Particle diffusion on a Galton board

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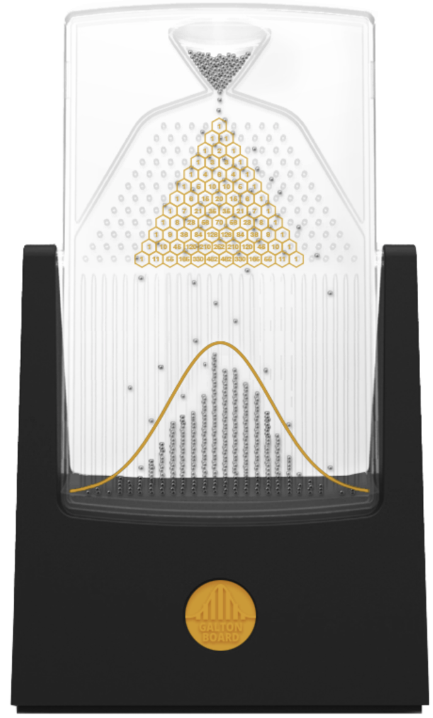
1 Initial remarks

Particle motions on a Galton board (Galton, 1894), also known as a quincunx or bean machine (Figure 1), have inspired the design of toys, descriptions of sediment particle motions (Einstein, 1937; Williams and Furbish, 2021) and theories of the statistical physics of a Lorentz gas (Moran et al., 1987; Chernov and Dolgopyat, 2008). The essential behavior of particles on a Galton board consists of their random walks in the transverse y direction as they successively encounter pegs during their downward motions, moving with some probability to the right or left past each peg, as with successive trials of a Bernoulli process (think coin flips). As the number n of peg rows increases, the collective transverse spreading of the particles increases. When the particles reach the bottom of the board, their distribution about the centerline is approximately Gaussian. Moreover, the variance of this distribution is linearly proportional to the number of peg rows through which the particles have moved.

This transverse particle spreading is in essence a macroscopic version of diffusion, and the linear increase in the variance of transverse particle positions is mathematically equivalent to Gaussian diffusion, also referred to as normal diffusion or Fickian diffusion. Descriptions of this diffusive behavior normally involve showing that with ideal particle motions — that is, mimicking Bernoulli trials — the distribution of transverse positions after n peg rows is a binomial distribution whose variance increases with n (Galton, 1894). The argument then appeals to the idea that with large n the binomial distribution can be well approximated by a Gaussian distribution.

The purpose of this brief essay is to present a simple geometric demonstration of the Gaussian



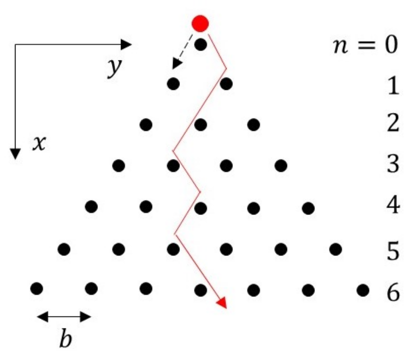


Figure 1: Desktop version of Galton board showing approximate Gaussian distribution of particles after moving through the pegs (from https://galtonboard.com/), and definition diagram of pegs with regular spacing b, rows $n = 0, 1, 2, 3 \ldots$, and a random walk through them.

diffusion of particles motions on a Galton board without appealing to the binomial distribution. For context and completeness, however, it starts with a summary of the usual presentation involving this distribution. The essay concludes with a brief description of the relevance of particle motions on a Galton board to sediment particle transport.

2 Formulation

2.1 Usual presentation

With reference to Figure 1, let x denote the downward coordinate axis and let y denote the transverse coordinate axis of a Galton board. Let b denote the regular spacing of the pegs on the board, and let $n = 0, 1, 2, 3, \ldots$ denote the number of the successive rows of pegs. Then let p denote the probability that a particle will move in the positive y direction following its collision with a peg so that $q = 1-p$ is the probability that the particle will move in the negative y direction. If $k = 0, 1, 2, 3, \ldots, n$ denotes the transverse position of the particle at the nth row, then the probability mass function of k is the binomial distribution (Galton, 1894). Namely,

$$f_k(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k} \,, \tag{1}$$

with variance $\sigma^2 = np(1-p)$. This variance directly reflects transverse diffusion associated with the random-walk motions of particles. In contrast, because all possible particle paths measured in the downslope direction are of equal length, an ideal Galton board involves negligible diffusion measured parallel to x — a point to which we return below. Also note that this description of the variance σ^2 is independent of time; it only reflects the increasing transverse variance with the number of pegs encountered during downward motion.

To recast this outcome in terms of dimensions of length and time, we start by noting that the variance of the transverse positions of particles measured about the mean position is $\sigma_y^2 \sim b^2 np(1-p)$, where we are neglecting numerical factors. Letting κ_y denote the transverse diffusivity, we then appeal to the Einstein-Smoluchowski equation to give

$$2\kappa_y = \frac{\mathrm{d}\sigma_y^2}{\mathrm{d}t} = b^2 p(1-p)\frac{\mathrm{d}n}{\mathrm{d}t} \,. \tag{2}$$

The left side leads to $\sigma_y^2 = 2\kappa_y t$, which is Gaussian diffusion, also referred to as normal diffusion to distinguish it from anomalous diffusion in which $\sigma_y^2 \sim t^\alpha$ with $\alpha \neq 1$. Letting $n_t = \mathrm{d}n/\mathrm{d}t$ denote the fixed rate at which particles encounter peg rows, then the right side of (2) leads to $\sigma_y^2 = 2\kappa_x t$ with diffusivity $2\kappa_x = b^2 p(1-p)n_t$. Below we reinterpret the rate $\mathrm{d}n/\mathrm{d}t$ with reference to mechanical dispersion.

2.2 Geometric description

With reference to Figure 1, imagine that N particles start at position $n = 0$. Half move left and half move right to the next row with two pegs at position $n = 1$. The number reaching each peg is

$$\frac{N}{2} \,. \tag{3}$$

In the same manner, half of the particles reaching each of the two pegs moves left and half moves right to the next row of three pegs at position $n = 2$. This means that the number reaching each of the outer two pegs is

$$\frac{1}{2}\frac{N}{2} = \frac{N}{2^2} \,. \tag{4}$$

The number of particles reaching the middle peg at position $n = 2$ from the two pegs above it is

$$\frac{1}{2}\frac{N}{2} + \frac{1}{2}\frac{N}{2} = \frac{2N}{2^2} \,. \tag{5}$$

Continuing to the next row of pegs, the number of particles reaching each of the two outer pegs at position $n = 3$ is

$$\frac{1}{2}\frac{N}{2^2} = \frac{N}{2^3} \,. \tag{6}$$

For outer pegs, it should now be apparent from (3), (4) and (6) that, in general, $N/2^n$. In turn, the number of particles reaching each of the two inner pegs at position $n = 3$ from the two pegs above it is

$$\frac{1}{2}\frac{N}{2^2} + \frac{1}{2}\frac{2N}{2^2} = \frac{N}{2^3} + \frac{2N}{2^3} = \frac{3N}{2^3} \,. \tag{7}$$

Perhaps one can recognize a second pattern emerging with the numbers in the numerators, namely, the binomial coefficients of Pascal's triangle. Let us do one more row. We now know that the number of particle reaching each of the outer pegs on row $n = 4$ is $N/2^n = N/2^4$. The number of particles reaching each of the adjacent pegs from the two pegs above is

$$\frac{1}{2}\frac{N}{2^3} + \frac{1}{2}\frac{3N}{2^3} = \frac{N}{2^4} + \frac{3N}{2^4} = \frac{4N}{2^4}\,. \tag{8}$$

The number of particles reaching the middle peg at position $n = 4$ from the two pegs above is

$$\frac{1}{2}\frac{3N}{2^3} + \frac{1}{2}\frac{3N}{2^3} = \frac{6N}{2^4}\,. \tag{9}$$

Now the pattern involving the binomial coefficients should be clear. Continuing this assessment through additional rows confirms the point.

We now need to calculate the variance in the position y of the N particles at each position n. If y_i denotes the position of the ith particle, then for an average position of zero the variance is

$$\sigma_y^2(n) = \frac{1}{N}\sum_{i=1}^{N} y_i^2\,. \tag{10}$$

With a peg spacing of b, then at position $n = 1$ half the particles are located at $y = -b/2$ and half are located at $b/2$. Thus,

$$\sigma_y^2(1) = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{N}{2}\left(-\frac{b}{2}\right)^2 + \frac{N}{2}\left(\frac{b}{2}\right)^2\right] = \frac{b^2}{4}\,. \tag{11}$$

At position $n = 2$,

$$\sigma_y^2(2) = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{N}{2^2}(-b)^2 + \frac{2N}{2^2}(0)^2 + \frac{N}{2^2}(b)^2\right] = \frac{2b^2}{4}\,. \tag{12}$$

Notice that the particles with $y_i = 0$ at the middle position do not contribute to the variance at position $n = 2$. At position $n = 3$,

$$\sigma_y^2(3) = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{N}{2^3}\left(-\frac{3b}{2}\right)^2 + \frac{3N}{2^3}\left(-\frac{b}{2}\right)^2 + \frac{3N}{2^3}\left(\frac{b}{2}\right)^2 + \frac{N}{2^3}\left(\frac{3b}{2}\right)^2\right] = \frac{3b^2}{4}\,. \tag{13}$$

This pattern continues with subsequent rows. So in general,

$$\sigma_y^2(n) = \frac{nb^2}{4}\,. \tag{14}$$

Comparing our work with the results of the preceding section, note that if $p = (1 - p) = 1/2$ then $p(1 - p) = 1/4$. Thus (14) is the result we started with in leading to (2).

3 Concluding thoughts

The probabilistic mechanics of particle motions on a Galton board have received significant attention (Lue and Brenner, 1993; Bruno et al., 2001; Rosato et al., 2004; Benito et al., 2009). Depending on details of the collisional behavior involved, a particle moving on an idealized Galton board within a constant gravitational field in the absence of friction is likely to accelerate indefinitely such that its velocity parallel to x steadily grows with time (Chernov and Dolgopyat, 2008). If friction balances gravity, and letting $n_x = 1/b$ denote the number of pegs encountered per unit distance parallel to x, then $\mathrm{d}n/\mathrm{d}t \sim n_x\langle u\rangle$ where $\langle u\rangle$ denotes an ensemble averaged particle velocity parallel to x. In this case (2) becomes $2\kappa_y \sim bp(1-p)\langle u\rangle$, or

$$\kappa_y \sim \delta\langle u\rangle\,, \tag{15}$$

where $\delta = bp(1 - p)$ denotes a characteristic length scale. In descriptions of transport and mechanical dispersion of particles within a porous medium, δ is equivalent to the transverse "dispersivity" (Phillips, 1991; Furbish, 1997). More generally, the result (15) indicates a fundamental point. Transverse diffusion requires downward particle motions parallel to x. That is, transverse diffusion does not occur in the absence of downward advection, as characterized by the average velocity $\langle u\rangle$. Conversely, downward motion cannot occur without transverse motions, indicating that these are fundamentally mechanically related.

Depending on the mechanical properties of the particles and pegs (e.g. their coefficients of restitution and stiffness), the design of the board (e.g. the peg radii and spacing) and the release of particles (e.g. one at a time or as a group), particle motions on a real Galton board generally do not satisfy the idealized behavior represented by Bernoulli trials with fixed probability p. As a consequence downward particle speeds vary and spreading (diffusion) measured parallel to x occurs. In addition, the shape and arrangement of pegs can be designed to change the probability p. A simple way to change p in the desktop Galton board shown in Figure 1 is to tip it sideways. This gives an average transverse drift to the particles.

The motions of sediment particles over a rough surface are analogous to particle motions on a Galton board, where surface bumps may be viewed as representing quasi-randomly arranged pegs.[1] For this reason Williams and Furbish (2021) describe surface roughness as having a bottom-up influence on randomization of particle motions. In contrast, particle angularity gives a top-down influence on this randomization, without or with surface bumps, due to the geometrical complexity of particle-surface collisions during the tumbling of the angular particles. Interestingly, Einstein (1937) appealed to particle motions on a Galton board in his probabilistic description of bed load transport, but he did not view the board as representing a rough surface. Rather, Einstein used the idea of Bernoulli trials — moving to the left or right past a peg — as mathematically representing either a particle step or a rest.

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[1] I appreciate continuing discussions with Rachel Glade and Sarah Williams regarding the mechanics of sediment particle diffusion.