The severe limitations of stress-based formulae for describing bed load transport

David Jon Furbish
Department of Earth and Environmental Sciences, Vanderbilt University, Nashville, Tennessee, USA

Consider the longstanding problem of predicting the flux of bed load particles in relation to quantities that are characteristic of a turbulent shear flow, for example, the space-time averaged stress on the bed or an averaged near-bed flow velocity. Our objective here is to explicitly consider all physical scales involved, and the assumptions that we make in moving between scales. The conclusion is straightforward. We have not yet identified suitable macroscopic state variables — if they exist — for bed load sediment transport. The macroscopic fluid-imposed bed stress in turbulent flows is an empirical heuristic and incorrect for many of the things we expect of it.

Consider the fluid, starting at the atomistic scale. At this scale we have closed the "box" of subatomic and quantum mechanical effects on atom behavior. We are assuming that the rules for describing how these effects are manifest at the atomic scale are specified and adequate. Fortunately, in envisioning atoms from the perspective of Chapman-Enskog theory, we are safe in closing the box without explicitly specifying any rules. That is, it is safe to assume that subatomic and quantum mechanical effects do not figure into our description of atom behavior; we may treat the atoms as classical Newtonian particles when inter-particle distances are much larger than the thermal de Broglie wavelength.

We now go to the continuum scale. At this scale we have closed the box on the atomistic scale, again assuming that the rules for describing the effects of atomistic behavior at the continuum scale are specified and adequate. In this case the rules are represented by macroscopic quantities that include the thermodynamic and dynamic pressures, and the fluid viscosity, depending on the atom species and number density. These rules allow us to ignore the details of atomistic behavior at smaller scales, yet fully reflect effects of this behavior at the continuum scale box. Moreover, so long as the continuum hypothesis is satisfied, the continuum box works at all scales up to the fluid system scale. Our description of the behavior of the box in the form of the Navier-Stokes equations in effect represents an emergent behavior from the atomistic scale. Analytical as well as numerical treatments of the Navier-Stokes equations for low Reynolds number flows are readily doable and well represent continuum motions.

Consider the transition from the continuum scale to the system scale with large Reynolds number. Unfortunately we cannot actually do much with our continuum description of fluid behavior unless we turn to direct numerical simulations down to the Kolmogorov scale or to large eddy simulations down to the kernel filtering scale. Momentarily setting these aside, consider Reynolds averaging where we acknowledge that this formally involves ensemble averaging, or, assuming ergodic behavior, that time averaging suffices. For illustration let us then appeal to Prandtl's mixing-length hypothesis concerning fluctuating motions, leading to the logarithmic velocity law for uniform flow that is steady in the mean. At this juncture we have closed the continuum scale box, replacing it with a new box at the same scale. Namely, in this new box the velocity field described by the logarithmic law *does not exist in a physical sense at any instant anywhere in the flow.* It is not a description of a continuum fluid nor of continuum behavior. It is a mathematical abstraction — a continuously differentiable function purporting to describe the ensemble expected (average) velocity state of the continuum. Although this law emerges from a description of fluctuating motions, it contains by itself no information regarding fluctuating motions. More generally, the Reynolds-averaged Navier-Stokes equations describe an imaginary fluid-like motion in which the velocity field represents statistically expected conditions, not actual conditions occurring in the continuum prototype. This Reynolds-averaged box then may be used to describe ancillary averaged quantities, for example, the averaged pressure field and the shear velocity or boundary stress, but nothing involving fluctuating quantities. We now set the fluid aside and turn to the particles.

Consider cohesionless sediment particles, starting at the particle scale. The atomistic scale box is closed and the rules that describe how effects of atom behavior are manifest at the particle scale include relatively crude quantities such as particle coefficients of restitution and the empirical Coulomb friction coefficient. Suppose that we continue up in scale, as we did above, envisioning a dense granular creeping behavior or a hydrodynamic-like flow behavior. Assuming (arguably) the existence of a continuum scale, then the rules for describing the effects of particle behavior are provided by, say, continuum-like $\mu(I)$ rheology in the case of a creeping granular material or by the granular analogue of the Navier-Stokes equations, coupled with the mechanical energy equation, in the case of dissipative hydrodynamic-like flows. In turn, the existence of Maxwell-demon-like behavior in dissipative granular gases mostly precludes appealing to a continuum description of such gases.

Turning to rarefied bed load transport — the condition that mostly occurs in natural channels — the idea of a hydrodynamic-like continuum scale is not relevant. Rather, particle motions are more akin to a rarefied granular gas, albeit strongly coupled with fluid motions as well as involving effects of particle-surface and particle-particle collisions, including collective entrainment. Because Reynolds-averaged conditions do not actually exist, these conditions provide no information on this coupling. That is, because sediment motions are inextricably coupled with fluctuating fluid motions, formulations that relate sediment motions to averaged system scale quantities such as the shear velocity or shear stress are purely empirical, despite the dimensional soundness of such formulations. Any formal averaging of the system consisting of a continuum fluid and rarefied particles in order to obtain a box that is analogous to the Reynolds-averaged box but which includes particle motions must incorporate this coupling in the averaging. This is a scary hard problem.

Consider the ideal gas law. Robert Boyle and Jacques Charles, although unaware of the existence of atoms, nonetheless got it right because of the simplicity of the physics of dense gas systems. Their wholly empirical relations between gas volume, temperature and pressure — system scale state variables — reflected clear invariant rules for how effects of atomistic behavior are manifest at the continuum scale, as subsequently revealed by statistical mechanics. By analogy, algebraic relations between system scale variables such as the shear velocity or the shear stress and the time-averaged bed load flux are essentially akin to an empirical gas law. But because such state variables are based on the Reynolds-averaged box, they are purely heuristic choices with no clear connection to particle scale behavior associated with fluid-particle coupling within the fluid continuum box that was closed upon averaging. Unlike developments following the work of Boyle and Charles, we have neither a statistical mechanical explanation of how these selected variables are related nor as yet a basis for choosing suitable system scale variables — if in fact they exist — for describing the particle flux with a predictive capability analogous to pressure and temperature in gas systems.

The essential source of the failure of stress-based formulae is this. Such formulae are implicitly based on two incorrect assumptions. The first is that quantities such as the local particle activity and average particle velocity, together with the bed stress, can be represented as continuously differentiable fields akin to a fluid continuum without reference to spatiotemporal fluctuations in these quantities or the effects of the fluctuations. The second assumption is that these state-like quantities are related according to multiplicative single-valued functions. Yet there is no theoretical justification for expecting such field-like quantities, or ancillary measures of fluctuating quantities, to be related in a simple algebraic manner. Moreover, the fluctuations are as important as the expected (average) conditions in characterizing the behavior of the coupled fluid-particle system, yet single-valued stress-based formulae entirely ignore the existence of covariances. Indeed, the particle flux is not merely equal to the product of the particle activity and the average particle velocity as defined for a continuum material, and the bed stress is a crude representation of the physics that determine the activity and average velocity, altogether neglecting fluctuations and particle-particle and particle-bed interactions. Time and effort may eventually yield a clear basis for algebraic expressions that relate state-like variables representing expected quantities as well as fluctuating quantities. But currently we are far from this state of affairs. Yes, more particles move, and with slightly higher velocities, at large stresses than at small stresses. Our mistake is not that we use stress-based algebraic formulations of bed load sediment transport. Rather, our mistake is that we expect far, far too much of them.