Let us invest in teaching our students about dimensions

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When I started teaching courses on transport phenomena, fluid dynamics, probability and statistics, and Earth-surface processes, I recall being surprised that many students did not have a clear understanding of dimensions and the requirements that dimensions impose on the mathematics we use. Here I am referring to the dimensions of variables, coefficients and parameters rather than the dimensionality of coordinate systems or objects. Over time I learned to not be surprised. Indeed, as an undergraduate geology student exposed to very little mathematics in my majors courses and who grasped only part of the material in my physics courses, I too was a latecomer to dimensions. I have thus learned to enjoy the experience of teaching our students about dimensions — to vicariously delight in their discovery and use of a newfound power contained in dimensions and dimensional analysis, including adding rigor to their own work as well as gaining the confidence to call out dimensional gobbledygook in the literature. So this essay is a simple plea. Let us invest in teaching our students about dimensions, particularly in the Earth and environmental sciences while recognizing that this is a cornerstone topic of *all* science. Below are examples of my favorite, recurring teaching experiences centered on dimensions.

The topic starts of course with the idea of fundamental dimensions, for example, mass [M], length [L], time [T] and temperature [Θ], versus derived dimensions such as velocity [L T^{-1}], momentum [M L T^{-1}] and energy [M L^2 T^{-2}], and the units we use to express these dimensions starting with the MKS and cgs systems. This is all straightforward, including exposure to the idea that, for each equation we write that purports to have physical meaning, there is an accompanying dimensional equation that we normally don't write, but sometimes should. Students are familiar with the idea of dimensional homogeneity, that each term in an equation purporting to have physical meaning must have the same dimensions, but few have heard it referred to as this. Their familiarity with dimensional homogeneity resides in using it as one method for checking their homework in physics and chemistry, specifically to ensure that the units in their calculations are correct. Faced with a generic case involving, say, a second-order polynomial,

$$z(t) = b_0 + b_1 t + b_2 t^2 \,, \tag{1}$$

students recognize that the coefficients b_0, b_1 and b_2 must have dimensions, not just the variables z and t. For example, if $z(t)$ [L] is a distance and t [T] is time, then these coefficients must have, respectively, the dimensions [L], [L T^{-1}] and [L T^{-2}]. Oh wait! That's a version of Newton's second law describing the vertical motion of a ballistic particle,

$$z(t) = z_0 + w_0 t - \frac{g}{2} t^2 \,, \tag{2}$$

with initial position $b_0 = z_0$ [L], initial vertical velocity $b_1 = w_0$ [L T^{-1}] and one-half the acceleration due to gravity $b_2 = -g/2$ [L T^{-2}]. Nice! But then here is where a student might offer a glitch. "This is well and good, but my problems are more complicated than ballistic motion. For example, I'm interested in the metabolic rate $R(v)$ of peccaries as a function of their average foraging speed v and I have a linear regression equation that suggests

$$R(v) = b_0 + b_1 v + \varepsilon \,, \tag{3}$$

with coefficients b_0 and b_1 and error ε. So what do I make of b_0 and b_1?" Yippee! Well, if the metabolic rate $R(v)$ is an energy per unit time (power) [M L^2 T^{-3}, e.g., watts], and if the formula is physically meaningful, then b_0 must be interpreted as an at-rest metabolic rate when $v = 0$. Yes? And b_1, which must have the dimensions of a force [M L T^{-2}], may be interpreted as a characteristic mass-dependent friction that resists peccary movement (physiological and environmental) such that $b_1 v$ is the metabolic expenditure — that is, the rate at which work is performed — to overcome this friction during foraging. Gosh! This represents the "hypothesis [due to Kram and Taylor (1990)] that the time course of generating force and the cost of supporting body weight during locomotion [are] the major determinants of the metabolic cost of running" (Farris and Sawicki, 2012). Lovely. Indeed, estimated coefficients in a regression equation may have dimensions; these are not just statistical quantities that arise from minimizing the sum of squared residuals. (*Sidebar:* This is an important point of discussion regarding the use of dimensions and dimensional analysis in data analysis, particularly statistics, as a key element of the principle of parsimony and the desideratum of physical interpretability.)

Students are familiar with transcendental functions. They have worked with functions such as $\cos(x)$, $\sin(x)$, e^{-x}, $\log(x)$ and so on. Thus, when I purposefully throw a curve ball at them and sketch on the chalkboard the function $\zeta(x) = A\cos(x)$ to represent a waveform with amplitude A that varies with position x, the students are right there with me. But then I ask, "If x denotes the position along the waveform, what are the dimensions of $\cos(x)$?" In 35 years I've never not been met with initial silence on this matter. This of course is not a bad thing, and it starts our discussion centered on the idea, not typically addressed in mathematics courses, that the argument x in this cosine function cannot have the dimension of length, and more generally that it must be a dimensionless quantity when appearing as the argument in any of the transcendental functions above in order for these functions to have physical meaning (Matta et al., 2011). This naturally extends to coverage of, for example, the appearance of the wavenumber $k = 2\pi/\lambda$ or the angular frequency $\omega = 2\pi/T$ in the functions $\cos(kx)$ or $\cos(\omega t)$, where now x and t indeed have the dimensions of length and time to go with the wavelength λ [L] and the period T [T]. Similarly, the discussion gives way to the idea of writing the exponential function as $e^{-x/L}$ or $e^{-t/\tau}$, and the natural logarithmic function as $\ln(x/x_0)$, where within context the parameters L [L], τ [T] and x_0 [L] each have physical meaning.

Students appreciate from their previous work in calculus that differentials such as dx and dt represent small increments of, say, space and time. Yet when used in derivatives and integrals they surprisingly seem to view these differentials as purely mathematical things that mathematicians use to define derivative and integral operations rather than seeing them as real things carrying dimensions. It seems that few have been explicitly taught that the operator d/dt has the dimension [T^{-1}] or that the operator d/dx has the dimension [L^{-1}]. Here it becomes fun to point out the rhyme and reason of Leibniz notion, for example, that the appearance of the 2 after the "d" but before the $x(t)$ in the numerator of $\mathrm{d}^2x(t)/\mathrm{d}t^2$ tells us to leave the dimension of $x(t)$ alone, whereas the appearance of the 2 after the t in the denominator reminds us to square the dimension of t giving the dimensions [L T^{-2}], hence an acceleration. This of course is the start of showing how derivatives and integrals take care of the dimensions for us when we properly apply them to functions, for example, how taking the derivative of (2) with respect to time maintains the proper dimensions and its physical soundness,

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = w(t) = w_0 - gt\,, \tag{4}$$

thence recovering Newton's second law for this problem involvng a particle with mass m, namely, $m\mathrm{d}w(t)/\mathrm{d}t = -mg$, or $F_z = ma_z$. In this vein one of my favorite recurring experiences is watching student Eureka moments when they take ownership of the idea that if the random variable x has the dimension, say, of length [L], then the probability density function $f_x(x)$ is a probability per unit length with the dimension $[\mathrm{L}^{-1}]$ so that $f_x(x)\mathrm{d}x$ $[\mathrm{L}^{-1}\ \mathrm{L}]$ is a probability and the cumulative distribution function,

$$F_x(x) = \int_{-\infty}^{x} f_x(u)\,\mathrm{d}u\,, \tag{5}$$

is, yes(!), a probability. The Eureka moments then continue: why $f_x(x)$ can have values greater than one — fixing their previous confusion over having incorrectly learned that $f_x(x)$ represents a probability — why $F_x(x)$ cannot, and why the probability of observing a specific value of x is precisely zero. All of this starting with an understanding of dimensions! (*Sidebar:* Students sometimes need coaching on the idea that u in (5) denotes the variable of integration since x appears as the upper limit of integration. There also is value in examining the history of Leibniz notation and it relation to other styles of notation.)

The basic ideas outlined above naturally extend to formal dimensional analysis, including Rayleigh's method and its generalization as the Buckingham π theorem as a valuable device in developing hypotheses and designing experiments. In addition these ideas provide the foundation of formal scaling analysis wherein we start by learning where those wonderful dimensionless numbers that seem to appear in so many problems — the Péclet number, the Reynolds number, the Damköhler number, the Froude number, and so on — actually come from and what they mean! In doing so we discover that this style of analysis is aimed at assessing the relative importance of terms in equations and the effects these terms represent, sometimes allowing us to simplify the mathematics as well as gain clarity on how a system works.

This is the power of dimensions. Please, let us invest in teaching our students about dimensions. Consider the idea of dedicating a course to dimensional analysis and scaling. The payoff is enormous.

Acknowledgment. I appreciate comments offered by Shawn Chartrand on an earlier draft.

References

[1] Farris, D. J. and Sawicki, G. S. (2011) The mechanics and energetics of human walking and running: a joint level perspective, *Journal of the Royal Society Interface*, 9, 110–118, doi: 10.1098/rsif.2011.0182, 2011.

[2] Kram, R., and Taylor, C. R. (1990) Energetics of running: a new perspective, *Nature*, 346, 265–267, doi: 10.1038/346265a0, 1990.

[3] Matta, C. F., Massa, L., Gubskaya, A. V., and Knoll, E. (2011) Can one take the logarithm or the sine of a dimensioned quantity or a unit? Dimensional analysis involving transcendental functions, *Journal of Chemical Education*, 88, 67–70,