On the mixed alluvial–bedrock channel problem

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1 Preamble

To create a valley, a river must downcut into bedrock. Flowing water cannot achieve this by itself.[1] The river flow must transport sediment particles that abrade the channel bed via particle–bed collisions. If the river transports too many particles then an alluvial cover develops and protects the bedrock from collisions. If the river transports too few particles then the frequency of particle–bed collisions, and thus abrasion, decreases. This is a quintessential Goldilocks-and-the-Three-Bears problem, wherein the river channel must transport "just the right" amount of sediment to achieve downcutting, and it therefore must be a mixed alluvial–bedrock channel.

Sediment transport and bedrock abrasion are stochastic processes, whether viewed at the short time scales of particle motions or at longer time scales during which significant downcutting occurs. In this situation it is unclear how to reconcile descriptions of transport and abrasion viewed at experimental time scales with the inherent variability that exists in the wild, and which cannot be empirically constrained with confidence owing to our limited ability to measure things over long time scales. The problem therefore is inherently probabilistic. We can only aim at the statistical likelihood of outcomes based on defensible probabilistic descriptions of transport and abrasion whose physics is suitably coarsened to the length and time scales of interest. Herein I offer a straightforward starting point for conceptualizing the mixed alluvial–bedrock part of the problem. Mechanistic descriptions of the abrasion part of the problem likewise require a statistical rethinking, although I only briefly comment on this point without elaboration.

2 Simple immigration and emigration of particles

Consider immigration and emigration of particles into and out of a control volume — a river channel segment. In the simplest case this is akin to the M/M/1 queuing problem, a Markov birth–death process. An M/M/1 queue is conceptually straightforward, and although it is a simplification of any real sediment system, the M/M/1 queue nonetheless serves to illustrate key points about expected values in relation to the mechanics of a system. Let $n(t)$ denote the number of particles within a control volume, where we do not distinguish between active or rest states of the particles. Assume that particles enter the volume as a Poisson process with fixed intensity λ. The indistinguishable particles are eventually re-entrained and leave the volume. The distribution of wait times between

[1] Aqueous chemical reactions can remove rock mass and weaken the bedrock surface, thereby increasing its susceptibility to mechanical abrasion. The process of plucking requires preconditioning by fractures, and likely hinges on transport with particle–bed collisions.

emigration events is exponential with mean $\mu_w = 1/\sigma$ and intensity σ. Thus emigration also is a Poisson process. Importantly, and for reference in the next section, neither the immigration rate nor the emigration rate depends on the state n of the system. Although simplistic, the M/M/1 queue is a conceptually reasonable description in which we imagine that the addition of particles to a control volume and the removal of particles from it involve like stochastic processes (Furbish and Doane, 2021).

Because of the competing processes of immigration and emigration the number $n(t)$ within the volume fluctuates. Under certain conditions the distribution $p_n(n)$ of the number state n is steady, independent of time. Here the temptation is strong to assume that an "equilibrium" condition exists if $\lambda = \sigma$, where over a sufficiently long period of time the number of particles leaving the volume is equal to the number entering the volume, with fluctuations about the equilibrium state, where the expected time-averaged difference is zero. But this is incorrect. If $\lambda/\sigma \geq 1$ the number $n(t)$ within the volume grows indefinitely. The distribution $p_n(n)$ is steady only if $\lambda/\sigma < 1$. We now demonstrate this point.

As a counting process (Feller, 1939) we assume that the probabilities of changes in state are

$$P[n(t + \mathrm{d}t) - n(t) = 1] = \lambda \mathrm{d}t(1 - \sigma \mathrm{d}t) + o(\mathrm{d}t)\,,$$

$$P[n(t + \mathrm{d}t) - n(t) = -1] = \sigma \mathrm{d}t(1 - \lambda \mathrm{d}t) + o(\mathrm{d}t) \qquad \text{and}$$

$$P[|n(t + \mathrm{d}t) - n(t)| > 1] = o(\mathrm{d}t)\,. \tag{1}$$

Notice that we do not specify an initial state. The quantities $(1 - \sigma \mathrm{d}t)$ and $(1 - \lambda \mathrm{d}t)$ only formally show that the probabilities P involve the joint occurrence of events and are actually unnecessary, as they lead to terms at order $o[(\mathrm{d}t)^2]$. For $n = 1, 2, 3, ...$ we then have

$$p_n(n, t + \mathrm{d}t) = p_n(n - 1, t)\lambda \mathrm{d}t + p_n(n + 1)\sigma \mathrm{d}t + (1 - \lambda \mathrm{d}t)(1 - \sigma \mathrm{d}t)p_n(n, t) + o(\mathrm{d}t)\,. \tag{2}$$

For $n = 0$,

$$p_n(0, t + \mathrm{d}t) = p_n(1, t)\sigma \mathrm{d}t + p_n(0, t)(1 - \lambda \mathrm{d}t) + o(\mathrm{d}t)\,. \tag{3}$$

We now expand (2) and (3), rearrange, then divide by $\mathrm{d}t$ and take the limit as $\mathrm{d}t \to 0$ to give

$$\frac{\mathrm{d}p_n(n, t)}{\mathrm{d}t} = -(\lambda + \sigma)p_n(n, t) + \lambda p_n(n - 1, t) + \sigma p_n(n + 1, t) \tag{4}$$

and

$$\frac{\mathrm{d}p_n(0, t)}{\mathrm{d}t} = -\lambda p_n(0, t) + \sigma p_n(1, t)\,. \tag{5}$$

The set of equations (4) and (5) for $n = 0, 1, 2, 3, ...$ represents a master equation describing the time evolution of the distribution $p_n(n, t)$.

There are several ways to solve (4) and (5) for the steady distribution $p_n(n)$. Here we take a direct route. Namely, we assume steady conditions and set the time derivatives in (4) and (5) to zero. Thus,

$$p_n(n + 1) = \frac{\lambda + \sigma}{\sigma} p_n(n) - \frac{\lambda}{\sigma} p_n(n - 1) \tag{6}$$

and

$$p_n(1) = \frac{\lambda}{\sigma} p_n(0)\,. \tag{7}$$

By starting with (7) and recursively stepping through $n = 1, 2, 3, ...$ we quickly discover that

$$p_n(n) = \frac{\lambda}{\sigma} p_n(n-1) \,. \tag{8}$$

This simple geometric progression decreases monotonically if $\lambda/\sigma < 1$ and is uniquely satisfied by the geometric distribution,

$$p_n(n) = \left(1 - \frac{\lambda}{\sigma}\right)\left(\frac{\lambda}{\sigma}\right)^n \qquad n = 0, 1, 2, 3, ... \tag{9}$$

with mean $\mu_n = (\lambda/\sigma)/(1 - \lambda/\sigma)$ and variance $\sigma_n^2 = (\lambda/\sigma)/(1 - \lambda/\sigma)^2$. Notice that the mean μ_n is undefined if $\lambda/\sigma \geq 1$. That is, as asserted above, an "equilibrium" condition does not exist if $\lambda = \sigma$. Also notice for reference below that $p_n(0) = (1 - \lambda/\sigma)$. Thus, as $n(t)$ fluctuates over a long period of time the proportion of this time for which $n = 0$ increases as the ratio λ/σ decreases.

Here are key points. First, the expected state $\mathrm{E}[n(t)]$ does not coincide with our usual view of an equilibrium condition in which the time-averaged rates at which particles enter and leave the volume are equal such that the time-averaged difference is zero. Indeed, this assumption leads to a result that is perhaps counterintuitive. Second, the extent to which the expected state can be connected to mechanistic reasoning is unclear. The Poisson input rate λ must derive from the physics of particle entrainment and the timing of downstream displacements that lead to the rate λ, but these physics are independent of anything occurring in the control volume. Likewise the exponential distribution of wait times and the timing of particle displacements must derive from physics that give the rate σ. Given just these rates, the expected state $\mathrm{E}[n(t)]$ is a probabilistic outcome. In any realization the state $n(t)$ stochastically fluctuates about the expected state but this expected state is not an attractor for such realizations in the sense of an Ornstein–Uhlenbeck process. Any mechanistic description of the expected state therefore would require a synthesis of the physics of the rates λ and σ. And swapping these rates with deterministic expressions of the flux that do not reflect the physics of λ and σ does not get closer to a mechanistic description, notably if such expressions hinge on or imply that $\lambda = \sigma$ under nominally steady conditions.

The M/M/1 queuing problem in fact represents a first-order version of an alluvial versus mixed alluvial–bedrock channel segment involving fluctuations in transport. A value of $\lambda/\sigma \geq 1$ represents the onset of alluvial conditions. With $\lambda/\sigma < 1$, then as the number of particles $n(t)$ fluctuates over time, periods with large $n(t)$ represent full alluvial cover. This is more likely with large λ/σ. Periods with small $n(t)$ represent limited cover and the possibility of downcutting into bedrock by abrasion as available particles move through the segment. This is more likely with $\lambda/\sigma \ll 1$. Real behavior is of course more complicated. For example, the input and output rates may not be Poissonian or homogeneous, and pertain only to periods of active transport. Output rates are not likely to be independent of the number $n(t)$ (see below). Nonetheless, this is an appropriate conceptual starting point, where we know that over long times bedrock downcutting requires $\lambda/\sigma < 1$, which is equivalent to saying that the time-averaged divergence of the particle flux must be negative over long times. On heuristic grounds the abrasion potential likely goes as $a \sim n(1 - n/n_0)$, where n_0 represents a state of complete alluvial cover (see below). The statistics of various attributes of this problem are well-known, for example, particle residence times, and crossing events defined as the time durations for which $n(t)$ is above or below a specified value; and in principle the effects can be time integrated.

3 State-dependent emigration

An emigration event is more likely to occur if a channel segment contains many particles than if it contains few particles. Let us therefore assume that the emigration rate depends on the state $n(t)$ (Allemand et al., 2023). As a counting process we assume that

$$P[n(t+\mathrm{d}t)-n(t)=1]=\lambda\mathrm{d}t(1-\sigma n\mathrm{d}t)+o(\mathrm{d}t)\,,$$

$$P[n(t+\mathrm{d}t)-n(t)=-1]=\sigma n\mathrm{d}t(1-\lambda\mathrm{d}t)+o(\mathrm{d}t)\qquad \text{and}$$

$$P[|n(t+\mathrm{d}t)-n(t)|>1]=o(\mathrm{d}t)\,. \tag{10}$$

Note that emigration now depends on the state n, where the rate constant σ must vary inversely with the size of the segment.

For $n=1,2,3,...$ we then have

$$p_n(n,t+\mathrm{d}t)=p_n(n-1,t)\lambda\mathrm{d}t+p_n(n+1)\sigma(n+1)\mathrm{d}t+(1-\lambda\mathrm{d}t)(1-\sigma n\mathrm{d}t)p_n(n,t)+o(\mathrm{d}t)\,. \tag{11}$$

For $n=0$,

$$p_n(0,t+\mathrm{d}t)=p_n(1,t)\sigma\mathrm{d}t+p_n(0,t)(1-\lambda\mathrm{d}t)+o(\mathrm{d}t)\,. \tag{12}$$

We now expand (11) and (12), rearrange, then divide by $\mathrm{d}t$ and take the limit as $\mathrm{d}t\to 0$ to give

$$\frac{\mathrm{d}p_n(n,t)}{\mathrm{d}t}=-(\lambda+\sigma)p_n(n,t)+\lambda p_n(n-1,t)+\sigma p_n(n+1,t) \tag{13}$$

and

$$\frac{\mathrm{d}p_n(0,t)}{\mathrm{d}t}=-\lambda p_n(0,t)+\sigma p_n(1,t)\,. \tag{14}$$

The set of equations (13) and (14) for $n=0,1,2,3,...$ represents a master equation describing the time evolution of the distribution $p_n(n,t)$.

Because we are interested in the steady distribution $p_n(n)$, we assume steady conditions and set the time derivatives in (13) and (14) to zero. Thus,

$$p_n(n+1)=\frac{\lambda+\sigma n}{\sigma(n+1)}p_n(n)-\frac{\lambda}{\sigma(n+1)}p_n(n-1) \tag{15}$$

and

$$p_n(1)=\frac{\lambda}{\sigma}p_n(0)\,. \tag{16}$$

By starting with (16) and recursively stepping through $n=1,2,3,...$ we quickly discover that

$$p_n(n)=\frac{\lambda/\sigma}{n}p_n(n-1)\,. \tag{17}$$

For $n\geq 1$ this progression decreases monotonically if $\lambda/\sigma\leq 1$ and it is non-monotonic if $\lambda/\sigma>1$. It is uniquely satisfied by a Poisson distribution,

$$p_n(n)=\frac{(\lambda/\sigma)^n}{n!}e^{-\lambda/\sigma}\,, \tag{18}$$

with mean $\mu_n = \lambda/\sigma$ and variance $\sigma_n^2 = \lambda/\sigma$. Notice that $p_n(0) = e^{-\lambda/\sigma}$. Thus, as $n(t)$ fluctuates over a long period of time the proportion of this time for which $n = 0$ increases as the ratio λ/σ decreases.

In contrast to the geometric distribution described above in relation to the M/M/1 queue, the mean and variance of the Poisson distribution are well defined for $\lambda/\sigma \geq 1$. Moreover, whereas it is incorrect to assume that $\lambda = \sigma$ represents a steady condition in the M/M/1 problem, here in fact we can write $\lambda = \sigma n$, take the ensemble average, and conclude that the expected value $\mathrm{E}[n(t)] = \lambda/\sigma$. In this problem the state-dependent emigration rate σn provides a negative feedback such that for a given immigration rate, emigration quickens with increasing n and it decreases with small n. Nonetheless, like the M/M/1 problem, in any realization the state $n(t)$ stochastically fluctuates about the expected state but this expected state is not an attractor for such realizations in the sense of an Ornstein–Uhlenbeck process.

4 Poisson fluctuations

If for illustration we take the abrasion potential as $a(n) = Cn(1 - n/n_0)$ with coefficient C, then this potential is a random variable with bounded probability distribution $p_a(a)$. According to the law of the unconscious statistician the expected value $\mathrm{E}(a)$ is

$$\begin{aligned} \mathrm{E}(a) &= C \sum_n n p_n(n) - \frac{C}{n_0} \sum_n n^2 p_n(n) \\ &= C\mathrm{E}(n) - \frac{C}{n_0}\left(\mathrm{Var}(n) + [\mathrm{E}(n)]^2\right) . \end{aligned} \tag{19}$$

The variance is

$$\begin{aligned} \mathrm{Var}(a) &= \sum_n \left[Cn - \frac{Cn^2}{n_0} - \mathrm{E}(a)\right]^2 p_n(n) \\ &= -2C\mathrm{E}(a)\mathrm{E}(n) + \left[C^2 + \frac{2C\mathrm{E}(a)}{n_0}\right] \mathrm{E}\left(n^2\right) - \frac{2C^2}{n_0}\mathrm{E}\left(n^3\right) + \frac{C^2}{n_0^2}\mathrm{E}\left(n^4\right) + [\mathrm{E}(a)]^2 . \end{aligned} \tag{20}$$

If $p_n(n)$ is a Poisson distribution, then $\mathrm{E}(n) = \mathrm{Var}(n) = \lambda/\sigma = m_1$. This gives

$$\mathrm{E}(a) = Cm_1 - \frac{C}{n_0}(m_1 + m_1^2) \approx Cm_1\left(1 - \frac{m_1}{n_0}\right), \tag{21}$$

assuming $1/n_0 \ll 1$. Turning to the variance $\mathrm{Var}(a)$, all moments of n in (20) can be expressed in terms of the first moment m_1 (Appendix A). We can therefore plot the expected value $\mathrm{E}(a)$ together with the variance $\mathrm{Var}(a)$ as these vary with the first moment $m_1 = \lambda/\sigma$ (Figure 1). This reveals a greater variability in the potential $a(n)$ at small and large values m_1/n_0 than at the value of m_1/n_0 coinciding with the maximum expected value $\mathrm{E}(a)$, and is reflected in the asymmetry of the bounded distribution $p_a(a)$ as it varies with m_1/n_0. The reason for this asymmetry is as follows. The variability in the state n associated with the Poisson distribution for small and large m_1/n_0 samples over a large range of $a(n)$ due to the steepness of this function. The variance of the Poisson distribution is larger for large m_1/n_0, so the effect increases. At a value m_1/n_0 coinciding with the maximum of $a(n)$, the Poisson distribution samples a smaller range of $a(n)$. As a consequence, time

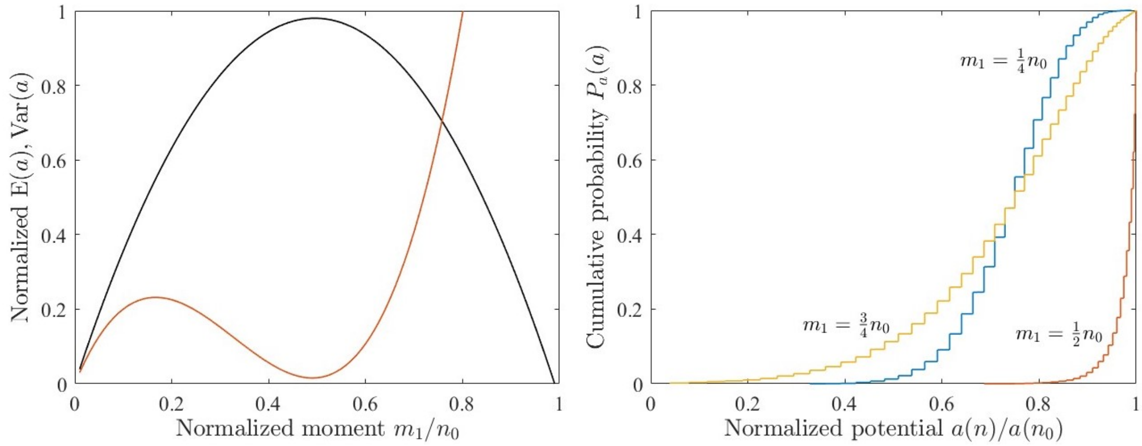

Figure 1: Plot of (left) normalized values of E(a) and Var(a) versus normalized moment m_1/n_0 and (right) cumulative distributions $P_a(a)$ for three values of the first moment m_1.

series of the state $n(t)$ with small and large m_1/n_0 experience greater variability in the potential $a(n)$.

The flux of particles out of one segment must equal the flux into the downstream segment. To illustrate this point we rewrite (11) for the ith segment in a series as

$$p_{ni}(n_i, t + \mathrm{d}t) = p_{ni}(n_i - 1, t)\sigma n_{i-1}\mathrm{d}t + p_{ni}(n_i + 1)\sigma(n_i + 1)\mathrm{d}t$$
$$+(1 - \sigma n_{i-1}\mathrm{d}t)(1 - \sigma n_i\mathrm{d}t)p_{ni}(n_i, t) + o(\mathrm{d}t)\,, \qquad (22)$$

showing that the ith segment is coupled to the state n_{i-1} of the upstream $(i - 1)$ segment. Time series of the number state $n(t)$ within three successive segments illustrate the fluctuations in this state about the expected value E$[n(t)]$ (Figure 2), where for simplicity of illustration I have selected

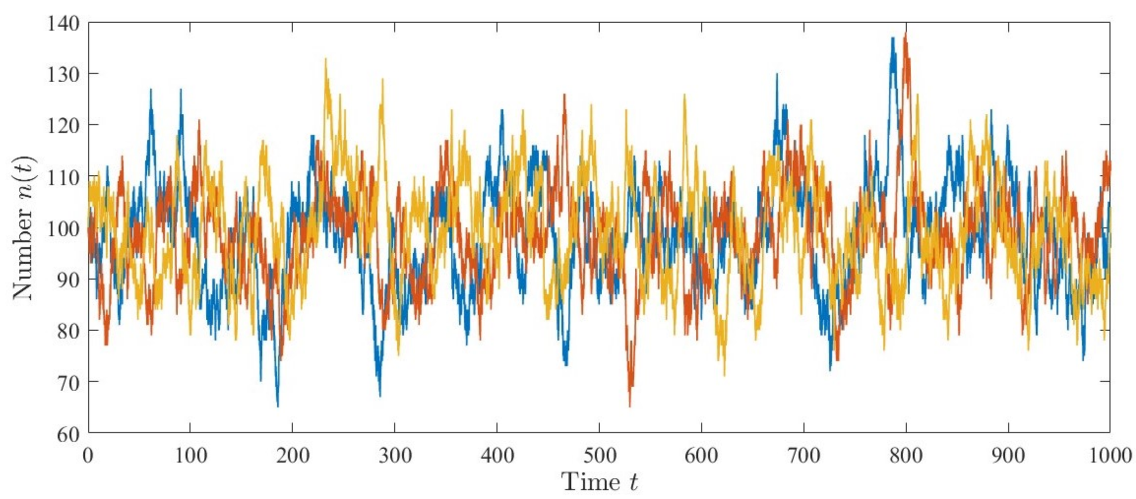

Figure 2: Plot of time series of number $n(t)$ for three successive segments with expected state E$[n(t)] = 100$. Time units scale with the upstream Poisson intensity.

rates that involve relatively small numbers. On this point, recall that the variance of the number state $n(t)$ increases with its expected value. Despite state-dependent emigration to downstream segments, the state n in each segment retains a Poisson character.

The time series of the abrasion potential $a(n)$ is in phase with the number series $n(t)$ when $m_1/n_0 = 1/4$; it is subdued and decreases with both positive and negative fluctuations in $n(t)$ when $m_1/n_0 = 1/2$; and it is out of phase and amplified when $m_1/n_0 = 3/4$ (Figure 3). This directly

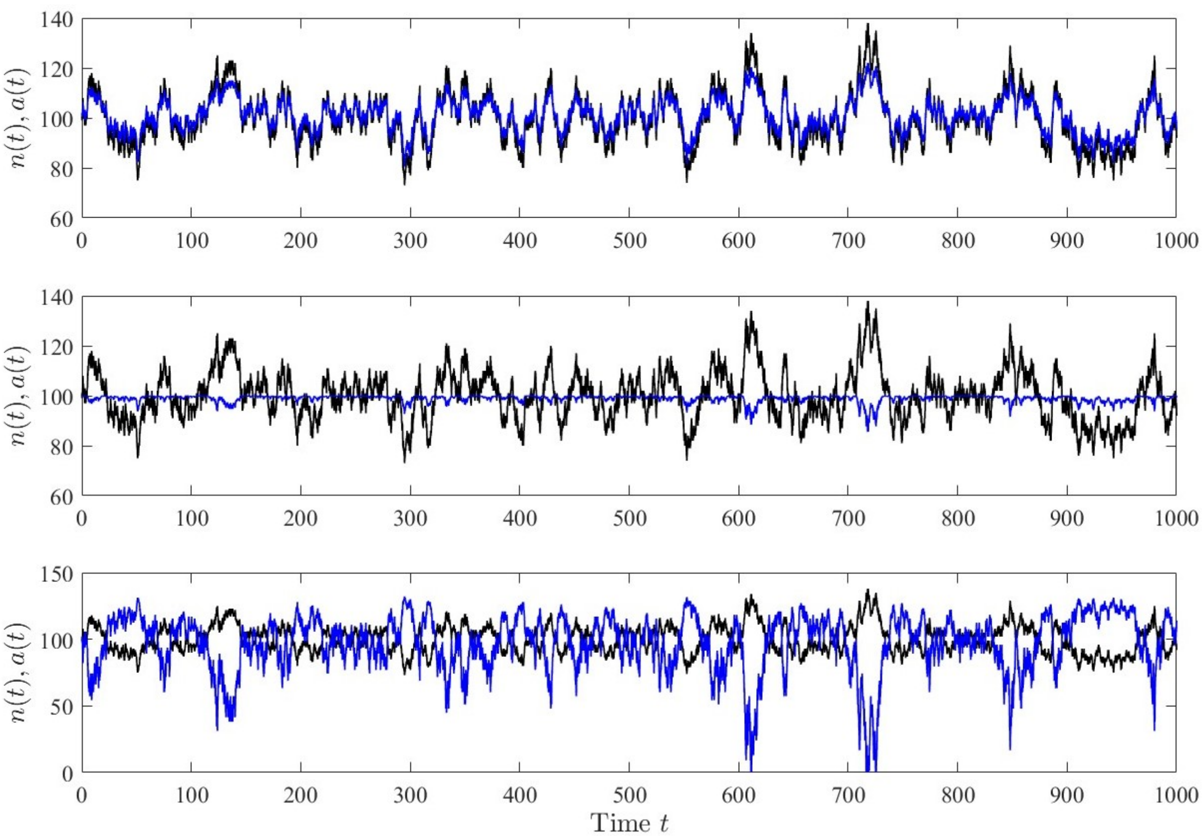

Figure 3: Plot of time series of number $n(t)$ (black lines) and potential $a(t)$ (blue lines) coinciding with $m_1/n_0 = 1/4$ (top), $m_1/n_0 = 1/2$ (middle) and $m_1/n_0 = 3/4$ (bottom). The levels of the $a(t)$ series are centered on the $n(t)$ series to highlight the phase differences.

reflects the effect of the differences in the variances $\mathrm{Var}(a)$ for each condition (Figure 1), the local slope of the function $a(n)$, and the effect of the upper bound of the distribution $p_a(a)$. In particular, for small m_1/n_0 the fluctuations in a are dominated by the leading term in $a(n) \sim n(1 - n/n_0)$ and thus track fluctuations in n. Particle are sparse and alluvial cover is minimal, so fluctuations in particle numbers are more important than fluctuations in alluvial cover. For large m_1/n_0 the fluctuations in a are dominated by the second-order term and thus respond inversely with n. Now particles are plentiful, so fluctuations in alluvial cover become more important. For $m_1/n_0 = 1/2$ the fluctuations in a reflect a subdued version of both effects.

Sediment transport in a mixed alluvial–bedrock channel occurs under rarefied conditions (Furbish et al., 2012; Furbish and Doane, 2021). In this situation it is essential to treat particle fluxes and conservation as counting process (Poisson, renewal, Lévy, etc.). The conceptualization offered above can be elaborated to incorporate, for example, inhomogeneous counting processes, mixtures of particle sizes, particle patchiness, and transitions between the active and rest states of particles. Such efforts, however, must be strongly guided by parsimony focused on uncertainty in considering time scales larger than experimental time scales — which often invites deterministic conjecture over falsifiability. Meanwhile, current mechanistic descriptions of sediment particle motions and abrasion mix deterministic quantities with random variables and manipulate these according to the ordinary rules of algebra, not recognizing or acknowledging that the algebra of random variables is unlike the algebra of ordinary variables, nor properly averaging the algebraic expressions involved. As a consequence it is difficult to decipher the mechanical meaning and implications of such descriptions, as they can involve hidden or missing information, or algebraic relations among quantities that do not correctly represent the physical situation. We therefore must await a clearer treatment of the abrasion part of the problem.

A Moments of the Poisson distribution

The moment generating function $M_n(t)$ of the Poisson distribution is

$$M_n(t) = e^{\lambda(e^t - 1)} \, , \tag{23}$$

for argument t. We obtain the moments of $p_n(n)$ by taking successive derivatives of $M_n(t)$ with respect to t and then setting $t = 0$. This leads to

$$\begin{aligned} \mathrm{E}(n) &= m_1 \, , \\ \mathrm{E}(n^2) &= m_2 = m_1^2 + m_1 \, , \\ \mathrm{E}(n^3) &= m_3 = m_1^3 + 3m_1^2 + m_1 \qquad \text{and} \\ \mathrm{E}(n^4) &= m_4 = m_1^4 + 6m_1^3 + 7m_1^2 + m_1 \, . \end{aligned} \tag{24}$$

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