The "uncertainty principle" of a Poisson point process

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The concept of a Poisson point process is a beautiful thing, with important applications throughout the sciences. This concept involves an "uncertainty principle," which, although not rivaling Heisenberg's in its importance, nonetheless is delightful in its implications.

Consider a finite interval $x = [0, s]$ of space or time and let $n = 0, 1, 2, 3, \ldots$ denote the possible number of Poisson events located within this interval for a fixed rate constant λ. The probability of the number of events n is described by a Poisson distribution, namely,

$$f_n(n; s, \lambda) = \frac{(s\lambda)^n}{n!} e^{-s\lambda}\,, \qquad (1)$$

with mean $\mu_n = s\lambda$ and variance $\sigma_n^2 = s\lambda$. Notice that this is like a mixed discrete-continuous joint distribution of n and λ (Figure 1). If we choose a number n, then we do not necessarily know the value of λ. Indeed, a specific number of events n within the interval s can occur with finite probability for any value of $\lambda > 0$. Thus, for specified s and n, we now let λ denote a continuous random variable with support $\lambda = (0, \infty)$ and rewrite (1) as

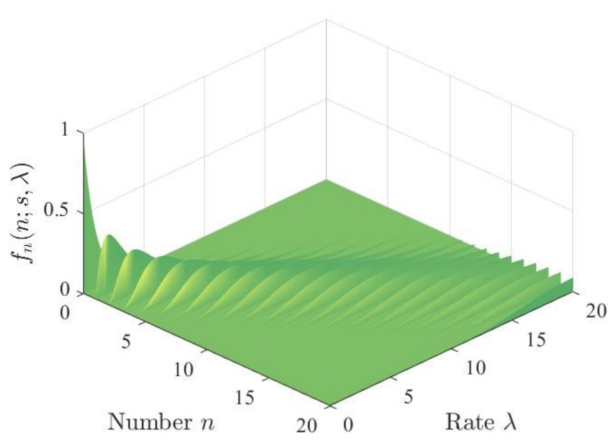

Figure 1: Joint variation of n and λ with Poisson distribution $f_n(n; s, \lambda)$.

$$f_\lambda(\lambda; s, n) = C\frac{(s\lambda)^n}{n!} e^{-s\lambda}\,, \qquad (2)$$

where C is a normalization factor. This distribution represents a probability per unit λ, so C must have the dimension of λ^{-1}. We then write the cumulative distribution function as

$$F_\lambda(\lambda; s, n) = \frac{C}{n!}\int_0^\lambda (su)^n e^{-su}\,\mathrm{d}u\,. \qquad (3)$$

Evaluating the integral,

$$F_\lambda(\lambda; s, n) = \frac{C}{sn!}\gamma(n+1, s\lambda)$$

$$= \frac{C}{sn!}[\Gamma(n+1) - \Gamma(n+1, s\lambda)]\,. \qquad (4)$$

Noting that $\Gamma(n+1) = n!$ and letting $\lambda \to \infty$ leads to the result that $C = s$. Thus,

$$f_\lambda(\lambda; s, n) = \frac{s(s\lambda)^n}{n!} e^{-s\lambda}\,, \qquad (5)$$

which is a gamma distribution with mean $\mu_\lambda = (n+1)/s$, mode $Mo = n/s$ and variance $\sigma_\lambda^2 = (n+1)/s^2$.

If for an interval s the Poisson rate λ is known then the number of events n occurring within s only can be specified with uncertainty according to (1). In contrast, if the number n is known then the rate λ only can be specified with uncertainty according to (5). This is the uncertainty principle of a Poisson point process. This principle of course must be taken into account in theoretical analyses involving Poisson processes and in certain sampling problems, including the canonical example of counting radioactive decay events in which s represents a time interval.