Probability is a physical thing, not a mathematical thing

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Consider the idea that probability, the measure p such that $0 \leq p \leq 1$, is a physical thing rather than a mathematical thing. To start, let us ponder what the clever 20^{th} century logician and philosopher Bertrand Russell had to say:

> "Probability is the most important concept in modern science, especially as nobody has the slightest notion what it means." [1]

Wow... and ouch. Indeed, there are several interpretations of probability, the principal ones being the classical, frequentist and subjectivist versions. Here is a brief nontechnical take on the matter, leaving out lots of detail.

Consider a six-sided die. The classicist, upon observing that the die is a cube, appeals to the Principle of Indifference (PoI). Namely, in the absence of information to suggest the die is unfair, one is indifferent to any of the six possible outcomes and therefore is justified in parsimoniously assuming that each is equally likely. Only then is mathematics used to translate this assumption into the probability $p = 1/6$ that any one of the six possible outcomes will occur. Note that mathematics has nothing to say about the PoI. This is a choice informed only by physics.

Each outcome is a possible state of the system, and these states are countable. The idea of "countable states" is key. The word "state" is a physics thing. The word "countable" is a mathematics thing. The probability p therefore is a physical quantity assigned to the system state, just as we might assign a volume to the cube. That is, probability in this situation is a physical thing, not a mathematical thing. Moreover, probability is precisely conserved, just as we insist that mass, momentum and energy are conserved. And, like these physics quantities, probability can be transported (e.g. advected and diffused).

Consider a probability mass function representing particle positions x. A great number N of particles moves quasi-randomly in one dimension. Each particle carries with it a probability equal to $h = 1/N$, just as it carries its mass. Conservation requires that the sum of h over N is equal to unity. The evolution of the probability mass function due to particle motions must precisely satisfy this statement of conservation of probability at all times, equivalent to conservation of mass in this example.

In passing from a discrete probability mass function to a continuous probability density function, one must be careful. Occupation of all positions on the real number line (e.g. by particles) is physically not possible, as we must not allow N to become infinite. (This is related to the physical meaning of real numbers.) A probability density function is merely a mathematical artifice — an abstraction related to the law of large numbers — just as Fick's first and second laws are approximations of molecular diffusion. But this is another topic, as is the question of how different views of probability are related.

Numerous so-called paradoxes purporting to demonstrate that the PoI must be entirely rejected have been proposed, the most famous being the Bertrand paradox [2]. Some of these challenges to the PoI seem to conflate ignorance with indifference, or imagine that any justification of the PoI must arise from mathematics. This is wrongheaded. The PoI belongs to physics. Indeed, the PoI is the foundational assumption of classical statistical mechanics [3]. In short, such paradoxes are much ado about very little. For example, see Edwin Jaynes's "The Well-Posed Problem" [4].

The title and first sentence of this short essay are purposefully provocative. In fact, the more accurate description of the matter is that probability is not a purely mathematical construct. Probability must have physical context; it is not independent of the construction of our physical description of a system. Mathematics is merely the language used to formalize the logic of probability in a given physical context. This is the material of a longer essay in preparation centered on the physical interpretation of a continuous probability density function.

References

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[3] Tolman, R. C. 1938. *The Principles of Statistical Mechanics*. Clarendon Press, Oxford.

[4] Jaynes, E. T. 1973. The well-posed problem. *Foundations of Physics*, 3, 477–493, https://bayes.wustl.edu/etj/articles/well.pdf