x) \wedge x) = \gamma_\beta(\delta_\beta(x)) \),
by (iv). Thus, \( \gamma_\beta \) is a nucleus.

Obviously, \( \delta_\beta(e) = \beta(e) \wedge e = e \), by the second property. The remaining two
properties of a kernel state that \( \delta_\beta(x)\delta_\beta(y) \) and \( \delta_\beta(x) \wedge y \) are elements fixed
by \( \delta_\beta \). It is easy to see that for every \( x \), \( \delta_\beta(x) = x \) if and only if \( x \leq \beta(x) \).
So, the remaining properties are equivalent to properties (iii) and (iv) of the
lemma. Additionally, \( \delta_\beta \) is an interior operator, since \( \delta_\beta(x) = \beta(x) \wedge x \leq x; \)
\( \delta_\beta(\delta_\beta(x)) = \beta(\beta(x) \wedge x) \wedge x = \beta(x) \wedge x = \delta_\beta(x) \), by (v); and if \( x \leq y \), then
\( \delta_\beta(x) = \beta(x) \wedge x \leq \beta(y) \wedge y = \delta_\beta(y) \). Thus, \( \delta_\beta \) is a kernel.

Finally, \( \beta_{(\gamma_\beta, \delta_\beta)}(x) = \gamma_\beta(\delta_\beta(x)) = \beta(\beta(x) \wedge x) = \beta(x) \).

**Claim 2:** If \( \delta \) is a kernel on \( L \) and \( \gamma \) a nucleus on \( L_{\delta} \), then the map \( \beta_{(\gamma, \delta)} \) is
monotone, idempotent and it satisfies the properties in the statement of the
lemma.

For the first property we have
\[
\beta(x)\beta(y) = \gamma(\delta(x))\gamma(\delta(y)) \leq \gamma(\delta(x)\delta(y)) \\
= \gamma(\delta(\delta(x)\delta(y))) \leq \gamma(\delta(xy)) \\
= \beta(xy).
\]

Also, \( \beta(e) = \gamma(\delta(e)) = \gamma(e) = e \), by Theorem 3.4(iii).
Since for every \( x \), \( x \leq \beta_{(\gamma, \delta)}(x) \) if and only if \( \delta_{(\gamma, \delta)}(x) = x \), properties (iii) and (iv) hold for \( \beta_{(\gamma, \delta)} \) if and only if the corresponding properties of a kernel hold for \( \delta_{(\gamma, \delta)} \). This is actually the case, since \( \delta_{(\gamma, \delta)} = \delta \), by Lemma 6.5.

The last property for \( \beta_{(\gamma, \delta)} \) is equivalent to \( \beta_{(\gamma, \delta)}(\delta_{(\gamma, \delta)}(x)) = \beta_{(\gamma, \delta)}(x) \), that is, \( \beta_{(\gamma, \delta)}(x) = \beta_{(\gamma, \delta)}(x) \), which follows from the idempotency of \( \delta \). \( \square \)

### 6.2 The morphism level

Let \( \text{GMV} \) be the category with objects GMV-algebras and morphisms residuated lattice homomorphisms. Also, let \( \text{LG}^* \) be the category with objects algebras \( \langle G, \beta \rangle \) such that \( G \) is an \( \ell \)-group and \( \beta \) is a core on \( G \) whose image generates \( G \); let the morphisms of this category be homomorphisms between these algebras.

**Theorem 6.9** The categories \( \text{GMV} \) and \( \text{LG}^* \) are equivalent.

**Proof.** For an object \( \langle G, \beta \rangle \) of \( \text{LG}^* \), define \( \Gamma(\langle G, \beta \rangle) = G_\beta \). For a morphism \( f \) of \( \text{LG}^* \) with domain \( \langle G, \beta \rangle \), define \( \Gamma(f) \) to be the restriction of \( f \) to \( G_\beta \).

Let \( \delta = \delta_\beta \) and \( \gamma = \gamma_\beta \). By Lemma 6.1 and Theorem 3.4, the algebra \( \Gamma(\langle G, \beta \rangle) \) is an object of \( \text{GMV} \). Actually, it can be easily seen that \( G_\beta = \langle G_\delta \gamma, \wedge, \vee, o_\gamma, \setminus_\delta, /_\delta, e \rangle \). To show that \( \Gamma(f) \) is a morphism of \( \text{GMV} \), we use the fact that \( f \) commutes with \( \beta \) – we use the same symbol for the cores in the domain and in the codomain.

First note that \( f \) commutes with \( \delta \) on \( L \) and \( \gamma \) on \( \delta(L) \). Indeed, by Lemma 6.5,

\[
\delta(f(x)) = \beta(f(x)) \wedge f(x) = f(\beta(x)) \wedge f(x) \\
= f(\beta(x) \wedge x) = f(\delta(x)).
\]

Moreover, \( \gamma(f(x)) = \gamma(\delta(f(x))) = f(\gamma(\delta(x))) = f(\gamma(x)) \). In particular, if \( x = \beta(x) \), then \( x = \gamma(x) = \delta(x) \) and \( f(x) = \delta(f(x)) = \gamma(f(x)) \).

We can now show that \( f \) preserves multiplication. For \( x, y \in \beta(G) \), \( x = \delta(x) = \gamma(x) \) and \( y = \delta(y) = \gamma(y) \), so

\[
\delta(xy) = \delta(\delta(x)\delta(y)) = \delta(x)\delta(y) = xy.
\]
Thus,
\[ f(x \circ \gamma y) = f(\gamma(xy)) = \gamma(f(xy)) \]
\[ = \gamma(f(x)f(y)) = f(x) \circ f(y). \]

Additionally,
\[ f(x/\delta y) = f(\delta(x/y)) = \delta(f(x/y)) \]
\[ = \delta(f(x)/f(y)) = f(x)/\delta f(y). \]

The proof for the other division is analogous. \( \Gamma(f) \) preserves the lattice operations, because they are restrictions of the lattice operations of the \( \ell \)-group, so \( \Gamma(f) \) is a homomorphism.

By Theorem 6.6, \( \Gamma \) is onto the objects of \( \text{GMV} \). Moreover, \( \Gamma \) is faithful, because if two morphisms agree on a generating set, they agree on the whole \( \ell \)-group.

To see that \( \Gamma \) is full, let \( g : M \to N \), be a morphism in \( \text{GMV} \). By Theorem 5.6, there exist \( \ell \)-groups \( K, H, K, H \) and nuclei \( \gamma \) on \( H^- \) and \( \pi \) on \( H^- \), such that \( M = K \times H \) and \( N = K \times H \). Moreover, by the proof of Theorem 6.6, there exist kernels \( \delta \) on \( K \times H \), \( \delta \) on \( K \times H \), and nuclei \( \gamma' \) on \( (K \times H) \) and \( \gamma' \) on \( (K \times H) \), such that \( \delta(k,h) = (k,h \wedge e) \), \( \delta(k,h) = (k,h \wedge e) \), \( \gamma'(k,h) = (k,\gamma(h)) \) and \( \gamma'(k,h) = (k,\gamma(h)) \), for \( h \in H, h \in H, k \in K \) and \( k \in K \). For the cores \( \beta = \gamma' \circ \delta \) and \( \beta = \gamma' \circ \delta \), there exist homomorphisms \( g_1 : K \to K \) and \( g_2 : H \to H \) such that \( g = (g_1, g_2) \); the reason for this is that invertible and integral elements are preserved under homomorphisms. By Theorem 4.10, there exists a homomorphism \( f_2^- : H^- \to H^- \) that extends \( g_2 \) and commutes with the \( \gamma \)'s. By the results in [2], there exists a homomorphism \( f_2^- : H \to H \) that extends \( f_2^- \). Let \( f : (K \times H, \beta) \to (K \times H, \beta) \) be defined by \( f = (g_1, f_2) \). It is clear that \( \Gamma(f) = g \). We will show that \( g(\beta(x)) = \beta(f(x)) \). Let \( (k,h) \in K \times H \).

\[
g(\beta(k,h)) = g(\gamma(\delta(k,h))) = g(k, \gamma(h \wedge e))
\]
\[
= (g_1(k), g_2(\gamma(h \wedge e))) = (g_1(k), \gamma(f_2^-(h \wedge e)))
\]
\[
= (g_1(k), \gamma(f_2(h) \wedge e)) = \gamma'(g_1(k), f_2(h) \wedge e)
\]
\[
= \gamma(\delta((g_1(k), f_2(h)))) = \beta(f(k,h)).
\]

Thus, by [23] (Theorem 1, page 93), \( \Gamma \) is an equivalence between the two categories. \( \square \)
7 Decidability of the equational theories

In this section, we obtain the decidability of the equational theories of the varieties \( \mathcal{IGMV} \) and \( \mathcal{GMV} \) as an easy application of the representation theorems established in the previous sections.

For a residuated lattice term \( t \) and a variable \( z \not\in \text{Var}(t) \), we define the term \( t_z \) inductively on the complexity of a term, by

\[
\begin{align*}
    x_z &= x \lor z \\
e_z &= e \\
(s \lor r)_z &= s_z \lor r_z \\
(s \land r)_z &= s_z \land r_z \\
(s/r)_z &= s/z \land r/z \\
(s \land r)_z &= s/z \land r/z \\
(s/r)_z &= s/z \lor r/z \\
(s/r)_z &= s/z \land r/z \\
(s/r)_z &= s/z \lor r/z \\
(s/r)_z &= s/z \land r/z \\
(s/r)_z &= s/z \lor r/z
\end{align*}
\]

for every variable \( x \) and every pair of terms \( s, r \).

For a term \( t \) and an algebra \( L \), we write \( t^L \) for the term operation on \( L \) induced by \( t \).

For a residuated lattice term \( t \), a residuated lattice \( L \) and a map \( \gamma \) on \( L \), we define the operation \( t^\gamma \) on \( L \), of arity equal to that of \( t \), by

\[
\begin{align*}
x_\gamma &= \gamma(x^L) \\
e_\gamma &= e^L \\
(s \lor r)_\gamma &= s_\gamma \lor r_\gamma \\
(s \land r)_\gamma &= s_\gamma \land r_\gamma \\
(s/r)_\gamma &= s_\gamma /r_\gamma \\
(s \land r)_\gamma &= s_\gamma \land r_\gamma \\
(s/r)_\gamma &= s_\gamma \lor r_\gamma \\
(s/r)_\gamma &= s_\gamma \land r_\gamma \\
(s/r)_\gamma &= s_\gamma \lor r_\gamma
\end{align*}
\]

for every variable \( x \) and every pair of terms \( s, r \).

Note that \( t_\gamma \) is obtained from \( t^L \) by replacing every product \( sr \) by \( \gamma(sr) \) and every variable \( x \) by \( \gamma(x) \); \( t_z \) is obtained from \( t \) by replacing every product \( sr \lor z \) and every variable \( x \) by \( x \lor z \). We extend the above definitions to every residuated lattice identity \( \varepsilon = (t \approx s) \) by \( \varepsilon_z = (t_z \approx s_z) \), for a variable \( z \) that does not occur in \( \varepsilon \). Moreover, we define \( \varepsilon_\gamma(\bar{a}) = (t_\gamma(\bar{a}) = s_\gamma(\bar{a})) \), where \( \bar{a} \) is an element of an appropriate power of \( L \).

**Proposition 7.1** An identity \( \varepsilon \) holds in \( \mathcal{IGMV} \) if and only if the identity \( \varepsilon_z \) holds in \( \mathcal{LG}^\gamma \), where \( z \not\in \text{Var}(\varepsilon) \).

**Proof.** We prove the contrapositive of the lemma. Let \( \varepsilon \) be an identity that fails in \( \mathcal{IGMV} \). Then there exists an integral GMV-algebra \( M \), and an element \( \bar{a} \) in an appropriate power, \( n \), of \( M \), such that \( \varepsilon(\bar{a}) \) is false. By Theorem 3.12, there exists an \( L \in \mathcal{LG}^\gamma \) and a nucleus \( \gamma \) on \( L \) such that \( M = L_\gamma \). By the definition of \( L_\gamma \), it follows that \( \varepsilon_\gamma(\bar{a}) \) does not hold in \( L \). Let \( p \) be the meet of all
products \( t_\gamma(\bar{a})s_\gamma(\bar{a}) \), where \( t, s \) range over all subterms of \( \varepsilon \) and \( u = \gamma(p) \). By Lemma 3.6, \( \gamma \) and \( \gamma_u \) agree on the principal filter of \( p \). Since the arguments of all occurrences of \( \gamma \) in \( \varepsilon_\gamma(\bar{a}) \) are of the form \( t_\gamma(\bar{a})s_\gamma(\bar{a}) \), where \( t, s \) are subterms of \( \varepsilon \), and \( t_\gamma(\bar{a})s_\gamma(\bar{a}) \) are in the principal filter of \( p \), we can replace, working inductively inwards, all occurrences of \( \gamma \) in \( \varepsilon_\gamma(\bar{a}) \) by \( \gamma_u \). Hence \( \varepsilon_\gamma(\bar{a}) = \varepsilon \gamma_u(\bar{a}) \). Thus \( \varepsilon \) fails in \( L \) and \( \varepsilon_\gamma \) is not a valid identity of \( L_{GMV} - \).

Conversely, if \( \varepsilon_\gamma \) fails in \( L_{GMV} - \), there exist an \( L \in L_{GMV} - \), \( \bar{a} \) in an appropriate power, \( n \), of \( L \) and \( u \in L \) such that \( (\varepsilon_\gamma)^L(\bar{a}, u) \) is false. Obviously, \( \gamma_u \) is a nucleus on \( L \), so \( L_{\gamma_u} \) is an integral GMV-algebra. Let \( \bar{b} \) be the element of \( L^n \), defined by \( \bar{b}(i) = \bar{a}(i) \lor u \), for all \( i \in \{1, \ldots, n\} \). Note that \( (\varepsilon_\gamma)^L(\bar{a}, u) = \varepsilon_{\gamma_u}(\bar{a}) = \varepsilon_{\gamma_u}(\bar{b}) = (\varepsilon_{\gamma_u})^L(\bar{b}) \) and \( u, \bar{b}(i) \in L_{\gamma_u} \), for all \( i \in \{1, \ldots, n\} \). So \( \varepsilon \) fails in \( L_{\gamma_u} \) and hence in \( IGMV \). \( \square \)

In view of Theorem 5.6 we have the following corollary.

**Corollary 7.2** An identity \( \varepsilon \) holds in \( GMV \) if and only if \( \varepsilon \) holds in \( LG \) and \( \varepsilon_\gamma \) holds in \( LG - \), where \( \gamma \notin Var(\varepsilon) \).

The variety of \( \ell \)-groups has a decidable equational theory by [19]. Based on this fact, it is shown in [2] that the same holds for \( LG - \). So, we obtain the following result.

**Theorem 7.3** The varieties \( IGMV \) and \( GMV \) have decidable equational theories.

Recall that a bounded GMV-algebra (also called a pseudo MV-algebra) is an expansion of a GMV-algebra by a constant \( 0 \) that satisfies the identity \( x \land 0 \approx 0 \). We denote the variety of all bounded GMV-algebras by \( bgMV \). Note that every bounded GMV-algebra is integral, as a consequence of Theorem 5.6.

For a term \( t \) in the language of residuated bounded-lattices and a variable \( z \notin Var(t) \), we define the term \( t_z \) inductively on the complexity of a term, by

\[
\begin{align*}
x_z &= x \lor z \\
e_z &= e \\
o_z &= 0 \\
(s \lor r)_z &= s_z \lor r_z \\
(s \land r)_z &= s_z \land r_z \\
(s/r)_z &= s_z/r_z \\
(s \setminus r)_z &= s_z \setminus r_z \\
(s \setminus r)_z &= s_z \setminus r_z \\
(s \setminus r)_z &= s_z \setminus r_z \\
\end{align*}
\]

for every variable \( x \) and every pair of terms \( s, r \). We use the same notation \( \varepsilon_z \) as before, since the two definitions agree if the equation \( \varepsilon \) does not contain any occurrences of the constant \( 0 \).

Minor modifications in the proof of Proposition 7.1 yield the following result.
**Proposition 7.4** An identity $\varepsilon$ holds in $b\mathcal{GMV}$ if and only if the identity $\varepsilon_z$ holds in $\mathcal{LG}^-$, where $z \notin \text{Var}(\varepsilon)$.

A careful analysis of the construction of an algebra in $\mathcal{LG}^-$ from an integral GMV-algebra shows that if the latter is commutative then so is the former. The same result is shown in [24]. So, the proof of Proposition 7.1 also shows the following.

**Proposition 7.5** An identity $\varepsilon$ holds in $\mathcal{MV}$ if and only if the identity $\varepsilon_z$ holds in $\mathcal{LG}^-$, where $z \notin \text{Var}(\varepsilon)$.

Consequently, we have the following result.

**Theorem 7.6** The varieties of of $\mathcal{MV}$-algebras and bounded $\mathcal{GMV}$-algebras have decidable equational theories.

**References**


