The "uncertainty principle" of a Poisson process: An example involving bed load transport

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In a separate essay[1] I describe the well known "uncertainty principle" of a Poisson point process. Here I offer a delightful example of this principle involving bed load transport, with important practical as well as theoretical implications. This example involves data that Madeline Allen and Shawn Chartrand are analyzing for a forthcoming paper.

Here is the essence of the uncertainty principle. Consider a finite interval $(0, t]$ of time and let $N = 0, 1, 2, 3, \ldots$ denote the possible number of Poisson events located within this interval for a fixed rate constant λ. The probability of the number of events N is described by a discrete Poisson distribution, namely,

$$f_N(N; \lambda, t) = \frac{(\lambda t)^N}{N!} e^{-\lambda t}\,, \tag{1}$$

with mean $\mu_N = \lambda t$ and variance $\sigma_N^2 = \lambda t$. This is the normal manner in which the Poisson distribution is presented. Namely, the Poisson rate λ and the interval t are precisely known, and the number N is then distributed according to (1).

Notice that (1) is like a mixed discrete-continuous joint distribution of N and λ (Figure 1). If instead of specifying the rate λ we specify the number of events N occurring within the interval t, then we do not necessarily know the value of λ. Indeed, a specific number of events N within an interval t can occur with finite probability for any value of $\lambda > 0$. As shown in the separate essay,[1] for a specified number of events N within t the rate λ is described by a continuous gamma distribution,

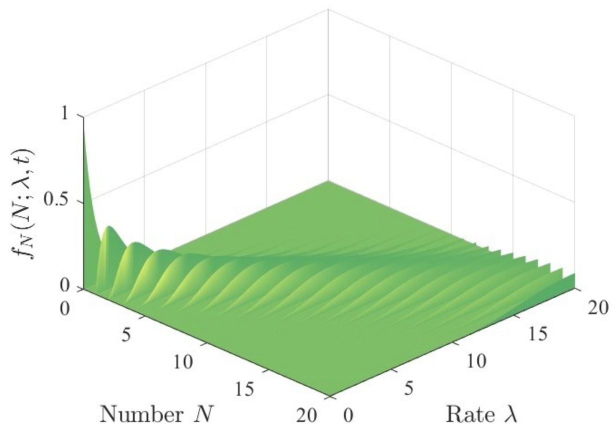

Figure 1: Joint variation of n and λ with Poisson distribution $f_n(n; s, \lambda)$.

$$f_\lambda(\lambda; t, N) = \frac{t(t\lambda)^N}{N!} e^{-t\lambda}\,, \tag{2}$$

with support $(0, \infty)$, mean $\mu_\lambda = (N+1)/t$, mode $Mo = N/t$ and variance $\sigma_\lambda^2 = (N+1)/t^2$.

If for an interval t the Poisson rate λ is known then the number of events N occurring within t only can be specified with uncertainty according to (1). In contrast, if the number N is known then the rate λ only can be specified with uncertainty according to (2). This is the uncertainty principle of a Poisson point process.

Consider an experiment in which gravel particles transported as bed load are counted as they cross the end of a flume. The number of particles $N(t)$ that cross by time t is a random variable. Suppose that we suspect the counting process $N(t)$ is Poissonian. Then, if the Poisson rate λ is specified, we ought to see the number $N(t)$ distributed according to (1) for any specified in-

[1] https://my.vanderbilt.edu/davidjonfurbish/files/2013/06/Uncertainty-Principle.pdf

terval t. But how do we know the rate λ? From an experimental perspective, we must estimate this rate; and the best estimate is the total number of crossing events $N(t)$ observed over the total time t of the experiment. Thus, we estimate $\lambda = N(t)/t$, which is the mode of (2).

But wait. We specified the total time t, then measured the number $N(t)$ to give $N(t)/t$. If we repeated the experiment in precisely the same manner a great number of times for exactly the same total time t, then almost certainly we would observe different values of $N(t)$ distributed according to (1). But from these we would calculate a distribution of empirical rates $N(t)/t$. So how do we know that the rate we estimated from our one experiment using a single value $N(t)$ is the correct underlying rate of the Poisson process? In fact, we don't.

Consider Figure 2. This shows a single measured realization of the number of crossing events $N(t)$ for one particle size in a gravel mixture under steady transport conditions. From this, Madeline and Shawn estimate a nominal Poisson rate λ. Note that the stochastic structure of this measured realization may not be strictly Poissonian (it likely represents an inhomogeneous renewal process or a compound Poisson process). Nonetheless, it suffices to illustrate the next few points. Figure 2 also shows a great number of simulated realizations of a Poisson process assuming λ is know with certainty and is equal to the experimentally estimated rate. Then compare these realizations for fixed λ with the great number of realizations for which the rate λ is not fixed, but instead is specified according to (2). The difference in the dispersion of the realizations with and without uncertainty about the expected rate, which actually is an estimate of the unknown underlying rate, is particularly clear in the box-and-whisker plots. Moreover, this uncertainty in the expected rate exists for other possible processes, not just Poisson processes. We still do not know the underlying true rate, and we can never know it.

Consider the practical problem of using a portable bed load sampler. Like transport in flume experiments, the stochastic structure of the number of particles $N(t)$ entering the sampler orifice is likely more complicated (noisier) than a Poisson process. Nonetheless, a measured realization during a specified interval t must be viewed as one of an infinite set of possible realizations, each entirely consistent with the physics involved, as depicted by the simulated realizations in Figure 2. This means that there can be decided uncertainty in estimates of the particle flux, $N(t)/t$. We address this topic in our forthcoming paper, and illustrate the uncertainty associated with specific sampling intervals.

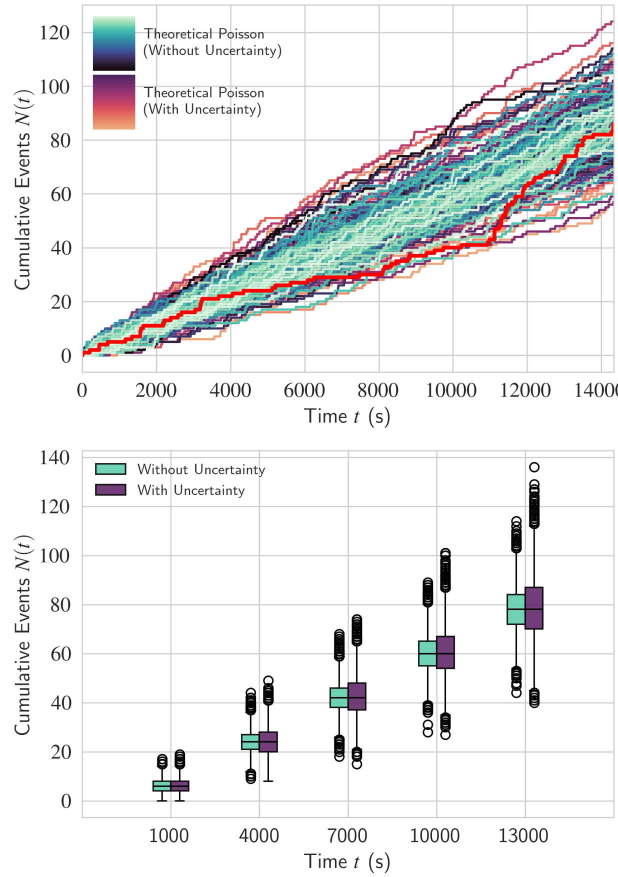

Figure 2: Plot of (red line) measured realization together with simulated realizations of a Poisson process with (blue-green) fixed rate λ and (red-purple) variable rates λ, and associated box-and-whisker plots based on 10 000 realizations showing larger dispersion about expected values with uncertain λ.