The joy of watching students demonstrate their intellectual ownership of course material

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My courses in Transport Processes, Fluid Dynamics, and Probability and Statistics involve take-home exercises and individual student projects. We discuss the idea that I am less interested in "correct" answers and far more interested in seeing how their thinking unfolds in demonstrating intellectual ownership of the material. Below is a fun example of what I mean by this. It's a bit technical, maybe... but I think it makes the point.

In the Probability and Statistics course we discuss why one must be particularly careful in working with a random variable defined as the reciprocal of a random variable. I then pose the following question as an exercise:

> Let u and v denote two random variables. We now appreciate that, in general, $\mathrm{E}(uv) \neq \mathrm{E}(u)\mathrm{E}(v)$. This also suggests that, in general, $\mathrm{E}(u/v) \neq \mathrm{E}(u)/\mathrm{E}(v)$. Indeed, forming the ratio of the averages of two random variables in order to estimate the average of the ratio of the random variables is a surprisingly common mistake. Moreover, except when absolutely necessary, statisticians typically avoid problems that involve a random variable in the denominator of a quantity.
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> **a)** With reference to the ratio above, let us define a new random variable as $w = 1/v$. Now, if v is distributed as a Gaussian distribution, how is w distributed? (Do not attempt to do this analytically! That said, simple numerics might be informative.) What can you say about the mean and variance of w? Sketches are welcome.
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> **b)** Suppose instead that v is distributed as an exponential distribution. How is w distributed? (Again, do not attempt to do this analytically.) What can you say about the mean and variance of w?
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> **c)** What condition(s) would you impose on the distribution of v in order to make the random variable $w = 1/v$ well behaved?
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> **d)** Name two distributions whose inverse distributions possess well defined first and second moments (I will tell you my two favorite in class), and two distributions whose inverse distributions have undefined moments. Note that "inverse" actually refers to the reciprocal of the random variable.

Two students popped into my office one afternoon and asked me to join them across the hall in the classroom. The students had produced a set of beautiful, carefully labeled sketches on the (large) chalkboard illustrating their responses to the questions above. By focusing on specific interval proportions they had properly mapped these proportions from the v number line to the $w = 1/v$ number line. This led to their discovery that the reciprocal Gaussian is a bimodal distribution, and that the inverse exponential is a weird distribution which, using LOTUS, has undefined moments. Lovely. We then looked at the Wikipedia description of inverse distributions. Boom! There it was: a brief discussion of the Gaussian and exponential examples, consistent with what the students had outlined on the chalkboard. But, oh my, it did bend their minds when we were looking at what appeared to be a well-behaved bimodal distribution on the chalkboard while Wikipedia was telling us that its moments are undefined, like the Cauchy distribution. Building from their results, we started a discussion of the implications of a random variable v with support $[a, b]$ where $a, b > 0$ versus $[0, \infty)$ versus $(0, \infty)$. Things thereafter started to click into place pretty quickly, and the students added a couple of sketches to their work.

I encouraged the students to add a few labels to their sketches then take photos and turn them in as their answer — which they did. Wonderful.