A Creative Destruction Perspective to Asset Pricing: Why Isn't There More Volatility?

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August, 2022

Abstract

A general equilibrium economy is studied in which growth through creative destruction has implications for asset pricing and returns. The resulting asset pricing formula contains a productivity-dependent survival function. New formulae for the risk premium and the Sharpe ratio are derived, which illustrate how unique factors can influence these measures. Asset returns contain a productivity-dependent hazard function. The risk premium and price volatility can be arbitrarily high for some assets. Although returns are influenced by the customary consumption-return correlation, other novel relationships are introduced. The risk premium of an asset can even switch from sign as the firm's productivity changes.

JEL codes: E0, G1, O4

Keywords: Asset Pricing, Risk Premium, Sharpe Ratio, Economic Growth, Creative Destruction

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†I have benefited from the comments of Robert Whaley, Atsushi Inoue and Jae Cho.

1 Introduction

A general equilibrium model, with autonomous creative destruction of firms, is studied which has predictions for the behavior of asset pricing, and rates of return. This framework is used to explore the relationship between the stochastic behavior of asset prices, and the underlying behavior of firm productivity. The equilibrium behavior of the model yields pricing formulae for a multitude of assets, which are unique relative to those in the existing literature. It is shown that the correlation of the asset payoff with consumption, or some other related factor, influences the asset price in a manner that is quite different from that seen in previous models, and this can contribute to our understanding of excess returns, as well as the volatility of those returns, for a multitude of assets. The model has the characteristic that there is a predictable component to asset returns or to the change in asset prices, even if the underlying technology of the firm does not have such a feature. The asset pricing function is shown to be the product of a standard pricing relationship, and a *survival function*, which reflects the expected discounted lifetime of the asset. Furthermore, another prediction is that a substantial fraction of assets should exhibit extremely volatile prices and returns and that this volatility should be highest for firms that have relatively low productivity.

It is well established that there is an organic nature to a growing economy in which new products and firms are continually born or introduced, while other products of firms perish. This concept was initially developed in the growth literature (see, for example, Aghion and Howitt [1], Grossman and Helpman [12]). It is natural to conjecture that the behavior of asset markets should reflect growth, and the factors that influence the rate of growth.[1] More recently, there have been a few papers that have sought to link this creative destruction concept with the accompanying implications for financial markets (see Kogan, Papanikolaou, and Stoffman [21], as well as Kung and Schmid [22]).[2] The current paper contributes to this literature by explaining how the creative destruction feature alone can have interesting implications for asset prices and returns, and how these relationships can depend on firm productivity. This analysis takes place within an environment in which both the production technology and preferences of individuals are both relatively simple or standard.

This topic has taken on added importance in recent years for promoting the understanding of trends in growth. For example, Huffman [17] documents that at the same time that there appears to be a growth slowdown in recent decades, there also seems to have been a perceptible reduction in the rates of entry and exit of firms or establishments, in the US and Canada. In other words, the reduction in the rates of creation and destruction has been accompanied by a slowdown in growth.

This paper builds on the work of Huffman [17], where the implications of a model of creative destruction for asset pricing are studied. However, in that model, the agents are risk-neutral and so the asset prices are of a rather primitive form. In contrast, in the current paper, there are risk-averse agents, and this introduces consumption risk into the model of asset pricing. It is then shown that there are three distinct elements or factors of risk: consumption, dividend, and exit or mortality risk. This model raises other questions, such as how it is possible to diversify away these separate elements of risk.[3]

[1] There are some papers at the intersection of this growth and finance literature, that feature firm exit as part of the growth process (for example, see Gomes, and Schmid [11]), although many of these papers have the exit at a constant, exogenously specified rate (Gomes, Kogan and Zhang [10], Corhay, Kung and Schmid [6]). However, these papers do not contain the key insight of this paper, which is that there is a linkage between firm productivity and the likelihood of exit (i.e. low productivity firms are more likely to exit), and that this has implications for financial market outcomes.

[2] Also, there may be factors in the environment that can influence this rate of exit.

[3] The approach adopted here is slightly related to the ideas explored by others who study the financial market

The model studied here will be valuable for enhancing our understanding of a variety of financial phenomena. First, it has been recognized for some time that asset prices and returns can display excess volatility, and that some returns can seem excessively large. The model studied below will provide a unique characterization of asset prices which then implies that some asset prices can exhibit extreme volatility. But the model is also relevant for enhancing our understanding of other apparent puzzles, such as the findings of Fama and French [8] that firm size or book-to-market value ratio seem to influence returns in a manner that is inconsistent with many existing models, such as the CAPM.[4] The model studied here will be valuable for enhancing our understanding of a variety of financial phenomena.

Second, there is evidence suggesting that there is a "size effect" where the volatility of prices or rates of return seem to fluctuate much more for larger (or "high-cap") firms, relative to smaller firms (e.g. see Banz [3] and Reinganum [26]). Also, average returns for small firms tend to be higher, but more variable than for larger, more established firms. More recently, İmrohoroğlu and Tüzel [18] also document that the variability of returns of firms in the low productivity deciles is higher than that in the high productivity deciles. Since these movements should naturally reflect similar changes in the underlying technology, one should expect to be able to see a similar size effect when measuring firm productivity. But there does not seem to be any evidence of this, and the model analyzed below can explain why this may be the case. In the analysis conducted below, it is shown that even if all firms are subject to the *same* standard deviation in the innovations to productivity, the low productivity firms can exhibit volatility in asset prices that is arbitrarily larger than that of high productivity firms. There is a non-linear relationship between firm productivity and volatility of the value of the firm. This increased volatility results even though the risk-free rate is the same for all assets, and *is also time-invariant*.

Third, there has been a great deal of research into the study of the risk or equity premium. The model can also shed some insight into why the US equity or risk premia is countercyclical, or higher at business cycle troughs than at peaks (see Harvey [15], Schwert [27], Chou, Engle and Kane [5], or Li [23]). The model can illuminate which factors can influence this premium because there is an inverse, non-linear relationship between the productivity of the firm, and the size of this premium. Also, there has been considerable work investigating how asset prices and rates of return should co-vary with consumption growth since dynamic economic models generally imply that there should be such a linkage. The model studied here will be used to show *new* avenues through which this covariance can influence asset prices and returns. This framework will yield a greater understanding of the risk premium, and the asset-specific factors that can help determine the unique risk premium for each asset. The risk premium depends on consumption risk in a unique manner. Additionally, it is shown that an asset may exhibit a risk premium even if its dividend payoff is uncorrelated with consumption growth, and this risk premium will be determined by a productivity-dependent *hazard function.* Another unique result is that in the model it is possible for the risk premium of an asset to "switch sign" (i.e. switch from being positive to negative) as the relative productivity of the asset changes *even though the underlying correlations and standard deviations have not changed*. This means that there is not some unique value for the risk premium for an asset, even if the underlying

implications of growth models. These include in Garleanu, Panageas, and Yu [9] as well as Kung and Schmid [22]. However, these papers do not explicitly address the how the endogenous exit or "destruction" feature can directly influence returns, and the models studied in these papers is quite different, and in most cases much more complicated, than the one proposed here.

[4] Gomes, Kogan and Zhang [10] make a serious contribution to explaining these observations using a general equilibrium model. However, the notion of "creative destruction" plays virtually no role in their analysis, since firms exit randomly at a constant rate (as discussed in section 4.1 below). In other words, in their model there is no linkage between firm productivity, and the exit probability.

parameters (i.e. variance and correlation) of the asset payoff are unchanged.

The asset pricing formula that results from the equilibrium of the model is unique in that it introduces new channels through which other factors, including consumption risk, can influence asset prices and returns. This new channel is represented by a productivity-dependent *survival function* that measures the likelihood and value associated with the asset exiting the market in the future - a result of the creative destruction process.[5] It is shown that the rates of return of these asset prices have a predictable component, which is based on the productivity of the firm. That is, changes in the prices of a firm's shares will have a predictable component that is a function of the productivity of the firm. This component is nearly constant for high-productivity firms. However, for firms with relatively low productivity, their asset prices can be *extremely sensitive* to changes in productivity. Here the price of the asset reflects the usual value of the correlation of the payoff of the asset with consumption. However, there is also an additional avenue through which sources of risk can influence the asset price because the ultimate shutdown of the firm is correlated with other factors, which might include aggregate consumption. For example, if the firm is very likely to cease operations when consumption is low, then the asset may exacerbate consumption risk. On the other hand, if the firm has reduced mortality (i.e. is unlikely to cease operations) when consumption is low then the asset provides insurance against consumption risk. This risk, and insurance capability, will be reflected in the asset's price and rate of return. The size of this new effect becomes more pronounced, the lower is the relative productivity of the firm. An additional source of risk, that is reflected in the risk premium of these assets, is captured by a *productivity-dependent hazard function*, that is conveniently characterized. One conclusion is that the risk-return frontier is a considerably more Byzantine territory than was previously thought, and it can depend on a multitude of factors and parameters.

To arrive at these results, the model does not rely on irrationality, asset bubbles, portfolio or borrowing constraints, externalities, or any unusual preferences, such as extreme risk-aversion, non-additively separable utility, or habit formation.[6] To facilitate the straightforward comparison of the findings here with results in the existing literature, the traditional risk-averse preferences will be employed. The behavior of the owners of the assets is passive or competitive and typical of those in representative agent models. These agents will have identical preferences and will hold identical, perfectly diversified portfolios of assets in the economy. Nor does the model require any unusual volatility for the equilibrium risk-free rate. In fact, *this rate is constant*. Furthermore, the model does not employ any complex features in the production function. The innovative features of the asset pricing equation will then be the result of the introduction of a new attribute into the model, which is a by-product of the creative destruction feature of the economy. The implication is that traditional models of the consumption-based asset pricing model have been omitting an important and simple factor that is capable of explaining much of the observed volatility in prices and returns. In other words, the strong focus on how consumption volatility could affect the random discount factor for asset payoffs may have been omitting some important features that can contribute to influencing prices and rates of return.

The remainder of this analysis will proceed as follows. First, a general equilibrium model

[5] Kogan, Papanikolaou, Seru, and Stoffman [20] develop an innovative method of quantifying some factors that can influence the degree of creative destruction. They find that the number of citations that a patent receives can be an important factor. This is (perhaps distantly) related to the findings here, because the results of Section 5 suggest that such a factor can show up in the asset price of (vulnerable) competitor firms.

[6] There are numerous papers that rely on non-additively separable preferences to study financial phenomenon (for example, see Kung and Schmid [22], or Jermann [19], who employs these preferences in addition to introducing production). Other papers, such as that of Kogan Papanikolaou and Stoffman [21] make use of the non-additively separable feature together with that of the agent caring about relative consumption as well.

of endogenous growth will be presented in which it is shown how new firms are created and eventually perish. Shareholders of these firms will then determine how the prices of these assets will be priced. This will give rise to a unique asset pricing relationship. More general pricing formulae will then be derived and studied in several steps. It is shown that if there is no mortality risk for the firm, the price of these shares looks very familiar. Then if this mortality or creative destruction risk is introduced, these prices and returns reflect this risk in a novel manner. Finally, a model is studied in which this creative destruction risk is influenced by time-varying factors, and this feature adds some novel features to the pricing formula for assets, as well as the rate of return. The level and volatility of asset prices and rates of return can be characterized as novel functions of the relative level of productivity, as well as other factors. It is also shown that prices and rates of return reflect a new correlation, which is not necessarily the same as the correlation of the asset return with aggregate consumption.

2 A Model of Endogenous Growth with Creative Destruction

The model is a variant of the general equilibrium framework developed in Huffman [16],[17] in which growth is produced through the continuous *creation* (or entry) of new technologies (or firms), concurrently with the *destruction* (or exit) of old technologies. The present model will build on these developments, to yield some insights into the behavior of financial markets. The "creative destruction" features of the model are quite primitive, since the focus here is not to study the complex factors that can influence the mechanics of entry and exit of new firms. Instead, the objective is to study the financial implications that this issue has for the pricing of assets of these firms.

This section will describe a basic economy in which some simple technologies and assets can be priced. The characterization of these prices will be presented in Section 4. But these assets will be a subset of a much broader, and more interesting class of assets that will be studied in Section 5.

There will be two groups of agents in the economy: managers and shareholders (or firm owners). The former will manage or run the firm, but will also decide when to quit operating the firm, at which point it ceases operations. The manager's payoff or consumption is proportional to the productivity of the firm. The shareholders are identical and own a perfectly diversified portfolio of shares of the firms, and receive dividends that are also proportional to the productivity of the firm.

Then there are a group of N managers, who have preferences that are linear in consumption. Each of these managers operates one, and only one firm. If the firm has a productivity of "z_t" then they receive income proportional to this productivity. They do not own the business, so much as they manage it, and therefore they share the income produced by the firm, with the owners or shareholders. If at any date t, the manager chooses to cease operating the business, then they bear a cost of $h(z) > 0$, and they can then begin a new business that has productivity Z_t. The manager must then decide at which point it will be optimal to shut down the old business and begin the new one. This evolution of business from new to old, entry to exit, or from higher to lower productivity has the flavor of a model of creative destruction. However, since the focus here is on asset pricing, the process by which new firms enter will be modeled in its most trivial form.

However, when these new businesses are initiated, they must be seeded with income or capital (or something), and this is done by the shareholders. Let this cost be εZ_t. Hence, if there are $\delta\Delta$ new firms created over any short span of time Δ, then this total seeding cost is $\varepsilon Z_t \delta\Delta$. Since the population of shareholders is normalized to unity, this is then also the cost per shareholder.

The shareholders are identical, and as such, they will each hold an identical portfolio of assets that contains all of the firms in the economy. Hence if $z_{t,i}$ represents the productivity of firm i at date t, and $G(\cdot)$ represents the distribution of these productivities, then at each moment t a shareholder will then consume the following:

$$\int_0^\infty z_{t,i} dG(z_{t,i}) - \varepsilon Z_t \tag{1}$$

It will be shown below that this quantity will be proportional to the productivity level Z_t.

2.1 A Stochastic Characterization of Technology

At any date $t \geq 0$, all new entrants or firms enter with a productivity of Z_t where

$$\frac{dZ_t}{Z_t} = \kappa_Z dt + \sigma_Z dW, \tag{2}$$

where W is a standard Brownian motion. The growth rate (κ) will be taken as a parameter, beyond the control of all agents. The goal here is not to provide an innovative theory of innovation and growth, but rather to study the implications that the features produced by growth will have for financial outcomes. Firms in each cohort are identical initially, but not for long! Following entry, the productivity of a firm that was a date-t entrant, but which is now of age a, evolves according to the following:

$$z_{t,a} = Z_t e^{(\kappa_1 a + \sigma_w w_a)},$$

where w_a is a standard Brownian motion that is independent across firms. It is assumed for now that w_a and W are uncorrelated, but in Section 5 this assumption will be relaxed to permit a consideration of a multitude of different assets, with different correlation properties.[7] Nevertheless, this current approach is useful because at any date t, all operational technologies at that date will be measured against the singular value of that of new entrants (Z_t), and this will serve as a unique reference point. Note that this assumption implies that an older firm benefits from technological change that has happened since it began operations, because $z_{t-a,a}$ is linked to Z_t. Once a firm ceases operations, $z_{t,a}$ becomes zero forever. So here κ_Z represents the growth rate of the technology for new entrants, while κ_1 will represent the growth rate of technology for incumbents.[8] If we consider t fixed then as the firm ages we have

$$\frac{dz_{t,a}}{z_{t,a}} = \kappa_1 da + \sigma_w dw_a. \tag{3}$$

Next, consider a firm that began operations at date $t - a$. It will be convenient to let s be an index of the productivity of a firm *relative to that of a new entrant* at date t. That is

$$s_t \equiv \ln\left(\frac{z_{t-a,a}}{Z_t}\right)$$

and so for this particular firm

$$ds = \kappa_s dt + \sigma_s dw_s$$

[7] Letting these processes be correlated does not change the behavior of the asset holders or managers. It only influences the value of the assets.

[8] In this way, there will be innovation, or productivity growth, by both incumbents and entrants, as is documented by Decker, Haltiwanger, Jarmin and Miranda [7]. However, only the productivity growth by entrants is determined by the decisions of agents.

where

$$\kappa_s = (\kappa_1 - \kappa_Z), \quad \sigma_s dw_s = \sigma_w dw_a + \sigma_Z dW$$

or

$$\sigma_s = \sqrt{\sigma_w^2 + \sigma_Z^2}.$$

Here $\kappa_s < 0$ denotes the trend growth of the productivity index of an incumbent firm, relative to that of new entrants. In the analysis below it will be convenient to assume that $\kappa_s < 0$, which means that that there may be growth in the productivity of incumbents, but that aggregate growth is driven by that of new entrants.[9]

Note that in the analysis below, the age (a) of the firm is not really relevant. That is, for two firms that are of different ages $a \neq a'$, if it turns out that $z_{t-a,a} = z_{t-a',a'}$, then these firms will be treated identically at that moment. This is important because this will mean that it will not be necessary to keep track of the distribution of technologies for all past entrants. Instead, at any date, it will only be necessary to know the distribution of firm's technologies at that particular moment.

2.2 Managers

There is a mass of managers, which is normalized to unity, who manage but do not own the firm. Each manager operates one firm and receives a flow of income proportional to the productivity (z) of the firm. The managers do not save or invest, and therefore do not make any decision concerning asset pricing. The manager is free at any time to shut down that firm and dispose of the technology, and this is their only decision. The cost of doing so will be $h(z)$. The manager can then obtain a new technology or firm, which has immediate productivity of Z_t. One might think of these managers as engaging in a non-market form of research and development that leads to innovative technologies or products.[10]

Managers have preferences that are linear in consumption:

$$E \int_0^\infty e^{-\hat{r}t} \left[c_t - h(z_t)\right] dt. \tag{4}$$

Here $\hat{r}$ is the rate of time preference for the manager, while $h(z_t)$ is the instantaneous disutility that a manager will bear if they decide to shut down their present firm, and seek to create a new one.[11] To maintain simplicity, will be assumed that $h(z_t) = h \cdot z_t$.[12] Momentarily, consider the value function $(W(\cdot))$ for such a manager of a firm with productivity z_t. It will be best to write this as a function of $s = \ln\left(\frac{z_t}{Z_t}\right)$. Hence if $s = 0$ then the manager's firm has productivity equal to that of a newly created firm (Z_t) at that date. Also let $\underline{s}$ denote the relative productivity of a firm that indifferent between continuing to operate, and alternatively shutting down. Let $Z_t W(s_t)$ denote the value function of a manager with a firm with relative productivity of s_t. The linearity of the preferences in equation (4), the assumed functional form for $h(z_t)$, and the assumption that consumption of the manager is proportional to productivity,

[9] In other words, innovation can be produced by both new entrants and incumbents. Decker, Haltiwanger, Jarmin and Miranda [7] document the roles of these firms in contributing to increased productivity. In a more sophisticated model, there could be innovation by both new entrants and incumbents. This would not affect the fundamental issue exploited below, which is that there is a relationship between firm productivity and their mortality.

[10] This feature is similar to the assumption, and intuitive idea, in Kogan, Papanikolaou and Stoffman [21] that market for ideas, and especially new ideas, is incomplete.

[11] That is, this cost is zero immediately before and after the manager shuts down the firm.

[12] It is important that this cost be a function of productivity (z_t) so that the costs grow in size as the productivity grows in the economy.

together imply homogeneity of the manager's problem. The value function for this problem will then be proportional to Z_t. The expression $W(s_t)$ might be termed the normalized value function. The HBJ equation for the manager's problem is then the following

$$\hat{r}W(s_t) = e^{s_t} + W'(s_t)\kappa_s + W''(s_t)\left(\frac{\sigma_s^2}{2}\right). \tag{5}$$

This problem has the following boundary condition:

$$W(\underline{s}) = W(0) - he^{\underline{s}}. \tag{6}$$

or

$$Z_t W(\underline{s}) = Z_t W(0) - h(z_t).$$

It is shown in the appendix that the manager's problem can be characterized as follows:

Proposition 1 *The solution to the manager's problem is the following value function:*

$$W(s) = \Phi_1 e^s + \left[\frac{\Phi_1 [1 - e^{\underline{s}}] - he^{\underline{s}}}{[1 - e^{-\underline{s}\phi}]}.\right] e^{(s-\underline{s})\phi}.$$

where

$$\Phi_1 = \left[\hat{r} - \kappa_s - \left(\frac{\sigma_s^2}{2}\right)\right]^{-1} > 0,$$

and the root ϕ of the characteristic equation must satisfy the following:

$$\phi = \frac{-\kappa_s - \sqrt{(\kappa_s)^2 + 2\sigma_s^2\hat{r}}}{\sigma_s^2} < 0.$$

Furthermore, there is a unique value of $\underline{s}$ such that satisfies the condition (6) that is given by the following:

$$\underline{s} = \ln\left(\frac{[1 - e^{-\underline{s}}] - (he^{\underline{s}}/\Phi_1)}{[1 - e^{-\underline{s}\phi}]}\right) + \ln(-\phi)$$

Proof. See Appendix ■

Here, as a more general interpretation, the function $(W(\underline{s}))$ represents an amalgam of all the factors, external to the firm, that can affect the firm's existence. Here the market forces that influence both the entry and exit decision are embedded in the payoff (e^s), which in turn influences the value function $(W(s))$. In a more detailed or complicated model, there could be many additional market factors that could influence the opportunity cost of operating a firm, or producing a product, and also for innovation. The point here is that factors that influence the incentives for innovation and the opportunity cost of operating a firm are intertwined (as can be seen by equations (5) and (6)), and policies or at parameters do not just influence one of these in isolation from the other.

The focus of this paper is not to develop a novel theory of endogenous growth, but instead to study the asset pricing implications of a model with endogenous entry and exit. In this framework it is the managers, and not the passive shareholders, who make decisions that ultimately influence the growth rate, while the shareholders determine the asset prices. Altering the incentives of these managers to innovate, or instead to not shut down their firms, will influence the growth rate.

2.3 The Distribution of Firms

Let the relative technology of a firm be denoted by $s = \ln(z/Z)$, so that if $s > 0$ then this particular firm has a productivity greater than that of new firms.[13] Let $\underline{s}$ denote the exit barrier so that $\underline{s}$=$\ln(\underline{z}/Z)$. The entry level of relative productivity is $s = 0$. The process for s has drift $\kappa_s < 0$, and standard deviation $\sigma\sqrt{\Delta t}$. We want to characterize the steady-state distribution of this process. To do this we must study the Kolmogorov forward equation, which is written as follows:

$$\frac{\partial f(s)}{\partial t} = -\kappa_s \frac{\partial f(s)}{\partial s} + \left(\frac{\sigma_s^2}{2}\right)\frac{\partial^2 f(s)}{\partial s^2}, \text{ for } x \in (\underline{s}, 0) \cup (0, \infty). \tag{7}$$

Note that for a stationary distribution the left side of this equation will be zero. Continuity of the distribution dictates that

$$\lim_{s \nearrow 0} f(s) = \lim_{s \searrow 0} f(s).$$

For $\underline{s}$ to be an absorbing barrier, and for $f()$ to be integrable it must be that the following boundary conditions must be satisfied

$$f(\underline{s}) = f(\infty) = 0. \tag{8}$$

It will be the case that the flow of firms exiting at the boundary will be

$$\frac{\sigma_s^2}{2}\left.\frac{\partial f(s)}{\partial s}\right|_{s=\underline{s}}. \tag{9}$$

Hence, for a stationary distribution we must have a quantity entering at the point $s = 0$. Using equation (8), as well as the fact that the left side of equation (7) is zero, and integrating this latter expression yields

$$0 = \frac{\sigma_s^2}{2}\left[\left.\frac{\partial f(s)}{\partial s}\right|_{s \nearrow 0} - \left.\frac{\partial f(s)}{\partial s}\right|_{s=\underline{s}} - \left.\frac{\partial f(s)}{\partial s}\right|_{s \searrow 0}\right].$$

This implies that the entry rate of firms is given by the following

$$\left.\frac{\partial f(s)}{\partial s}\right|_{s=\underline{s}} = \left.\frac{\partial f(s)}{\partial s}\right|_{s \nearrow 0} - \left.\frac{\partial f(s)}{\partial s}\right|_{s \searrow 0}. \tag{10}$$

This equates the rate of entry to the rate of exit. Now that since the left side of equation (7) is zero, the root of this differential equation is the following

$$\delta = \frac{2\kappa_s}{\sigma_s^2} < 0. \tag{11}$$

It is then possible to show that the steady-state distribution of *relative* technologies is characterized as follows:

$$f(s) = \begin{cases} \left(\frac{1}{-\underline{s}}\right)\left(1 - e^{\delta(s-\underline{s})}\right) & \text{for } s \in (\underline{s}, 0) \\ \left(\frac{1}{-\underline{s}}\right)\left(1 - e^{-\delta\underline{s}}\right)e^{\delta s} & \text{for } s \in (0, \infty) \end{cases}. \tag{12}$$

Note also that

$$f'(s)\big|_{s=\underline{s}} = \left(\frac{1}{\underline{s}}\right)\delta > 0.$$

[13] This section follows the analysis of Harrison[14] and Luttmer[24].

2.4 Observations

There are a few interesting properties of the equilibrium that can be noted at these points. These are listed as follows, and the proofs are listed in the Appendix.

Lemma 2 *Aggregate output is proportional to* Z_t*.*

Lemma 3 *The average lifetime of a firm, which is the expected first passage time from* $s = 0$ *to* $s = \underline{s}$ *is*

$$E(T) = \frac{\underline{s}}{\kappa_s}. \tag{13}$$

and the standard deviation of this time is given by

$$\frac{-\underline{s}\sigma_s}{\sqrt{2|\kappa_s|}}.$$

3 Shareholders or Firm-Owners

There will be a population of shareholders, which is also normalized to unity. To show that none of the results derived below rely on any unusual assumptions about preferences of shareholders, the typical CRRA preferences will be employed. This will permit the easy comparison of the results with those in the existing literature.[14]

Therefore, the environment will be one in which each shareholder has the following preferences:

$$E_0 \int_0^\infty e^{-rt} \left(\frac{c_t^{1-\theta} - 1}{1-\theta} \right) dt \tag{14}$$

with $\theta \geq 0$. To show that the results do not depend on any peculiar features of the asset market, it will be assumed that all of the shareholders will hold the same perfectly diversified portfolio of assets issued by all of the firms. In this way, there is no further portfolio insurance that is missing from this economy. The budget constraint for these agents is simple. At each date, they can buy/sell a portfolio of assets (x_τ), which are indexed by $\tau \in [0,1]$. Let the price of these assets be denoted by V_τ, and let the instantaneous dividend be π_τ. At each moment t the budget constraint for such an agent can then be written as follows:

$$c + \int_0^1 V_\tau \dot{x}_\tau d\tau = \int_0^1 \pi_\tau x_\tau d\tau \tag{15}$$

In a steady-state of the equilibrium, since the shareholders are identical, and the supply of assets is time-invariant, it will be the case that $c = \int_0^1 \pi(\tau) x(\tau) d\tau$.

The shareholders are identical and will each hold a perfectly diversified portfolio of assets. Because lemma 2 states that aggregate output, and therefore consumption is proportional to Z_t, in what follows it will be assumed that the consumption of shareholders will be equal to Z_t.

[14] In the interest of clarity, the present analysis abstracts from these intricate formulations, and instead employs the standard, yet simple, CRRA formulation. This will be expedient because this yields a standard and simple pricing kernel that facilitates a comparison with other pricing formulae. This contrasts with the common practice in many recent articles to employ somewhat exotic or complicated specifications of preferences in order to account for certain facts. For example, Kung and Schmid [22] as well as Corhay, Kung and Schmid [6] employ Epstein-Zin preferences. Then there are papers, such as Garleanu, Panageas, and Yu [9], that use non-additively separable preferences together with an external habit formulation so that individuals care about "relative consumption". Additionally there are papers such as Kogan, Papanikolaou, and Stoffman [21] which use the Duffie-Epstein preferences together with agents caring about consumption relative to the aggregate.

The consumption of all managers is measured by equation (1), which is their income minus net of costs. This is proportional to Z_t. The shareholders collectively consume an amount equal to the first term in this equation. So total output or consumption is equal to the net amount in equation (1), plus the addition again of the first term of this equation.[15]

Before proceeding to an analysis of the price of each specific asset, it will be useful to price a few other benchmark assets which can then be used as reference points.

It will be helpful in the following analysis to adopt the notation of $\kappa_q = \kappa_1, \sigma_q = \sigma_w$.

3.1 Pricing of Some Benchmark Assets

It is of interest to then price an asset that has a payoff proportional to that of aggregate consumption. It is then straightforward to establish the following result.

Lemma 4 *The price of an asset that has a payoff equal to that of aggregate consumption is*

$$P(Z_t) = \frac{Z_t}{r - \kappa_Z (1-\theta) - \frac{\sigma_Z^2 (1-\theta)^2}{2}}.$$

The instantaneous expected rate of return on this asset is given by

$$\left(r + \theta \kappa_Z + \frac{\theta (2-\theta) \sigma_Z^2}{2} \right), \tag{16}$$

and the risk-free rate of return is

$$R^{rf} = \left(r + \kappa_Z \theta - \left(\frac{\sigma_Z^2}{2} \right) \theta^2 \right). \tag{17}$$

Of course, the instantaneous percentage change in the price is then given by

$$\frac{dP(Z)}{P(Z)} = \left[\kappa_Z + \left(\frac{\sigma_Z^2}{2} \right) \right] dt + \sigma_Z dW.$$

This establishes the following conventional results.

Lemma 5 *For an asset that has a payoff proportional to aggregate consumption, the risk premium is $\theta \sigma_Z^2$, and the Sharpe ratio for this asset is $\theta \sigma_Z$.*

These typical results are what one would normally expect in such an environment. This is useful because the novel results established below then are an outcome of an environment that is otherwise rather traditional.

It is important to note that, because of the random-walk nature of consumption, the risk-free rate (17) will be constant. This in turn is determined by the process described in equation (2). Therefore, the high sensitivity or volatility of some assets, which is shown below, *cannot* be attributable to movements of this rate.

[15] The value of ε is not particularly relevant here, since this merely reduces the level of consumption of mangers, but does not change the fact that their aggregate consumption is, like that of the share-holders, proportional to Z_t. That is, the value of ε will not show up in the formulae for asset prices or returns. .

4 Pricing the Value of any Single Firm

It is then appropriate to price the value of the firms or assets in this economy. First, note that it is straightforward to show that consumption is proportional to Z_t. To do this we will once again make use of the fact that for the shareholders, $c_t \simeq Z_t$. Note that this is determined by equation (2).

In this environment the dividend paid by the term is synonymous with the productivity of the firm (z_t). Therefore, let the dividend or payoff from any one particular firm be denoted by $z_t = e^{q_t}$, while keeping in mind that the firm exits when $q_t = \underline{q}_t$. Let $\tilde{V}\left(Z_t, q_t, \underline{q}_t\right)$ denote the utility value function that measures the value of the asset with current payoff $z_t = e^{q_t}$, with exit threshold $e^{\underline{q}_t} = Z_t\left(e^{\underline{s}}\right)$, when current consumption is $c_t = Z_t$.The HBJ equation for this problem is then written as follows:

$$r\tilde{V}\left(Z_t, q_t, \underline{q}_t\right) = \left(Z_t\right)^{-\theta} e^{q_t} + \mathcal{A}\tilde{V}\left(Z_t, q_t, \underline{q}_t\right) \tag{18}$$

and where the drift term is determined as follows:

$$\begin{aligned}\mathcal{A}\tilde{V}\left(Z, q, \underline{q}\right) &= \kappa_Z \tilde{V}_1\left(Z, q, \underline{q}\right) + \sigma_Z^2 \tilde{V}_{11}\left(Z, q, \underline{q}\right)/2 + \kappa_q \tilde{V}_2\left(Z, q, \underline{q}\right) + \sigma_q^2 \tilde{V}_{22}\left(Z, q, \underline{q}\right)/2 \quad (19)\\ &\quad + \kappa_Z \tilde{V}_3\left(Z, q, \underline{q}\right) + \sigma_Z^2 \tilde{V}_{33}\left(Z, q, \underline{q}\right)/2 + \sigma_Z^2 \tilde{V}_{13}\left(Z, q, \underline{q}\right).\end{aligned}$$

The last term involving $\tilde{V}_{13}$ arises here because Z_t and $\underline{q}_t$ are perfectly correlated.[16]

In this environment, the asset continues to payoff until $q_t = \underline{q}$, at which point the asset is worthless. Therefore, a necessary boundary condition for this price is the following

$$\tilde{V}\left(x, \underline{q}_t, \underline{q}_t\right) = 0.$$

It is shown in the appendix that the solution to this problem can be shown to be of the following form.

$$\tilde{V}\left(x, q_t, \underline{q}_t\right) = \left(e^{(-\theta x_t)}\right) B\left[1 - e^{(\beta-1)\left(q_t - \underline{q}_t\right)}\right]\left(e^{q_t}\right).$$

It then follows that the price of the asset measured in consumption units is given by the following.

Proposition 6 *The real price or value of the asset current payoff $z_t = e^{q_t}$, with exit threshold $e^{\underline{q}_t}$, when current consumption is $c_t = Z_t$ is given by the following:*

$$V\left(q_t, \underline{q}_t\right) = B\left(e^{q_t}\right)\left[1 - e^{(\beta-1)\left(q_t - \underline{q}_t\right)}\right]. \tag{20}$$

where

$$B = \left[r + \kappa_Z \theta - \left(\frac{\sigma_Z^2}{2}\right)\theta^2 - \kappa_q - \left(\frac{\sigma_q^2}{2}\right)\right]^{-1}. \tag{21}$$

$$\beta = \frac{-b - \sqrt{b^2 + 4a\left(r + \kappa_Z(\theta - 1) + \left(\frac{\sigma_Z^2}{2}\right)\left(1 - \theta^2 + 2\theta\right)\right)}}{2a}, \tag{22}$$

$$a = \left(\frac{\sigma_q^2}{2}\right) - \left(\frac{\sigma_Z^2}{2}\right), \quad b = \kappa_q - \kappa_Z + \sigma_Z^2(1 + \theta).$$

and

$$\underline{q}_t = \underline{s} + \ln\left(Z_t\right). \tag{23}$$

[16]The terms involving V_{12} and V_{23} are absent in this expression because it is assumed that the processes for each asset's productivity growth (w_a) is uncorrelated with that for consumption.(W).

Proof. See Appendix. ■

There is then the following special case to consider.

Corollary 7 *In the special instance in which $a = 0$ then it is the case that*

$$\beta = \frac{r + \kappa_Z(\theta - 1) + \left(\frac{\sigma_Z^2}{2}\right)\left(1 - \theta^2 + 2\theta\right)}{\kappa_q - \kappa_Z + \sigma_Z^2(1 + \theta)}.$$

Despite the simplicity of the equation, there are several features embedded in the pricing equation in Proposition 6 that are important to note.

In the case in which $\underline{q}_t = -\infty$, the price is merely $B(e^{q_t})$, which is the discounted value of all future payoffs for an asset with the current payoff of (e^{q_t}), *when the asset is assumed to continue to pay off forever.* But the asset may not continue to pay off forever. When the first passage time of $q_t = \underline{q}_t$ is reached then the asset has zero value. Therefore, this equation shows that the mortality risk of the firm is capitalized into the price of the firm's shares. When the current payoff of the asset is (e^{q_t}), the discounted value of the loss of the remaining dividends is then $B\left[e^{(\beta-1)\left(q_t - \underline{q}_t\right)}\right](e^{q_t})$. It will be useful to refer to this additional feature as "*mortality risk*" since it reflects that the likely death of the value of the asset, and it should be distinguished from the risk inherent in the payoff of the asset, or consumption risk.[17]

The term β has a complicated interpretation. Note first that the exit time (when $q_t = \underline{q}_t$) is a random variable in this environment. Hence the discount factor associated with these exit dates is also random. The term $e^{(\beta-1)\left(q_t - \underline{q}_t\right)}$ in equation (20) is then the expected value of this random discount factor, when the current payoff is (e^{q_t}) while the exit barrier is $(e^{\underline{q}_t})$. With this in mind, β acts as a type of discount rate, but it is not a discount rate with respect to time. Instead, it is the discount factor with respect to relative *productivity.* Note also that in principle, it is possible that different assets (or firm shares) would have different values of β associated with each of them. Since it acts as a type of discount rate, it is best to consider the case in which $\beta < 0$.

Obviously the price of the asset in equation (20) is a function of the productivity of the asset (q_t), and the relative productivity $\left(q_t - \underline{q}_t\right)$. It will be shown below that the rate of return and the volatility of the price will depend on these features as well. That is, the productivity of the asset will help predict the price change of the asset.

There are other natural questions that arise for this economy. For example, one might inquire about the relationship between the rate of firm destruction (or exit) and the volatility of asset prices. Such a relationship is not easy or natural to study, since parameters that affect these outcomes do so in a complicated manner. That is, the values of β and $\underline{s}$ are complicated non-linear functions of the underlying parameters. However, one can conduct some superficial, or first order analysis. To do this, let us momentarily let equation (13) be an inverse measure of firm destruction, since it is the average length of time a firms exits. To simplify the analysis, let us assume that $\underline{s}$ is fixed. Then a change in the volatility of returns for each asset (σ_q) does not change the rate of exit in equation (13). The rate of exit would be influenced by $\kappa_s = \kappa_q - \kappa_Z$, which would influence the average (but not the standard deviation of the) growth rate. This in turn would influence the value of β. This would not affect the standard deviation of asset prices with high relative productivity $\left(q_t - \underline{q}_t\right)$, which is influenced by σ_q, but would influence the volatility of returns for low-productivity assets.

This pricing relationship can also yield some insight as to observed co-movement in asset prices and returns. Consider two different assets in this environment that have payoffs $(z = e^q)$

[17] Obviously this is analogous to default risk.

that are uncorrelated. Notice from equation (20) that the corresponding asset prices, and therefore returns may exhibit co-movement because they both have a common factor of $Z^{\beta-1} = e^{(\beta-1)\underline{q}}$.[18] In other words, as long as the productivity of the assets (q) is not too high, the prices and returns of the assets can exhibit co-movement even if the payoffs of the assets themselves are independent. This can explain some of the important findings of Fama and French [8], as well as Gomes, Kogan and Zhang [10].

Further analysis of this pricing equation, and the implications for rates of return, is certainly in order. But since the equations presented above are special cases of more general pricing expressions, it seems proper to develop the general asset pricing equation first. This will be presented in Section 5.

4.1 The Importance of Creative Destruction (or Non-random Exit)

It is essential to note the importance of Creative Destruction, or the role of endogenous firm exit, is to the pricing of an asset. Equation (20) shows that the price of the firm's assets is a function of the firm's productivity, and when this productivity reaches some lower bound (or absorbing barrier), the firm exits and the value of the asset is zero. Here it is vital to understand the importance of the endogeneity of this relationship. One could consider an alternative framework in which firms exited (or died) with some constant, time-invariant probability ϕ, irrespective of the level of productivity. Then, it is easy to show that the price of a share in the firm would be

$$(e^{q_t}) \left[r + \phi + \kappa_c \theta - \left(\frac{\sigma_c}{2} \right) \theta^2 - \kappa_q - \left(\frac{\sigma_q^2}{2} \right) \right]^{-1}. \tag{24}$$

Now contrast this expression with equation (20). In equation (24), the probability of exit lowers the value of the asset for *all* levels of productivity. However, the expected change in the price, as well as the variance of the price are the same for all levels of productivity. This is quite different from that in the model with creative destruction. This shows that not only is exit important, but in order to fully understand the behavior of asset prices it is critical to understand and capture the reasons for this exit. In a more comprehensive analysis of asset returns it would be imperative to investigate what factors can influence this exit decision.

4.2 Diversification

In this environment, there are three distinct sources of risk. First, there is the systemic risk associated with consumption. Shareholders cannot eliminate this risk, but in this environment where consumption follows a random walk, this does not change over time. Secondly, there is the idiosyncratic, or asset-specific risk associated with (e^{q_t}). It is possible to diversify one's portfolio to reduce this risk by holding many similar assets. In particular, if one were to hold a portfolio of assets with similar values of productivity (q_t), then one could certainly reduce this risk.

But next, and uniquely, there is the mortality risk that is *inherent in all assets* - but perhaps to differing degrees. Since this risk is inherent in all assets, it cannot be fully diversified away. In particular, consider a portfolio of assets with similar values of productivity (q_t). Then this type of diversification can reduce the mortality risk, but can never eliminate it.

[18] This operates through equation (23).

5 A More General Analysis of Asset Prices

The assets studied in the previous section have very well-defined characteristics that are specific to that environment. However, those assets are a special case of more general assets that will be characterized and studied next. In particular, within the context of the environment studied in Section 4, it is possible to price a multitude of assets, with a rich set of covariance properties, that have zero net supply in this economy. It will be of considerable interest to investigate how the properties of these assets will influence the prices and rates of return.[19]

It will be useful to preserve as much of the environment of Section 4 as possible, and to merely change some of the characteristics of the assets under study. With this in mind, it will be assumed that the stochastic process for consumption will be the same as that used in Section 4, and so similar to that of equation (2). However, it will be convenient to distinguish between consumption and the process for Z_t in this section, so consumption will be assumed to obey the following process

$$\frac{dc_t}{c_t} = \kappa_c dt + \sigma_c dW_c.$$

Note then that this implies that in the subsequent analysis, the risk-free rate is given by equation (17) where κ_Z and σ_Z are replaced by κ_c and σ_c respectively. Next, we will consider hypothetical assets that have payoffs of ($e^{q_t} = z_t$), where this process is given by equation (3). Additionally, a variety of processes will be considered for the exit barrier ($e^{\underline{q}_t}$). However, another useful assumption will be to let it follow the process used in Section 4, Therefore, the following assumptions will be made about these processes:

$$dq_t = \kappa_q dt + \sigma_q dW_q, \quad and \quad d\underline{q}_t = \kappa_Z dt + \sigma_Z dW_Z. \tag{25}$$

This is intended to embody the idea that there can be threats to a firm's existence (i.e. exit) other than merely innovators or competitors.[20] This will be explored more below. Of course, another case considered below is where $\kappa_Z = \sigma_Z = 0$.[21]

It will also be of interest to study the cases in which these processes are correlated. Therefore, it will be useful to employ the following notation:

$$\rho_1 = corr(W_c, W_q), \quad \rho_2 = corr(W_c, W_Z), \quad \rho_3 = corr(W_q, W_Z).$$

In the special case of the economy studied in Section 4, $\rho_1 = \rho_3 = 0$, while $\rho_2 = 1$. In much of the existing literature, the focus is usually on ρ_1, which is the correlation of the asset payoff with contemporaneous consumption growth. However, as will be shown below, these separate correlations will appear in the expressions for asset prices and returns. It will be important to highlight the distinction and the importance of each of these correlations, and how they each contribute to understanding excess returns of assets.

[19] That is, in this section the pricing of new assets will be conducted. These assets are similar to those present in Section 4, but not identical since they may have different volatility or correlation properties. It would seem possible to amend Section 3 and 4 to make assets with these different covariance properties appear endogenously, since nothing substantial in that section rested on the independence of these assets.

[20] As was indicated above, it would also be possible to present a simpler version of the model, along the lines of that presented in Luttmer [24] in which the firm exits or ceases operations when it fails to cover its fixed costs. One could then suppose that these fixed costs vary over time, and then $\underline{q}_t$ is the productivity necessary to cover the fixed costs.

[21] This environment is not inconsistent with that of Sections (3) and (4). Previously consumption of the asset holders was perfectly correlated with Z_t, whereas now these features are distinct. One might make this more palatable by assuming that there is some non-capital income on the right side of the budget constraint, and this income equals $\int_0^1 \pi(\tau) x(\tau) d\tau - c_t$.

It is appropriate to pause and consider the fact that $\underline{q}_t$ need not be constant, and think about the factors that might influence this process within the context of a more detailed model. Clearly, $\underline{q}_t$ acts as a type of threat to the existence of the benchmark asset under consideration. The higher is the growth rate (κ_Z), the greater is this threat, and the lower is likely to be the resulting price $\left(V\left(q_t, \underline{q}_t\right)\right)$. Similarly, variability in the process for $\underline{q}_t$ is likely to add to the volatility of the price.

First, note the fact that $\underline{q}_t$ is time-varying is consistent with the benchmark model described above. If the "outside option" of the manager were to vary over time, then so would $\underline{q}_t$ (see equation (23)).

But what sort of phenomenon or policies are likely to influence the process for, or level of $\underline{q}_t$? Any government policy that advantages new entrants, or innovation for new entrants, at the expense of incumbents, is likely to raise the level or growth rate of $\underline{q}_t$. Similarly, policies that reduce the innovation of outsiders will lower the level of $\underline{q}_t$. These are important considerations because that one would like to have some knowledge regarding the appropriate values for parameters such as $\rho_2, \rho_3, \kappa_Z, \sigma_Z$, as well as the factors that might influence these parameters, since these parameters will have an important impact on the value of the underlying asset. For example, to the extent that $\underline{q}_t$ reflects innovation by outsiders or potential new entrants, one would like to know how this is correlated with aggregate consumption (ρ_2) or with the payoff with the benchmark asset (ρ_3).

But additionally, there may be factors unrelated to the productivity of outside innovators that may affect the viability or mortality of the benchmark asset or firm, and hence could be embodied in the process for $\underline{q}_t$. For example, the sustainability of a firm may depend tenuously on a single key employee, and their future tenure with the firm would certainly affect the value of the firm through $\underline{q}_t$. Similarly, there could be other factors internal to the firm that do not directly impinge on contemporaneous productivity that could be important in determining its future value. Examples of this would be the makeup of the firm's debt, the status of licenses or patents, management expertise, intangible capital, or government regulation. Additionally, the behavior or status of foreign or domestic competitors, or even foreign government policies, can also pose a threat to the viability of a domestic firm. Lastly, exchange rate risk can also pose a problem for the existence of domestic firms. All of these factors could affect the current price of the asset, but may not directly influence current productivity (q_t).

In subsequent sections the actual decisions that give rise to the behavior of $\underline{q}_t$ will not be studied or modeled explicitly, since the goal here is merely to study the behavior of the price of the benchmark asset. Instead, the process described in equation (25) will be assumed.

5.1 The Asset Pricing Equation and the Survival Function

In this environment, where the consumption of shareholders follows the process described above, it is possible to price arbitrary assets, which may have zero net supply in the economy. To this end, consider pricing an arbitrary benchmark asset that has a continuous payoff (e^{q_t}), and the process for ($\underline{q}_t$) is as described above. Using same notation as above, let $V\left(q_t, \underline{q}_t\right)$ denote the price of the asset, measured in units of current consumption. An additional constraint is that $V\left(\underline{q}_t, \underline{q}_t\right) = 0$. It is shown in the appendix that the price can be characterized as follows.

Proposition 8 *The real price or value of the asset current payoff $z_t = e^{q_t}$, with exit threshold*

$e^{\underline{q}_t}$, *is given by the following:*[22]

$$V\left(q_t, \underline{q}_t\right) = B\left(e^{q_t}\right)\left[1 - e^{(\beta-1)\left(q_t - \underline{q}_t\right)}\right], \tag{26}$$

where

$$B = \left[r + \kappa_c\theta - \left(\frac{\sigma_c^2}{2}\right)\theta^2 - \kappa_q - \left(\frac{\sigma_q^2}{2}\right) + \sigma_c\sigma_q\rho_1\theta\right]^{-1}, \tag{27}$$

$$\beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{28}$$

where

$$\begin{aligned} a &= \left[\left(\frac{\sigma_q^2}{2}\right) - \left(\frac{\sigma_Z^2}{2}\right) - \left(\sigma_q\sigma_Z\rho_3\right)\right] \\ b &= \left[\kappa_q - \kappa_Z + \sigma_Z^2 - \left(\sigma_c\sigma_q\rho_1\right)\theta + \left(\sigma_c\sigma_Z\rho_2\right)\theta + \left(\sigma_q\sigma_Z\rho_3\right)\right] \\ c &= -\left[r + \kappa_c\theta - \left(\frac{\sigma_c^2}{2}\right)\left(\theta^2\right) - \kappa_Z + \left(\sigma_c\sigma_Z\rho_2\right)\theta + \left(\frac{\sigma_Z^2}{2}\right)\right]. \end{aligned} \tag{29}$$

Proof. See Appendix. ■

Corollary 9 *In the special instance in which $a = 0$ then it is the case that*

$$\beta = \frac{\left(r + \kappa_c\theta - \left(\frac{\sigma_c^2}{2}\right)\theta^2 - \kappa_Z + \left(\frac{\sigma_Z^2}{2}\right) + \left(\sigma_c\sigma_Z\rho_2\right)\theta\right)}{\kappa_q - \kappa_Z + \sigma_Z^2 - \left(\sigma_c\sigma_q\rho_1\right)\theta + \left(\sigma_c\sigma_Z\rho_2\right)\theta + \left(\sigma_q\sigma_Z\rho_3\right)}.$$

There are many unique cases to consider in the analysis below. However, in the interests of focusing the investigation, unless it is stated otherwise, the analysis will study the case in which $\beta < 0$.

5.1.1 Unique Features of the Pricing Equation

It is important to pause to consider some of the unique properties that are intrinsic to this pricing relationship. This pricing relationship has the potential to enhance our understanding of various asset pricing puzzles. For example, it may yield insight into why two seemingly very similar assets could have radically different values, or could behave quite differently.

First, note that equation (26) can be written as $V\left(q_t, \underline{q}_t\right) = B\left(e^{q_t}\right) S\left(q_t - \underline{q}_t\right)$, where

$$S\left(q_t - \underline{q}_t\right) = 1 - e^{(\beta-1)\left(q_t - \underline{q}_t\right)}. \tag{30}$$

This latter function has the form of a productivity-dependent *survival function*, and so it is related to the likelihood of the asset "surviving", or maintaining $q_t > \underline{q}_t$. But this is a unique survival function in that it involves not just the likelihood of surviving at all future dates, but then discounting those exit times back to the current date. The influence of this survival function will be shown to arise in the expected returns, which will be explored below.[23] While

[22] This formula, and analysis, is easily generalized to having multiple absorbing barriers ($\underline{q}$). This could be a situation in which there were multiple threats, or factors that contribute to the firm's existence.

[23] The term "Survival function" is indeed appropriate here since there are cyclical and low-frequency movements in the longevity of firms. A study by McKinsey (Hillenbrand, Kiewell et al, [25]) finds that the median age of firms in the S&P has fallen from 85 years in 2000 to 33 years in 2018. The average lifespan fell from 61 years in 1958 to 18 years in 2016. These are remarkable movements, and it would be even more incredible if these changes were not reflected in asset prices. The variation in the age of firms seem to be evidence not just of creative destruction, but also the increasing pace thereof.

the price is the product of these two separate functions, these functions are not completely divorced from each other, since some (but not all) parameters, such as κ_q and σ_q, will influence both of these factors.

Note that for the case of random exit, described in section 4.1, the survival function takes the form of $S = 1 - (\phi/B)$ (where B is described in equation (27)), and this is independent of both q_t and $\underline{q}_t$.

It is entirely conceivable that changes or variations in the price of the asset, as well as in the rates of return on the asset, can primarily reflect changes in the expected value of the survival function, rather than merely involving the current payoff (e^{q_t}) (although these terms will be correlated). For assets with relatively low values of productivity, this will certainly be the case. In other words, movements in asset prices that ostensibly cannot be explained by expected changes in the dividend must then be attributed to changes in factors that influence the survival function.[24] This is most easily seen in the case where q_t is constant, but where $\underline{q}_t$ exhibits variation.

One very important feature of this analysis is that like most such models, the optimization condition (18) that gives rise to the asset pricing equation is "locally linear", the ultimate solution for the price (through 26 or 30) is anything but linear. These non-linearities will play an important role in studying rates of return below.

There are many other unique properties of this pricing function. For example it is straightforward to verify that

$$\lim_{q_t \searrow \underline{q}_t} \left[\frac{\partial \ln \left(V \left(q_t, \underline{q}_t \right) \right)}{\partial q_t} \right] = 1 - \beta.$$

Therefore, for low levels of relative productivity $\left(q_t - \underline{q}_t \right)$, the response of the asset price to a change in productivity will depend on the value of β. It is entirely conceivable that different assets could have quite different values of β. For example, a low value of σ_q could certainly raise the value of $|\beta|$. But if the value of $|\beta|$ is high, then even assets with slight differences in productivity could have quite different prices.[25] Similarly, assets with nearly identical productivity could have quite different prices if they have quite different growth prospects (κ_q), since this would also imply a different value of β. This could help explain why assets with different growth properties could have different returns.

Another way to look at this same issue is to note that

$$\lim_{q_t \searrow \underline{q}_t} \left[\frac{\partial \ln \left(V \left(q_t, \underline{q}_t \right) \right)}{\partial \beta} \right] = +\infty.$$

This means that assets with low levels of relative productivity $\left(q_t - \underline{q}_t \right)$, a slight variation in β, perhaps produced by a difference in the underlying parameters in equation (28) such as κ_q or σ_q, could, in turn, produce dramatic differences in the price of the asset. In other words, it is conceivable that two assets with seemingly similar levels of productivity could have quite different values.

Next, consider the unique correlation terms in the asset pricing equation (26). The usual correlation of consumption with the asset payoff (i.e. ρ_1) is present in equation (27). However, the pricing equation is affected by ρ_1, ρ_2, and ρ_3 through β in equation (28). To understand this

[24] One additional feature to note is that a survival function of this type is entirely forward-looking.

[25] It is not difficult to produce examples with reasonable parameter values which imply that β is in the range of -50 - -20.

relationship, note that in equation (26), for a fixed level of $\left(q_t - \underline{q}_t\right)$, this price is decreasing in the value of β. Next, consider the impact of ρ_1 on the price through the effect on β for the case in which $b > 0$. As is customary, a higher value of ρ_1 means that the asset payoff is positively correlated with consumption growth. However, here a higher value of productivity (q_t) implies that the asset will also have a *longer lifespan*, and therefore a higher future value. A higher value of ρ_1 will lower the value of b and therefore raise the value of $|\beta|$. The upshot of this is that an increase in the level of productivity (q_t) will extend the asset's lifetime and will be coincident with higher consumption growth. In other words, the asset will exhibit a higher price, again reflecting the extension of the asset lifespan, when consumption growth is high. This type of asset does not provide insurance against consumption risk but instead *exacerbates* it. Because the asset does not provide insurance against consumption risk, this results in a lower price for the asset. In other words, a higher value of ρ_1 will make the capital gain portion of the asset return exhibit a higher correlation with consumption growth. Furthermore, this effect is magnified, the closer is the level of productivity to the threshold level $\left(\underline{q}_t\right)$. Of course, the opposite is the case if $\rho_1 < 0$.

It is important to contrast this result with the traditional effect that is present in equation (27). Through equation (27), if $\rho_1 > 0$ then the dividend payoff is positively correlated with consumption growth and therefore does not provide insurance against consumption risk. But the effect of ρ_1 in equation (29) acts through an entirely different mechanism. Here a change in the dividend of the asset changes the price of the asset (or capital gain) through altering the expected lifespan of the asset. This effect will influence the correlation of the capital gain portion of the asset with consumption growth.

Similarly, as can be seen in equation (29), the effect of ρ_2 on the asset price is the opposite sign as that of ρ_1, in the same equation. To facilitate the understanding of this effect, consider, as a benchmark, a situation where $\underline{q}_t$ represents some outside factor, perhaps operating through the productivity of an outside competitor, or predatory firm. The higher is the value $\underline{q}_t$, the more precarious is the lifespan of the benchmark asset. In other words, a higher value for $\underline{q}_t$ poses a threat to the benchmark firm. Next consider the case in which $\rho_2 > 0$, so that the threat that $\underline{q}_t$ poses to the benchmark firm, is positively correlated with consumption growth. This will mean that the capital gain portion of the asset is likely to be *negatively correlated* with consumption growth. But this will *enhance* the insurance that the asset will provide against consumption risk. Because of this added insurance feature, the asset will have a higher value, and this shows up as a *higher* price for the asset.

Finally, consider the unique features introduced by the presence of ρ_3. If $\rho_3 > 0$ this implies that the dividend payoff is positively correlated with the outside threat posed by $\underline{q}_t$. This will reduce the correlation between the dividend payoff and the capital gain portion of the asset and perhaps even make this correlation negative. In this case, the asset itself has some *built-in insurance* in its return. This feature will tend to raise the value of b in equation (29) which in turn will lower β and raise the price of the asset. That is, the asset will have a higher value because it has this extra insurance feature. It is important to note that the parameter θ is not relevant here since this attribute is not material to consumption risk.

This last example is illustrative for another reason. Consider the case in which the dividend itself has no risk associated with it $(\sigma_q = 0)$. Then even though the dividend payoff is devoid of risk and the risk-free rate is constant, the asset itself can certainly be risky since there is risk introduced through σ_Z via $\underline{q}_t$, which shows up in the survival function.

On a closely related point, the asset pricing equation (26) not only highlights the threat posed by $\underline{q}_t$ but it also reflects the uncertainty associated with this threat. The presence of terms involving σ_Z in the formula for β shows that this uncertainty is important. This is true even if all the correlations are zero. As an interesting illustration of this feature, consider the instance where all the correlations are zero and there is an increase in the value of σ_Z. Through raising the value of σ_Z in equation (29) this can lower the value of β and for a fixed level of $\left(q_t - \underline{q}_t\right)$ this will *raise* the price of the benchmark asset.

Next, note that there are other subtle issues inherent in this asset pricing equation. The asset price (26) can explicitly reflect features that may seem entirely extrinsic to the asset, or at least to its current payoff (q_t). To see this, again suppose that the behavior of $\underline{q}_t$ acts as a threat to the benchmark firm or asset, and that perhaps this hazard is posed by some competitor firm. Now an increase in $\underline{q}_t$ will directly reduce the price of the benchmark asset, as is evident from equation (26). Through this mechanism, one would naturally expect the price $V\left(q_t, \underline{q}_t\right)$ and $\underline{q}_t$ to be negatively related. But now note that this effect may be partially offset through the effect of ρ_2 and ρ_3 on β. If, for example, an increase in $\underline{q}_t$ is positively correlated with the dividend q_t, then this would alter the correlation between $V\left(q_t, \underline{q}_t\right)$ and $\underline{q}_t$. In other words, *the price of the benchmark asset will inherit some of the properties of its competitor firm that are not directly related to the benchmark firm's productivity.* In a sense, the benchmark asset may have some built-in or intrinsic diversification inherent in its structure, in that it reflects the behavior of competitor firms. *The asset price will then reflect features that are extrinsic to the firm's dividend payoff, but intrinsic to the firm's lifespan.*

As indicated earlier, there is no reason to restrict this analysis solely to that of assets that represent payoffs by firms, as it could equally be used to study bonds as well, including government-issued securities. Consider, for example, an infinitely-lived bond that has a well-defined (i.e. certain) stream of payments. In this case, any movements in the price of this security would only reflect changes in the survival function alone. For these particular assets, *pricing functions are indeed survival functions.*

This analysis can also yield some insight into the value puzzle, and in particular why high growth firms may exhibit lower than average returns. Note that if, in equation (26) that there were no survival function (or $\underline{q}_t = -\infty$), then κ_q would affect all assets in the same manner, and so returns of different assets would be influenced in the same manner. However, with the creative destruction feature, and the survival function, the growth rate of the asset (κ_q) affects the asset price and return in a complicated, non-linear manner. It is certainly conceivable that an increase in this growth rate can lower expected returns.

There is another important implication of this model that may warrant further exploration. In a typical model of asset pricing one might ascribe different valuations of an asset by various agents to different opinions or expectations about the future expected payoff of the asset (e^{q_t}). But in this environment similar to the one studied here, one could imagine where different agents could have complete agreement about the future expected payoff of the asset, but nevertheless have different valuations. This is not a sign of irrationality. Instead, it could be that the agents have different expectations about the future behavior of the outside factor ($\underline{q}_t$).

The conclusion from all this is that the price of an asset should reflect all features that are intrinsic to the future asset payoffs. But furthermore, the asset should also reflect all qualities associated with factors that might pose a negative or productive influence over the asset in the future. While these factors may be naturally difficult to quantify, it seems apparent that understanding this relationship is the obvious avenue necessary to understanding the behavior of asset prices.

A Digression on Prices and Growth Rates Typically in such infinite-horizon models, assets for which the growth rate of its dividend is significantly greater than the risk-free rate will not have a well-defined price. One can see this in equation (21) above. Here if the rate of growth of the dividend ($\kappa_q + \left(\frac{\sigma_q^2}{2}\right)$) is greater than the risk-free rate ($r + \kappa_c\theta - \left(\frac{\sigma_c^2}{2}\right)\theta^2$), then the asset price would not seem to be well defined. However, in a model of creative destruction, of the sort studied here, this need not be a problem. It is conceivable that the asset price can still be finite even with such an elevated growth rate, if the growth rate of the exit threat ($\underline{q}_t$) is sufficiently high. To see this in the simplest form, consider the simple case in which all of the correlations (ρ_i) are zero. Next, consider the pricing equation (26) for the case in which $\beta > 1$, so that the survival function is now negative. Now suppose that the growth rate of the asset payoff is greater than the risk-free rate so that $B < 0$. However, the price itself is still well-defined, because it is the product of two negative terms. Furthermore, the pricing function then has the reasonable property that $dV\left(q_t, \underline{q}_t\right)/dq_t > 0$. Under the assumption that the correlations are zero, a sufficient condition for this latter condition to hold is that

$$\kappa_q + \left(\frac{\sigma_q^2}{2}\right) < \left[\kappa_Z + \left(\frac{\sigma_Z^2}{2}\right)\right] - \sigma_Z^2.$$

The left side of this equation is the growth rate in q_t, while the term in brackets on the right is the growth rate of $\underline{q}_t$. Assets with these properties can have elevated growth rates of the payoff (e^{q_t}), and this is fine as long as the growth rate of the factor ($\underline{q}_t$) that will ultimately will destroy them is even higher. These assets are like meteorites that will ultimately crash and burn.

This is an unusual case. As was stated above, most of the remaining analysis will focus on the case in which $\beta < 0$.

5.2 Characterizing Rates of Return

The (instantaneous) rate of return on any asset then consists of the dividend-price ratio, plus the capital gain, or

$$\frac{(e^{q_t})}{V\left(q_t, \underline{q}_t\right)} + \frac{dV\left(q_t, \underline{q}_t\right)/dt}{V\left(q_t, \underline{q}_t\right)}. \tag{31}$$

Using the pricing equation above the first term can be re-written as

$$\frac{\left[r + \kappa_c\theta - \left(\frac{\sigma_c^2}{2}\right)\theta^2 - \kappa_q - \left(\frac{\sigma_q^2}{2}\right) + \left(\sigma_c\sigma_q\rho_1\right)\theta\right]}{\left[1 - e^{(\beta-1)\left(q_t - \underline{q}\right)}\right]}. \tag{32}$$

Clearly this expression $\rightarrow +\infty$, as $q_t \rightarrow \underline{q}_t$. Then the remaining term in equation (31) is again given by:

$$\begin{aligned}
\frac{dV\left(q_t, \underline{q}_t\right)}{V\left(q_t, \underline{q}_t\right)} = & \left[\left[\frac{1 - \beta e^{\left(q-\underline{q}\right)(\beta-1)}}{1 - e^{\left(q-\underline{q}\right)(\beta-1)}}\right]\kappa_q + \left[\frac{1 - \beta^2 e^{\left(q-\underline{q}\right)(\beta-1)}}{1 - e^{\left(q-\underline{q}\right)(\beta-1)}}\right]\frac{\sigma_q^2}{2}\right] dt \\
& + \left[\left[\frac{(\beta-1)\, e^{\left(q-\underline{q}\right)(\beta-1)}}{1 - e^{\left(q-\underline{q}\right)(\beta-1)}}\right]\kappa_Z + \left[\frac{(\beta-1)^2 e^{\left(q-\underline{q}\right)(\beta-1)}}{1 - e^{\left(q-\underline{q}\right)(\beta-1)}}\right]\frac{\sigma_Z^2}{2}\right] dt \\
& + \left[\left[\frac{1 - \beta e^{\left(q-\underline{q}\right)(\beta-1)}}{1 - e^{\left(q-\underline{q}\right)(\beta-1)}}\right]\sigma_q\right] dW_q + \left[\left[\frac{(\beta-1)\, e^{\left(q-\underline{q}\right)(\beta-1)}}{1 - e^{\left(q-\underline{q}\right)(\beta-1)}}\right]\sigma_Z\right] dW_Z \\
& + \left[\frac{\sigma_Z\sigma_q\rho_3}{\left[1 - e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2}\right]\left[1 - \beta e^{\left(q-\underline{q}\right)(\beta-1)}\right]\left[(\beta-1)\, e^{\left(q-\underline{q}\right)(\beta-1)}\right] dt
\end{aligned} \tag{33}$$

The first line of this expression derives from the drift in the price of the asset's dividend payoff alone, while the second term is the drift in the termination barrier $\left(\underline{q}_t\right)$. The third terms are the standard deviation of this payoff (q_t) as well as the standard deviation of $\left(\underline{q}_t\right)$. The last expression is the drift that derives from the joint correlation of the termination barrier and the dividend payoff. Obviously, this depends on the correlation ρ_3. Since $(\beta - 1) < 0$, this term will be positive if $\rho_3 < 0$.

The expected value of equation (33) will be either $\pm\infty$ as $q_t \to \underline{q}_t$ because the denominators in the expressions approach zero. The (instantaneous) variance of the change in the asset price is characterized as follows:

$$var\left(\frac{dV(q_t)}{V(q_t)}\right) = \left[\frac{1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}}{1-e^{\left(q-\underline{q}\right)(\beta-1)}}\right]^2 \sigma_q^2 + \left[\frac{(\beta-1)\,e^{\left(q-\underline{q}\right)(\beta-1)}}{1-e^{\left(q-\underline{q}\right)(\beta-1)}}\right]^2 \sigma_Z^2 + \frac{2\left[(\beta-1)\,e^{\left(q-\underline{q}\right)(\beta-1)}\right]\left[1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}\right](\sigma_Z\sigma_q)\,\rho_3}{\left[1-e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2}. \tag{34}$$

Note that for large values of relative productivity $\left(q_t - \underline{q}_t\right)$, this last expression converges to σ_q^2. But as $\left(q_t - \underline{q}_t\right) \searrow 0$ this expression $\nearrow +\infty$ because the denominator converges to zero.[26]

This property should be linked back to the discussion immediately following Proposition 8. For large values of $\left(q_t - \underline{q}_t\right)$ the mortality risk in the price is essentially zero, and so equation (34) reflects payoff or dividend risk σ_q^2. But as the mortality risk becomes more pronounced and so the remaining terms in equation (34) become more important. In fact, they can become so important that they overwhelm the effect of σ_q^2 and drive the variance to $+\infty$.

The presence in equation (34) of the term involving ρ_3 means that the variance can be increased or reduced, depending on the size of this term, which again represents whether the capital gain and dividend payoff are negatively or positively correlated.

The value of β is important here. In what follows the value of β is also a function of the parameters: $\sigma_c, \sigma_q, \sigma_Z, \rho_1, \rho_2, \rho_3, \theta$. That is, the value of β is *specific to the asset*, but is independent of the particular value of productivity q_t.

6 Analysis of the Properties of Asset Prices and Returns

The asset pricing equation developed in the previous section, together with the rates of return, has many unique characteristics. To develop the properties of these, it is illuminating to proceed in stages, and study several cases in order, beginning with the simplest case. This will illustrate the distinctive features that influence asset prices and returns, and how these features may be related to the creative destruction process.

6.1 The Standard Case of $\underline{q} = -\infty$

Since $\underline{q}$ plays such a vital role in this analysis it will be best to first consider the primitive case where it is constant, and then to focus on the implications that arrive when $\underline{q}$ varies over time.

[26] Zhang [28] also studies a model in which the volatility of returns can depend on firm size. However, his model is much more complicated than that studied here. Among other things, his technology employs capital accumulation along with asymmetric adjustment costs.

In the case in which $\underline{q}= -\infty$, equation (33) then becomes

$$\frac{dV(q_t)}{V(q_t)} = \left[\kappa_q + \left(\frac{\sigma_q^2}{2}\right)\right] dt + \sigma_q dW_q.$$

The standard deviation of the price change is σ_q. The rate of return on the asset is then the sum of the expected price change and the dividend price ratio, or

$$\left[r + \kappa_c\theta - \left(\frac{\sigma_c^2}{2}\right)\theta^2 + (\sigma_c\sigma_q\rho_1)\,\theta\right].$$

Since the risk-free rate is given (the analogous version of) equation (17), the excess return on the over the risk-free rate is then $(\sigma_c\sigma_q\rho_1)\,\theta$. The Sharpe ratio is then given by $(\sigma_c\rho_1)\,\theta$.

It is important to note here in this case that since the asset payoff follows a random walk, the dividend and the capital gain portion of the asset return are perfectly correlated. This is a feature that is present in many models of asset pricing that seems to go unmentioned. It will be shown below that the asset returns can be much more interesting when this linkage is broken.

6.2 Analysis of the Case of a Time-Invariant $\underline{q} > -\infty$

Next, consider the case in which $\underline{q}$ is constant and finite. Equation (33) is then written as follows:

$$\begin{aligned}\frac{dV(q_t)}{V(q_t)} &= \left[\left[\frac{1-\beta e^{(q-\underline{q})(\beta-1)}}{1-e^{(q-\underline{q})(\beta-1)}}\right]\kappa_q + \left[\frac{1-\beta^2 e^{(q-\underline{q})(\beta-1)}}{1-e^{(q-\underline{q})(\beta-1)}}\right]\left(\frac{\sigma_q^2}{2}\right)\right] dt \\ &\quad + \left[\left[\frac{1-\beta e^{(q-\underline{q})(\beta-1)}}{1-e^{(q-\underline{q})(\beta-1)}}\right]\sigma_q\right] dW_q\end{aligned} \tag{35}$$

The standard deviation of the price change is

$$\left[\frac{1-\beta e^{(q-\underline{q})(\beta-1)}}{1-e^{(q-\underline{q})(\beta-1)}}\right]\sigma_q. \tag{36}$$

Again, this measure of volatility $\nearrow +\infty$ as $\left(q-\underline{q}\right) \searrow 0$ if $\beta < 0$.

The rate of return on the asset is then the sum of the expected price change and the dividend price ratio, or

$$\frac{\left[r + \kappa_c\theta - \left(\frac{\sigma_c^2}{2}\right)\theta^2 + (\sigma_c\sigma_q\rho_1)\,\theta\right]}{\left[1-e^{(\beta-1)\left(q_t-\underline{q}\right)}\right]} + \left[\left[\frac{\beta e^{(q-\underline{q})(\beta-1)}}{1-e^{(q-\underline{q})(\beta-1)}}\right]\kappa_q + \frac{\left[\beta^2 e^{(q-\underline{q})(\beta-1)}\right]}{\left[1-e^{(q-\underline{q})(\beta-1)}\right]}\left(\frac{\sigma_q^2}{2}\right)\right].$$

The excess return over the risk-free rate is then

$$\left[\frac{1-\beta e^{(q-\underline{q})(\beta-1)}}{1-e^{(q-\underline{q})(\beta-1)}}\right](\sigma_c\sigma_q\rho_1)\,\theta. \tag{37}$$

As $\underline{q} \to -\infty$, this risk premium $\to [(\sigma_c\sigma_q\rho_1)\,\theta]$. But as $\left(q-\underline{q}\right) \searrow 0$ this expression approaches $+\infty$ if $\rho_1 > 0$, and $-\infty$ if $\rho_1 < 0$.

It is easily seen that the Sharpe ratio for this asset is still $(\sigma_c\rho_1)\,\theta$. The explanation for this is that as productivity (q_t) changes, the expected rate of return and the standard deviation of this return are affected in exactly the same proportion.

Also, note that the term in square brackets in equations (35) and (37) is the hazard function that is derived from taking the derivative of the logarithm of the survival function in equation (30). The hazard function acts as a scaling factor in determining the magnitude of the influence the correlation (ρ_1) has on the excess rate of return. This hazard function has some important properties that will be described in more detail below. Of course, this mortality risk is dependent on the value of q_t for the firm. Another way to measure this excess risk is by the size of β. Notice that in equation (37) the closer is β to unity, the less sensitive is the risk premium to the level of productivity q_t. However, if $\beta < 0$, and then the size of $(\beta - 1)$ measures the sensitivity of the risk premium to a change in productivity q_t.

As discussed above, when $\underline{q} = -\infty$, the dividend and the capital gain portion of the asset return are perfectly correlated. This resulted in a constant risk premium and a time-invariant variance for the change in the asset price. However, here when $\underline{q}$ is finite, this is not the case. In this instance, the dividend portion of the return is proportional to the asset payoff (e^q), but the capital gains portion is not. In fact, equation (37) shows that this is a complicated non-linear function of the payoff.

6.2.1 Quantitative Analysis

It will be useful to proceed beyond an analytical exploration of the model, to gain an understanding of the quantitative impact that various parameters can assert over asset prices and returns in this economy. Furthermore, it is also of interest to know what fraction of the firms in the model may exhibit any unusual behavior. It is possible to use this economy as a vehicle to study the behavior of other asset prices or returns, even if the underlying asset does not play an important role in the economy.

Further characterization of the magnitude of these effects is in order and is best conducted by viewing a few parameterized examples. To do this, let us adopt the following parameterization for the benchmark economy:

$$\theta = 5, \rho_i = 0, \kappa_c = .02, \sigma_c = .01, \kappa_q = .01, \sigma_q = .05, h = 2.0.$$

The value of r is chosen to ensure that, given the other parameters, the risk-free rate is 5%.[27] The value of θ is modest when compared with much of the literature on the risk premium, but it will evident in the subsequent analysis how alterative values would affect the outcomes.[28] The parameters characterizing consumption mimic the behavior in the US. The parameters for the asset generate the necessary larger volatility for returns than for the consumption, while simultaneously still generating a finite asset price. For these experiments, the focus will be on the expected change in the asset price, which is given by the first part of equation (33), as well as the instantaneous standard deviation of the price change, which is given by the second part of this equation.

It is important to note that, the impact of a change in consumption on asset prices is the same for all assets. Because of the random-walk nature of consumption, a shock to consumption would have no impact on the asset prices, irrespective of the level of productivity. However, because each asset may have different values for some parameters (i.e. κ_q and σ_q), and these parameters interact with the parameters for the consumption process (κ_c and σ_c) through equation (28), the consumption process will then influence the asset-specific value for β.

[27] This value is a little higher than would be typically chosen. However, this ensures that the price is well-defined when considering various other values for the parameters of the model, such as kappa or theta.

[28] This specific value for θ is not as high as that employed in some of the finance literature. A higher value will then raise the magnitude of the effect of ρ_1 and ρ_2 in the expressions for the rates of return calculated below. However, the message of the analysis is that even with this high value for θ can be overwhelmed by other important features of the asset price.

Figure 1 shows the effect on the expected instantaneous change in the asset price (equation (35)), for different values of σ_q. In this illustration, the horizontal axis is the point in the distribution of firms.[29] In this case, increasing the volatility of the firm-specific shocks lowers the expected change in the asset price, for low levels of firm productivity. This effect operates directly through the change in the value of σ_q, and also indirectly through the change in the value of β. Since the chosen values of σ_q do not seem unreasonable, this figure shows that there are a substantial fraction of assets for which the expected change in price is clearly a non-trivial function of the level of productivity.

Figure 2 shows the effect on the standard deviation of the change in the asset price, for different values of σ_q. It is not surprising that this effect can be more pronounced for larger values of σ_q. More volatility in the innovations to payoffs yields more volatility in prices. Once again, this effect operates directly through the change in the value of σ_q, and also indirectly through the change in the value of β. It is important to recognize the magnitudes of these effects, as characterized by the scale of the vertical axis. While the standard deviation at the extreme right tail would approach σ_q, the standard deviation at the opposite end eventually approaches $+\infty$. Even firms at the median of the distribution have a standard deviation of price change that is several orders of magnitude greater than that of the high productivity firms. For higher levels of productivity (q_t) the asset will primarily reflect risk directly associated with the dividend payoff. However, for low levels of productivity (q_t), this risk will be overwhelmed by the risk associated with asset mortality or exit, inherent in the survival function. Viewed from this perspective, one might wonder why measures of volatility of actual securities are not much greater.

Figure 3 shows the effect on the expected instantaneous change in the asset price, for different values of risk aversion (θ), holding the other parameters constant. For this economy, raising this parameter actually *lowers* the expected price change. This effect operates primarily through changing the risk-free rate in the economy, which in turn influences the parameter β.

It should be emphasized that the extraordinary volatility shown in the last few figures is not the result of any unusual behavior in the risk-free rate. In this instance, as was shown in equation (16), the risk-free rate is time-invariant in this model. This suggests that the pursuit of explanations for how consumption risk could affect the discount rate for assets may have been a futile quest. Similarly, this analysis suggests that focusing on risk aversion (θ), or other features of preferences to explain the observed volatility in asset prices and returns, may also be fruitless. This illustration shows that increasing the level of risk aversion may actually reduce the volatility of prices. But more importantly, this figure shows that a change in the productivity of the firm can have a significantly greater effect on volatility than increasing the level of risk aversion.

Figure 4 illustrates the effect on the standard deviation of the change in the asset price, for different values of risk aversion (θ). Higher values of this parameter seem to *lower* the standard deviation of the change in the price of the asset. Once again, the reason is that a change in θ will influence the resulting value of β.

It is appropriate to pause and reflect on the high volatility of prices and returns in this environment. An outside observer, who was not aware of the necessity of considering the importance of firm mortality or exit (or where $\underline{q}= -\infty$.), might observe the high volatility of the asset prices, and the high sensitivity of some of these asset prices to changes in the dividend. This observer might then be tempted to attribute this behavior up to some sort of irrationality on the part of shareholders. This is obviously wrong.

It should be noted that for this economy if $\rho_1 = 0$, if one were to change the value of θ but

[29] In other words, the point 0.20 denotes a firm with productivity that is greater than 20% of all other existing firms.

to also change the value of r so that the risk-free rate was unchanged, then this would have no impact on the value of β, and hence not alter the effects in equation (33). However, if $\rho_1 \neq 0$, this would not be the case, and so the value of β would be affected.

The goal of this analysis is not to match up the distribution of firms to that of any actual economy, but instead to see how the mechanisms or features described here can influence asset prices generally. That is, the goal is to study asset prices generally, and not just those specific to the model economy described above. For this reason, in the illustrations presented below, the horizontal axis will not feature the percentage in the distribution of firms, but instead the relative productivity $\left(q_t - \underline{q}\right)$ of the firm. This will permit an assessment of how the firm-level productivity can influence asset returns, even for firms that may not yet exist within this economy.

6.2.2 A New View of the Effect of the Correlation With Consumption

Since the correlation of the asset payoff with consumption occupies so much importance in the existing literature, it is important to see how significant a role this correlation plays in this model.

It is essential to note where the correlation of the payoff with consumption (ρ_1) appears in equation (26). As was indicated earlier, this correlation appears in equation (21) in determining the value of B. *A change in ρ_1, operating only through B affects all assets in the same manner, irrespective of their level of productivity q_t.* However, here the parameter ρ_1 also influences the asset price through the value of β. *This effect will be different, depending on the firm's relative productivity.*[30] A high value of ρ_1 means the payoff of the asset is highly correlated with consumption growth and therefore does not provide much insurance against consumption risk. In particular, the firm is unlikely to shut down when consumption is high but has a higher likelihood of this happening when consumption is low. Hence the capital gains portion of the return actually does not provide insurance against fluctuations in consumption. This is why the price $\left(V\left(q_t, \underline{q}\right)\right)$ will be low.

On the other hand, if $\rho_1 \approx -1$, then the payoff of the security is *negatively* correlated with consumption growth. This results in a larger value of $|\beta|$. This will tend to raise the price of the asset, holding the value of $q_t - \underline{q}$ constant. The reason for this is that the firm is less likely to cease operating when consumption falls. This means that it provides some insurance against consumption risk. This makes the asset more valuable and raises its price or value $\left(V\left(q_t, \underline{q}\right)\right)$.

Figure 5 shows the expected instantaneous change in the asset price, for different values of ρ_1, as a function of relative productivity $\left(q_t - \underline{q}\right)$. This is interesting as the effect here acts solely through the change in the value of β. Reducing the value of ρ_1 lowers the value of β (but raises $|\beta|$), and thereby makes the price more sensitive to changes in productivity (q_t). In most models of asset pricing this correlation coefficient would not affect the expected percentage change in the asset price.

In summary, it appears that there are several types of "risk" in this model. First, there is the usual risk of the asset having its payoff correlated with consumption. The higher is this correlation, the less insurance the asset provides against consumption risk.

Additionally, a unique feature of this model is that there is an additional risk, which might be termed *termination or mortality risk*, (or, alternatively, the *creative destruction component*) and this can be either positively or negatively correlated with consumption. This is the risk that the firm will cease operations, and therefore cease paying future dividends altogether. Since this event may occur when consumption growth is relatively high or low, there is the possibility

[30] This is relevant because it may help explain the findings of Fama and French [8] that firms of similar size, but in different industries, may exhibit related movements in asset returns.

that this contributes to, or reduces consumption risk. Of course, if this correlation is negative, then there is a lower likelihood that the firm will cease operation when consumption is low, and therefore this particular firm will provide some insurance against consumption risk.

Another way to state this is to note the following:

$$\frac{\partial^2 \ln V\left(q_t, \underline{q}_t\right)}{\partial B \partial q_t} = 0. \tag{38}$$

This implies that through the traditional channel the percentage change of the price of a change in productivity is the same, irrespective of the value of q_t.

Next, note that

$$\frac{\partial^2 \ln V\left(q_t, \underline{q}_t\right)}{\partial \beta \partial q_t} = \frac{-\left[1 - e^{(\beta-1)\left(q_t - \underline{q}_t\right)}\right] - \left(q_t - \underline{q}_t\right)(\beta - 1)\, e^{(\beta-1)\left(q_t - \underline{q}_t\right)}}{\left[1 - e^{(\beta-1)\left(q_t - \underline{q}_t\right)}\right]^2}. \tag{39}$$

It is easily seen that this expression $\to -\infty$ as $\left(q_t - \underline{q}_t\right) \searrow 0$. Through this new channel, the effect of a change in productivity is infinitely large (but negative) as $q_t \searrow \underline{q}_t$.

Yet another way to state this is as follows. The effect of the value of ρ_1 operating through B influences the *level of the price*. In contrast, the effect of the value of ρ_1, operating through β, influences the *magnitude of the change in price* in reaction to a change in dividend (q). The former effect should not affect the volatility of the asset, but the latter effect will. The former effect will affect all assets in the same manner. The latter will have a more pronounced effect on low-productivity assets, with this effect being infinitely large as $\left(q_t - \underline{q}_t\right) \to 0$.

6.2.3 The Risk or Equity Premium for the Case of Constant $\underline{q}$

As was indicated above, in the environment in which $\underline{q}$ is constant but finite, the risk premium takes the following form:

$$RP(q) = \left[\frac{1 - \beta e^{\left(q - \underline{q}\right)(\beta-1)}}{1 - e^{\left(q - \underline{q}\right)(\beta-1)}}\right] \left[\sigma_q \sigma_c \rho_1 \theta\right] \tag{40}$$

Once again, this premium is illustrated in figure 6, for two different values of ρ_1. This expression makes it clear that the risk premium for each asset is a non-linear function of the firm's underlying level of productivity if $\rho_1 \neq 0$. If $\rho_1 > 0$, the risk premium ranges from $+\infty$ to $[\sigma_q \sigma_c \rho_1 \theta]$, as q ranges from $\underline{q}$ to $+\infty$. If one were to observe different risk premia for different assets, it would be natural to conclude that this was attributable to different correlations (ρ_1) of variability (σ_q) in the returns for these assets. This analysis shows that this need not be the case, and that the risk premium on an asset can become arbitrarily high, even though these parameters have not changed. In fact, as the figure illustrates, it is possible to have two assets that are identical in their underlying parameters, but whose relative productivity $(q_t - \underline{q})$ is slightly different, and consequently these assets can have quite different risk premia.

Additionally, because of the convex nature of figures such as 1-4, it can certainly be the case that the risk premium of a collection or portfolio of assets can be substantially larger than that of an "average" firm within the collection.

6.2.4 Consumption Risk and the Risk-Premium: The Effect Through β

The effect that consumption risk can have on the risk premium in this environment can be rather complex. There is the obvious impact shown on the right side of equation (40). But then there is the impact that consumption risk, as well as other factors have through the value of β. For high levels of relative productivity $\left(q-\underline{q}\right)$ the effect of a change in β in this expression is trivial. On the other hand, consider what happens for low levels of productivity. To see this note that equation (40) can be used to show that[31]

$$\lim_{(q\searrow\underline{q})}\left[\frac{\partial RP(q)}{\partial\beta}\right]=\lim_{(q\searrow\underline{q})}\left[\left(\frac{-e^{\left(q-\underline{q}\right)(\beta-1)}}{1-e^{\left(q-\underline{q}\right)(\beta-1)}}\right)\sigma_q\sigma_c\rho_1\theta\right]=\begin{cases}+\infty \text{ if } \rho_1\theta<0\\ -\infty \text{ if } \rho_1\theta>0\end{cases}. \quad (41)$$

This illustrates that a change in any parameter that results in variation in the value of β will then have a large impact on the value of the risk premium, for assets at the low end of the productivity spectrum. Furthermore, while a change in the value of ρ_1 or σ_c has the usual impact on the risk premium on the right side of equation (40), these parameters are also embedded in the determination of β. As this last expression shows, a change in the behavior of consumption risk can have a substantial impact on the behavior of low productivity assets.

6.3 The Case of an Time-Varying $\underline{q}_t$

Next, it is important to study the behavior of hypothetical assets that may have a time-varying exit barrier $\underline{q}_t$. Once again, one should consider q_t to be the date-t payoff of the benchmark asset, while $\underline{q}_t$ is the threshold value of the payoff which will trigger the shutdown of the benchmark firm. Within the context of related models or actual economies, one could imagine a multitude of factors that could influence this barrier, and which could make it uncertain. Many of these factors would be outside the control of the shareholders, or even the employees of the benchmark firm with payoff q_t. One could consider that R&D efforts of competing firms, or external individuals who are not yet engaged in production could influence $\underline{q}_t$ because these efforts will influence the likelihood that there could be an innovation that poses a threat to asset q_t.[32] Similarly, there could be foreign firms that could be engaged in discovering or developing products that could pose a threat to the benchmark firm. On the other hand, it is possible that the benchmark firm under study (i.e. whose shares are being priced) could be engaged in R&D efforts that would make it more productive relative to its competitors, and these efforts would then *lower* the value of $\underline{q}_t$. In an economy in which there was a firm that used intermediate goods in production, innovation could occur in some critical supply chain or some related complementary good which could reduce the value of $\underline{q}_t$. On the other hand, the innovation could be in a commodity that is a close substitute, and this would raise the value of $\underline{q}_t$. In short, there could be many reasons why $\underline{q}_t$ could be varying over time, and depend on a multitude of factors. It is certainly conceivable that the value of $\underline{q}_t$ could change over time in a somewhat unpredictable manner, and this risk will certainly influence the properties of the price of the asset under study (q_t).

Then there is the issue, explored below, of how q_t and $\underline{q}_t$ might be correlated. If external factors influenced the productivity of the benchmark firm (q_t) as well as competitors in the same manner, then one might expect these variables to be positively correlated. On the other hand, perhaps a natural reason why q_t and $\underline{q}_t$ would be negatively correlated is that a low level of productivity (q_t) would signal to predatory, competing firms that the incumbent firm was vulnerable, and this would invite further innovation that raises the level of $\underline{q}_t$.[33] From a positive perspective, some forms of R&D spending might enhance one at the expense of the

[31] This derivative is not as trivial as it appears.

[32] This is where the findings of papers such as Kogan, Papanikolaou, Seru, and Stoffman [20] can come into play. It can be that an important patent by one firm can then show up as having a negative impact on a competing firm.

[33] There are other situations in which it is possible that these correlations could be different. Consider a situa-

other, and so the correlation between these could be negative. It could be that in industries where there was intense innovation during periods of accelerated growth, which then caused so much competition that many existing firms would exit. This could cause q_t and $\left(q_t - \underline{q}_t\right)$ to be negatively correlated.

The goal here is not to directly study or model all of the conceivable factors that could influence $\underline{q}_t$, but instead to investigate how this feature will influence asset prices.[34] Therefore, in what follows it will be assumed that this process follows the following geometric Brownian motion described in equation (25), and then proceed to study how this would influence the price $V\left(q_t, \underline{q}_t\right)$. It will then be possible to study how the parameters determining the behavior of $\underline{q}_t$ will then influence asset prices and returns.

It is straightforward to show that the price change for some benchmark asset is determined by equation (33). Therefore, the expected change in price for this asset is given by the following expression

$$\left[\frac{1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}}{1-e^{\left(q-\underline{q}\right)(\beta-1)}}\right]\kappa_q + \left[\frac{1-\beta^2 e^{\left(q-\underline{q}\right)(\beta-1)}}{1-e^{\left(q-\underline{q}\right)(\beta-1)}}\right]\left(\frac{\sigma_q^2}{2}\right) + \left[\frac{(\beta-1)\, e^{\left(q-\underline{q}\right)(\beta-1)}}{1-e^{\left(q-\underline{q}\right)(\beta-1)}}\right]\kappa_Z + \left[\frac{(\beta-1)^2 e^{\left(q-\underline{q}\right)(\beta-1)}}{1-e^{\left(q-\underline{q}\right)(\beta-1)}}\right]\left(\frac{\sigma_Z^2}{2}\right)$$
$$+\left[\frac{\sigma_Z\sigma_q\rho_3}{\left[1-e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2}\right]\left[1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}\right]\left[(\beta-1)\, e^{\left(q-\underline{q}\right)(\beta-1)}\right] \tag{42}$$

Also, the variance of the price change is given by the following:

$$var\left(\frac{dV\left(q,\underline{q}\right)}{V\left(q,\underline{q}\right)}\right) = \frac{\left[1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2\sigma_q^2}{\left[1-e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2} + \frac{\left[(\beta-1)\, e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2\sigma_Z^2}{\left[1-e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2} + \frac{2\left[(\beta-1)\, e^{\left(q-\underline{q}\right)(\beta-1)}\right]\left[1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}\right](\sigma_Z\sigma_q)\,\rho_3}{\left[1-e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2}. \tag{43}$$

This last expression is not necessarily monotonic in the level of productivity (q_t). This variance then has the seemingly unique property that, if $\rho_3 > 0$, it could be lower at intermediate levels of productivity than for higher or lower levels.

The important features of the behavior of price and the rates of return can be encapsulated in the following:

Proposition 10 *The expected excess rate of return, over the risk-free rate, is given by the following:*

$$\frac{-\left[e^{\left(q-\underline{q}\right)(\beta-1)}\right](1-\beta)}{\left[1-e^{\left(q-\underline{q}\right)(\beta-1)}\right]}(\sigma_c\sigma_Z\rho_2)\,\theta + \left[\frac{1-\beta\left[e^{\left(q-\underline{q}\right)(\beta-1)}\right]}{1-e^{\left(q-\underline{q}\right)(\beta-1)}}\right](\sigma_c\sigma_q\rho_1)\,\theta \tag{44}$$

Furthermore, the Sharpe ratio for such an asset is given as follows:

$$\frac{-\left[e^{\left(q-\underline{q}\right)(\beta-1)}\right](1-\beta)(\sigma_c\sigma_Z\rho_2)\,\theta + \left[1-\beta\left[e^{\left(q-\underline{q}\right)(\beta-1)}\right]\right](\sigma_c\sigma_q\rho_1)\,\theta}{\left(\left[1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2\sigma_q^2 + \left[(\beta-1)\, e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2\sigma_Z^2 + 2\left[(\beta-1)\, e^{\left(q-\underline{q}\right)(\beta-1)}\right]\left[1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}\right](\sigma_Z\sigma_q)\,\rho_3\right)^{1/2}}. \tag{45}$$

Proof. See Appendix. ■

It is essential to pause and reflect on a few important features in these last two expressions, as well as in the asset pricing formula presented in Proposition 8. For all the assets under consideration by these formulae, the parameters σ_c, and κ_c are common to all of them. On the

tion in which there is a small economy that relies on foreign actors to influence its domestic technological frontier. Then domestic decisions influence domestic consumption and therefore ρ_1, but foreign decision determines the behavior of Z_t, and therefore ρ_2.

[34] It is possible to modify the analysis of Section (2.2) to have separate stochastic factors influencing $\underline{q}_t$.

other hand, aside from the relative productivity level $\left(q_t - \underline{q}_t\right)$, the parameters ($\kappa_q$, κ_Z, σ_q, σ_Z, ρ_1, ρ_2, ρ_3) are all asset-specific. That is, it is possible to have two seemingly very similar assets, but which differ only in one (or a few) of these parameters. These assets could then display quite different pricing and return features. This shows how the productivity-dependent mortality risk can magnify the impact that a slight change in productivity, or in some parameter, can have on asset prices or returns.

These formulations of the rates of return suggest a reason why the assets of firms with similar productivities (or size), but in rather different industries, may exhibit similar co-movement. Suppose there are two firms that have payoffs (q) that are uncorrelated, but which have threshold levels ($\underline{q}$) that are correlated. They would then have correlated returns, and the lower was the relative payoff $\left(q_t - \underline{q}_t\right)$, the more correlated would be the returns.[35]

The formula for the Sharpe ratio is also noteworthy for its unique characteristics. In particular, it is a function of the level of relative productivity $\left(q_t - \underline{q}_t\right)$. By contrast, in many models the Sharpe ratio is a constant, and this is usually because factors or parameters will have an identical impact on the excess return as well as its standard deviation. This is certainly not the case here, at least in general. The one extreme case where this does hold is when $\left(q_t - \underline{q}_t\right) \nearrow +\infty$, and so the Sharpe ratio converges to the standard formula: $(\sigma_c \rho_1)\,\theta$. Otherwise, a change in the relative productivity of the asset (or firm) does not have a proportional effect on both the numerator and denominator of this expression. In fact, it is possible to construct examples in which a change in productivity would raise the numerator but lower the denominator in equation (45). This will be explored more below using numerical methods.

The apprehensive reader may wonder why the risk premium in equation (44) seems so different from the typical expression derived from the discrete-time euler equation (for the consumption-based CAPM). While the typical expression ($\sigma_c \sigma_q \rho_1 \theta$) is a special case of (44), one might be skeptical that this could be derived from the usual optimization condition. In the appendix it is shown that if the asset-pricing equation (26) is used in the discrete-time euler equation, then equation (44) is exactly the result. What is missing from the typical euler equation is the presence of the survival function, which is instrumental in determining the excess returns.

6.3.1 An Analysis of the Risk Premium, Sharpe Ratio, and the Hazard Function

It is appropriate to consider the different factors that contribute to the risk premium, as shown in equation (44). This formulation illustrates the asset-specific factors that can help determine the unique risk premium for each asset. There are three correlations or components to this expression, and only two of them are directly influenced by the risk aversion parameter θ. These three components reflect the three different types of portfolio insurance (or lack thereof) that can be inherent in an asset's price and return.

The last term in equation (44) is the standard correlation of consumption growth with the asset payoff (ρ_1). Once again, in this instance, this term is adjusted or multiplied by a factor involving relative productivity $\left(q_t - \underline{q}_t\right)$ because the risk or volatility of this asset increases as $\left(q_t - \underline{q}_t\right)$ approaches zero.

[35] At first this might seem like an odd circumstance: where the payoffs of two assets may be uncorrelated, but where the values of $\underline{q}_t$ are correlated. Consider two different firms that are engaged in two different industries, and so their payoffs are uncorrelated. Now suppose that both these firms are vulnerable to a common factor such as a change in a government policy, the status of patent protection, legal regulation, or a change in trade policy. These firms might have values for $\underline{q}_t$ that then move in tandem. This suggests a reason why the cross-industry behavior of asset prices and returns can be quite different from the cross-industry behavior of production.

Next, consider the remaining terms. The term involving ρ_2 is present because consumption growth can be correlated with the exit threshold $\left(\underline{q}_t\right)$. This term represents the *growth* in the mortality risk of the firm derived from the change in $\underline{q}_t$, and the rate of return must be adjusted for this risk.[36] The greater the growth in $\underline{q}_t$, the larger is the expected fall in the price of the asset. This means that consumption risk can be correlated with the capital gains risk of the asset (as opposed to the dividend-price risk). This risk premium is also directly related to σ_c and σ_Z. If $\rho_2 > 0$ then this asset bears some insurance against consumption risk and this insurance feature adds value to the asset so that the individual would be willing to pay more for it. But this raises the price for the asset and lowers the rate of return. Alternatively, if $\rho_2 < 0$, the asset exacerbates consumption risk.

It is also important to note here that if $\rho_2 > 0$, then an increase in risk aversion (θ) can *reduce* the risk premium. One typically thinks of increases in θ as raising the risk premium of an asset (as long as $\rho_1 > 0$), but here this is not necessarily the case here. This issue will be explored below.

Next, consider the impact of the value of ρ_3. This term does not explicitly enter the expression for the risk premium, but instead enters indirectly through influencing the size of the parameter β. In other words, this correlation will influence how sensitive the risk premium is to changes in relative productivity $\left(q_t - \underline{q}_t\right)$.

The magnitude of the impact of ρ_3 will be studied below, but it is useful at this point to pause and reflect on the importance of this term. First, this is an important feature because it demonstrates that there may be factors that influence the risk premium that are not related to consumption risk. One can think of reasons why this correlation, should not be zero. It does not require much imagination to think that there can be instances of movements in asset prices that do not seem to be fully explained by changes in payoffs, or interest rates. Secondly, one could think of the current payoff of the benchmark asset (q_t) as being determined by the current productivity of the asset or firm. On the other hand, the capital gain portion of the return would reflect the expected *future productivity or viability*, and there could be many other factors that could influence this productivity. In this simple framework $\underline{q}_t$ will be one such factor (or encapsulate a variety of factors) that can influence this capital gain, which is not directly linked to the current payoff of the asset. Nevertheless, it is conceivable that the current values of (q_t) and $(\underline{q}_t)$ could be correlated and hence $\rho_3 \neq 0$. The next question is how the factors that might influence the value of $(\underline{q}_t)$, such as innovation by competing firms, might be correlated with the current payoff. At this point, it is not entirely clear what the default or natural answer to this should be.

It may be enlightening to re-write the risk premium (44) in the following manner:

$$\frac{\left[e^{\left(q-\underline{q}\right)(\beta-1)}\right](1-\beta)}{\left[1-e^{\left(q-\underline{q}\right)(\beta-1)}\right]}\left[\left(\sigma_c\sigma_q\rho_1\right)\theta-\left(\sigma_c\sigma_Z\rho_2\right)\theta\right]+\left(\sigma_c\sigma_q\rho_1\right)\theta. \tag{46}$$

This expression is distinctive because the last term is the usual expression involving the correlation of the dividend with consumption. This is independent of the firm's productivity level q_t, whereas the other terms are related to the level of productivity. For reasonable parameter values, the last term in equation (46) can be overwhelmed by the remaining terms as well. In summary, the traditional term that characterizes the risk premium can be the least important part of the above expression.

[36] To be clear, this term is not "the mortality risk", but instead it is the *growth in the mortality risk*. This is part of the rate of return to holding the asset. Of course, if all firms perished at a constant, exogenous rate, then this term would not be present.

Now notice that this asset-specific productivity factor in this last equation also has the following property:

$$\frac{\left[e^{\left(q-\underline{q}\right)(\beta-1)}\right](1-\beta)}{\left[1-e^{\left(q-\underline{q}\right)(\beta-1)}\right]}=\frac{d\ln\left[1-e^{\left(q-\underline{q}\right)(\beta-1)}\right]}{dq}>0. \tag{47}$$

This is the derivative of the logarithm of the survival function, equation (30), and the term in equation (47) could be termed a productivity-dependent *hazard function*. Therefore, the returns in equations (44) and (45) must reflect the *changes in the survival function, or survival probabilities in the future*. The survival function is present in the pricing function (26), and captures the degree of future firm survival (i.e. the opposite of mortality) that is embedded in the current asset price. Therefore, equation (47) measures the sensitivity of the survival function in response to a percentage change in the productivity of the firm. The higher is this elasticity (or sensitivity), the greater will be the importance of the first set of terms in equation (46) relative to that of the last term. If the firm's mortality was not sensitive to productivity, then this term would be zero.

Another way to view equation (46) is as follows. The rate of return (or the excess rate of return) of an asset can be broken down into two parts: the rate of return on the asset if it were expected to continue to pay off forever, and secondly, the *change* in the survival function. Any change in the return that is not obviously linked to a change in the payoff or dividend, could then also be ascribed to a change in the survival function.

At this point it is important to re-emphasize the importance of the survival function in the asset pricing equation (26), and its interaction with the remainder of that expression. The presence of the expression in equation (47) in the risk premium (46) is attributable to the presence of the survival function in the asset price (equation (26)). *The correlation terms that are multiplied by the hazard function in equation (46) are present only because of the interaction (or correlation) between the survival function and the permanent component of the asset price* $(B(e^{q_t}))$ *in equation* (26).

Similarly, consider the variance expression in equation (43), which contains terms involving the mortality risk (47). When the mortality risk is close to zero ($\left(q_t-\underline{q}_t\right)\nearrow+\infty$) the variance is merely σ_q^2. But as the mortality risk grows the variance approaches $+\infty$.

Note that the term involving ρ_2 is new relative to the existing literature. One could also consider a slightly more complicated environment where there were multiple factors ($\underline{q}_{i,t}$) that could each pose threats to the existence of the benchmark firm. For example, there could be multiple firms attempting to innovate and consequently threaten the existence of the benchmark firm. In this case, the pricing function (equation (26)) would become more complicated since it would have to reflect each of these different factors (and there would be multiple absorbing barriers). However, the formulae for the risk premium and the Sharpe ratio would then have additional expressions for the correlations of each of these additional factors. In other words, there would be many more correlations involving terms like ρ_1 and ρ_2. From this perspective, one can see that there could be a multitude of factors that could contribute to these rates of return.

This analysis illustrates how the productivity-dependent mortality risk can have a dramatic effect on asset prices and returns for certain assets. Furthermore, this mortality risk can magnify the impact that a slight change in productivity, or in some parameter, can have on these asset values.

6.3.2 The Risk Premium and Risk Aversion

In this environment, risk aversion has some unique effects on asset prices and rates of return that are not present in more standard economies. The examples presented earlier illustrate how the productivity (q_t) of the asset could affect the returns and prices. This example will illustrate why even the *sign* of the risk premium may change as productivity changes, even though the underlying asset will not have changed its fundamental features. Consider the case in which $\rho_1 > 0$, and $\rho_2 > 0$. As is evident for equation (44), as $\left(q_t - \underline{q_t}\right) \nearrow +\infty$, then changes in $\underline{q_t}$ have very little impact on the price or rate of return on the asset, and the excess return will approach $\theta\sigma_c\sigma_q\rho_1$. Since the correlation of the payoff (q_t) with consumption is positive, there will be a positive risk premium. However, as $\left(q_t - \underline{q_t}\right)$ becomes smaller, changes in $\underline{q_t}$ will have a more pronounced impact on the price or rate of return on the asset. In particular, positive movements in the dividend-price ratio are offset by the capital gain portion of the return, and so this asset now provides some financial self-insurance. Because of this extra insurance inherent in the asset at this stage, the risk premium can switch from being positive to being negative.

Risk aversion can also have an unusual influence on the risk premium in this environment. It is possible to see that for high values of relative productivity $\left(q_t - \underline{q}_t\right)$, an increase in risk aversion (θ) can raise the risk premium, while for low values of productivity, an increase in risk aversion can lower the risk premium.

6.3.3 Further Analysis of the Sharpe Ratio

It is instructive to study further how the various factors contribute to the Sharpe ratio in equation (45). This formula delivers the usual formula $(\sigma_c\rho_1\theta)$ as $\left(q_t - \underline{q}_t\right) \nearrow \infty$, when $\sigma_Z = 0$. Of course, this latter formula is independent of the many other parameters that are present in β. This means that in this typical formula, factors that influence the risk premium itself, perhaps through a channel such as β, would also affect the standard deviation of the return *in a proportional manner*. However, more generally when $\sigma_Z > 0$, then this is not the case. Consider, for example, the simple case where $\sigma_Z > 0$, but where $\rho_2 = \rho_3 = 0$. Equation (45) then simplifies to the following

$$\frac{\left[1 - \beta\left[e^{\left(q-\underline{q}\right)(\beta-1)}\right]\right](\sigma_c\sigma_q\rho_1)\,\theta}{\left(\left[1 - \beta e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2\sigma_q^2 + \left[(\beta-1)\,e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2\sigma_Z^2\right)^{1/2}}. \tag{48}$$

In this case, there are factors that directly influence the variance of the excess return that do not influence the expected excess return in the same manner. In particular, the most notable feature of this formula is that it is explicitly dependent on σ_q, σ_Z, and $\left(q_t - \underline{q}_t\right)$. For example, changes in relative productivity $\left(q_t - \underline{q}_t\right)$ will alter the expected excess return, and the standard deviation of this return to *different degrees*, or in different magnitudes. Note that if $\sigma_Z > 0$, then a change in σ_q will not have a proportionate effect on the numerator and denominator of this equation. Similarly, a change in σ_Z will not have a direct influence on the expected excess return, but will certainly affect the variance of this return. To see this, consider the case in which $\left(q_t - \underline{q}_t\right)$ approaches zero. Then equation (48) approaches

$$\frac{(\sigma_c\sigma_q\rho_1)\,\theta}{\left(\sigma_q^2 + \sigma_Z^2\right)^{1/2}}. \tag{49}$$

This ratio is less than the traditional formulation $(\sigma_c \rho_1 \theta)$. This means that for relatively low values of $(q - \underline{q})$, a change in σ_Z can have an impact on the variance of the return that does not appear in the expected value of the return. Also, an increase in σ_q will have a larger proportionate effect on the numerator than on the denominator. The channel through which this effect operates is through the capital gain portion of the asset return. As $\left(q_t - \underline{q}_t\right)$ falls the capital gain portion of the return becomes more volatile, but also more sensitive to change in parameters, or movements in $\left(q_t - \underline{q}_t\right)$.

In addition to these direct effects that these factors have on the standard deviation and the expected value of the excess return, there are indirect effects. These effects operate through changing the value of β.

Lastly, by studying equation (45) it is possible to imagine parameter changes that could move the numerator and denominator in opposite directions.

The effect of parameter changes on the Sharpe ratio will be studied further below.

6.3.4 Analysis of Different Cases

It is important to gain a greater understanding of the role that each of these separate factors play in determining the risk premium (44) and the Sharpe ratio (45). Since the effect that various parameters can play in these equations can be somewhat Byzantine, it is best to proceed in a sequence of simple cases that will explore the role that these parameters play.

Analysis: $\rho_1 = \rho_2 = \rho_3 = 0$. The introduction of a variable process for $(\underline{q}_t)$ influences the asset price even if this is not correlated with anything else. A change in either κ_Z or σ_Z affects the change in price in equation (42) directly, but also indirectly through the change in β.

Figure 7 shows the expected change in the asset price (equation (42)), for different parameter values, as a function of the level of productivity. As can be seen, increasing the value of the growth in $\underline{q}_t$, which is κ_Z, or reducing the standard deviation σ_Z, lowers the expected value of the change in the price when the process for $\underline{q}_t$ is uncorrelated with other variables. Raising κ_Z lowers the expected value of the change in the price because there is then a greater likelihood (or at least an earlier prospect) that the asset will soon hit the exit barrier, where its value will be zero. Hence this will lower the price, and lower future price changes. Lowering σ_Z has very much the same effect. This reduces the likelihood of the asset extending its lifetime before reaching the exit barrier and lowers the value of the asset.

As can be seen from equation (43), raising the standard deviation (σ_Z) raises the standard deviation of the change in price when $\rho_3 = 0$. In this case, there are two obvious sources of risk in this formula: changes in q_t, and $\underline{q}_t$. Raising the variability of either of these is likely to raise the variability of the price.

To the extent that the process for $\underline{q}_t$ is not observable, if one were to witness an unexplained increase in the variability of the price, one might be tempted to attribute this to any number of external factors.

Analysis: $\rho_2 = \rho_3 = 0, \rho_1 \neq 0$. Here we will focus on the role played by the standard correlation of the asset payoff with consumption (ρ_1). Changing this parameter directly impinges on the features of equation (42), but also does so indirectly through the change in β. Raising the correlation lowers the value of β (or raises its absolute value). This has the effect of modestly raising the expected price change in equation (42).

In the standard equilibrium model without exit of firms, the risk premium is given by

$(\sigma_c \sigma_q \rho_1)\,\theta$. However, in this model the risk premium is now given by

$$\left[\frac{1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}}{1-e^{\left(q-\underline{q}\right)(\beta-1)}}\right](\sigma_c \sigma_q \rho_1)\,\theta,$$

where both q and $\underline{q}$ are time-varying. This expression is identical to that given in section 6.3.1. However, here β is a function of such parameters as κ_Z or σ_Z, which are usually absent in the analysis of risk premia. Figure 8 shows how the risk premium is influenced by these parameters for the case of $\rho_1 = .90$. Once again, for reasons similar to those given in the previous section, the risk premium is increasing in both κ_Z and σ_Z. The presence of $\left(\underline{q}_t\right)$ is an important feature as it poses a mortality threat to the asset under consideration. And to the extent that $\left(q_t - \underline{q}_t\right)$ is small, this risk can be substantial.

Lastly, the Sharpe ratio for this particular asset is now given by equation (48), Here the factor $\underline{q}_t$ poses a risk to the mortality of the asset under consideration. The standard deviation in the denominator of this formula is not necessarily monotonic in q_t. Frequently this formula for the standard deviation is convex in q_t. More will be said about this below.

Figure 9 shows how the Sharpe ratio behaves as a function of relative productivity $\left(q_t - \underline{q}_t\right)$, for different values of ρ_1. Here, as $\left(q_t - \underline{q}_t\right) \nearrow +\infty$, then this ratio approaches $(\sigma_c \rho_1)\,\theta$. But as $\left(q_t - \underline{q}_t\right) \searrow 0$, this ratio approaches the expression in equation (49). In the figure, for the case where $\rho_1 = 0.90$, the Sharpe ratio is increasing only because $\sigma_Z < \sigma_q$, and if this inequality were reversed then the Sharpe ratio would be a decreasing function of $\left(q_t - \underline{q}_t\right)$.

Analysis: $\rho_1 = \rho_3 = 0, \rho_2 \neq 0$. Here we will focus on the role played by (ρ_2), which is the correlation of consumption with the exit threshold $(\underline{q}_t)$. To the extent that external features such R&D might influence $\underline{q}_t$, this correlation might reflect how consumption might be correlated with external R&D spending.

In the standard equilibrium model without exit of firms, the risk premium is given by $(\sigma_c \sigma_q \rho_1)\,\theta$, and so in this case it would be zero. However, in this specific case, the risk premium is now given by

$$\frac{\left[e^{\left(q-\underline{q}\right)(\beta-1)}\right](1-\beta)}{\left[1-e^{\left(q-\underline{q}\right)(\beta-1)}\right]}\left[-\sigma_c \sigma_Z \rho_2 \theta\right].$$

Figure 10 shows how the risk premium is influenced by these parameters, for the case in which $\rho_2 = -.90$. Raising the value of σ_Z can raise the risk premium for low values of relative productivity $\left(q_t - \underline{q}_t\right)$, but not necessarily for higher values. Once again, the presence of $\left(\underline{q}_t\right)$ is important in illustrating the mortality threat to the asset under consideration, and to the extent that $\left(q_t - \underline{q}_t\right)$ is small, this risk can be substantial.

Analysis: $\rho_2 = 0, \rho_1 \neq 0, \rho_3 \neq 0$. Here we will focus on the unique role played by the correlation of the asset payoff with the threshold $(\underline{q}_t)$, which is measured by ρ_3. The correlation ρ_3 does not explicitly affect either the risk premium or the Sharpe ratio. However, it does influence the value of β, which does appear in these formulae. The reason for this is because the asset return consists of two parts: the capital gain and the dividend-price ratio. If these two parts are positively correlated (i.e. $\rho_3 < 0$) then they reinforce each other and magnify the risk of the asset. This increased risk means that individuals will have to be offered a higher return in order to hold the asset. Therefore, the risk premium will be positive.

But if they are negatively correlated (i.e. $\rho_3 > 0$) then these two parts of the return partially offset each other. In this case the asset offers a type of "built-in" portfolio insurance. This relationship is illustrated in figure 11 for the case where $\rho_1 = .90$.[37] A higher value of ρ_3 will lower the risk premium of the asset.

Lastly, the Sharpe ratio for this particular asset is now given by the following:

$$\frac{\left[1-\beta\left[e^{\left(q-\underline{q}\right)(\beta-1)}\right]\right]\left(\sigma_c\sigma_q\rho_1\right)\theta}{\left(\left[1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2\sigma_q^2+\left[(\beta-1)\,e^{\left(q-\underline{q}\right)(\beta-1)}\right]^2\sigma_Z^2+2\left[(\beta-1)\,e^{\left(q-\underline{q}\right)(\beta-1)}\right]\left[1-\beta e^{\left(q-\underline{q}\right)(\beta-1)}\right]\left(\sigma_Z\sigma_q\right)\rho_3\right)^{1/2}}$$

Note again that the denominator in this formula is related to the standard deviation of the asset return. It is noteworthy that if $\rho_3 > 0$ then it is possible that the standard deviation of the return may not be monotonic in q_t. As $\left(q-\underline{q}\right) \nearrow +\infty$ the term involving σ_q will dominate this expression, while as $\left(q-\underline{q}\right) \searrow 0$ the term involving σ_Z can also become important. However, since $\beta < 0$, for intermediate values of productivity, the two sources of uncertainty can offset each other so much that the variance may attain a minimum in q_t.

The behavior of the Sharpe ratio, as a function of relative productivity $\left(q-\underline{q}\right)$, is shown in figure 12, for $\rho_1 = .90$, for two different values of ρ_3. As can be seen, this ratio is far from constant, and in fact can exhibit quite unusual behavior. Clearly the level of relative productivity $\left(q-\underline{q}\right)$ influences the numerator and denominator of the Sharpe ratio in a manner that is not proportional to each other.

In the previous analysis and examples, it was shown that a reduction in relative productivity $\left(q-\underline{q}\right)$ can raise the excess return, as well as the volatility of the return, in a non-linear manner. But figure 12 illustrates something much more. It shows that a change in productivity does not affect the excess return, as well as the volatility of the return, in tandem. Furthermore, the value of ρ_3 can have an important influence over how productivity can influence the excess return, as well as the volatility of the return.

This section, together with the previous section where $\rho_2 \neq 0$, truly illustrates the important nature of the creative destruction mechanism in the characterization of rates of return in this environment. Here there are factors external to the asset payoff and the discount factor (i.e. the risk-free rate) that can influence the longevity and the price of assets. The case in which $\rho_2 \neq 0$ shows that this factor that is influencing the price of the benchmark asset need not even be correlated with the payoff or dividend of that asset.

[37] A curious circumstance can arise when ρ_3 becomes sufficiently negative. In this case, the risk premium can suddenly "flip" from being positive to being negative. The reason for this is that the value of β "flips" from being negative to being positive. This is a feasible circumstance. To understand this situation keep in mind that in these models the variance terms contribute to the drift process. But then the correlations across variables also influence the drift processes. So the variance in one variable can influence the growth rate in another. For example, if $\rho_3 < 0$ then the variance in $\underline{q}_t$ can influence the drift in q. Next, if $\rho_3 < 0$, and σ_Z and σ_q are sufficiently large, then it is possible that a further reduction in ρ_3 can cause the risk premium to flip from being positive to negative. In this case, the rise in the correlation has the effect of magnifying the correlation between the capital gain, and the dividend-price portion of the return. It does this to such an extent that it kicks the growth rate slightly above the discount rate. This does not make the price explode to infinity since it is still finite for β between zero and one (see equation (26)). However, what this does is, for some values, make the price a decreasing function of the contemporaneous dividend. This happens because an increase in the dividend (q) lowers the price (instead of raising it), and because of this it raises the expected value of the lost dividends when the asset life is terminated. That is, an increase in $\underline{q}$ reduces the value of the asset by shrinking expected payoff horizon. In this case, the capital gain and the dividend portion of the return are now negatively correlated. Hence the asset is now providing some financial self-insurance. This is an attractive property of an asset, and individuals are willing to sacrifice some expected return to get this insurance. Hence, the expected excess return will then be negative.

Analysis: $\rho_3 = 0, \rho_2 \neq 0, \rho_1 \neq 0$. It is of interest to study another example because of the unique features that it presents. The earlier examples illustrated how productivity (q_t) of the asset could affect returns and prices. This example will illustrate why even the *sign* of the risk premium may change as productivity changes, even though the underlying asset will hot have changed its fundamental features or parameters. Consider the case in which $\rho_1 > 0$, and $\rho_2 > 0$. As is evident from equation (44), as $\left(q_t - \underline{q}_t\right) \nearrow +\infty$, changes in $\underline{q}_t$ have little impact on the price or rate of return of the asset, and there will be a positive risk premium. However, as $\left(q_t - \underline{q}_t\right)$ becomes smaller, changes in $\underline{q}_t$ movements in the dividend-price ratio can be offset by the capital gain portion of the return. The result is then that the risk premium can now become negative.

This is illustrated with a specific example in figure 13. Here $\rho_1 = .90$, for alternative values of ρ_2. Of course, for this "switching" of sign to take place, it must be that σ_Z is large relative to σ_q.

This example illustrates an important phenomenon that is related to that mentioned in Section (6.2.1). There it was shown that depending on the level of relative productivity, the asset price variability might reflect different types of risk. Exit or mortality risk may be overwhelmingly important for assets with lower levels of productivity (q_t). But here, in this example and as illustrated in figure 13, we have another related feature that comes into view. For relatively high levels of productivity the correlation of the dividend with productivity (ρ_1) can be important, but for lower levels of productivity this correlation can be overwhelmed by the correlation (ρ_2) of the consumption with $\underline{q}_t$.

6.3.5 Summary

Many researchers have become inured to the idea that explanations of the risk premium must entail studying the relationship between the payoff of the asset, and consumption risk (or, changes in the intertemporal marginal rate of substitution). Considerable ink, computational and intellectual resources have been spent investigating what preferences could be capable of explaining this premium. The analysis presented here shows why this view offers only a partial explanation of this phenomenon, but it points to other important factors that can influence this return. While consumption risk is not necessarily unimportant, this analysis suggests that other types of risk that can potentially be much more important and deserve greater consideration.

Figures 12 and 13 suggest that a focus on these two-dimensional relationships can be misleading. Here there are several sources of risk: asset payoff, consumption, and at least one source of exit risk. So diversification of these types of risk is a multi-dimensional concept.

One might consider how a hypothetical risk-averse might seek to hold an optimally-diversified portfolio of assets in this environment. If, as in the benchmark model, the firms all have the same, independent productivity parameters, then a well-diversified portfolio is likely to involve reducing the holdings of assets that have low or falling productivity, since these returns are likely to have higher risk.

7 The Role of Aggregate Volatility

It is of interest to inquire into the role that aggregate volatility plays in this framework, relative to more standard models of asset pricing and returns. First, consider what happens if the standard deviation of consumption (σ_c) were to rise. Consider the standard model in which $\underline{q} = -\infty$, in equation (26). In this case, if $\rho_1 = 0$, then an increase in σ_c would raise the asset price if agents are risk-averse ($\theta > 0$) because it lowers the discount rate by raising the growth rate. The magnitude of this effect is independent of the level of productivity q_t. That is, the value of σ_c will affect the level of the price, but not how sensitive the price is to changes in the payoff q.

However, if $\underline{q} > -\infty$ then a change in σ_c will also influence the parameter β. In this case, an increase in σ_c could raise or lower the value of β. This will change the sensitivity of the asset price to changes in the payoff q. It can be shown that

$$\frac{d\ln\left(V(q,\underline{q})\right)}{d\beta} = 1 + \frac{\left(q-\underline{q}\right)e^{(\beta-1)\left(q-\underline{q}\right)}}{1-e^{(\beta-1)\left(q-\underline{q}\right)}}.$$

And as $\left(q-\underline{q}\right) \searrow 0$, this derivative converges to $1-(\beta-1)^{-1}$ which is greater than unity since $\beta < 0$. In other words, if β were to increase (i.e. get closer to zero), then the asset price becomes more sensitive to changes in the payoff.

So what is the reasoning behind why a change in σ_c could influence the price $(V(q,\underline{q}))$ through the value of β? A change in σ_c influences the growth rate of consumption (and marginal utility) in such a way as to change the discounted value of dividends forgone or lost because the asset will cease being productive when $q = \underline{q}$. [38] If $\rho_i = 0, i = 1,2,3$, and $a > 0$ in equation (28), then it is possible to establish that an increase in σ_c will raise the value of β, and change the volatility of the asset price.

If $\rho_1 > 0$, then an increase in σ_c will lower the price of the asset through the traditional channel of lowering B. However, when $\rho_1 > 0$, then an increase in σ_c can also change the size of β, and thereby influence the sensitivity of the price to a change in the payoff q. This does not necessarily work in tandem, or the same direction, as the previously described effect of an increase in σ_c.

Next, consider what happens to the risk premium if σ_c were to rise. To do this, focus on equation (44). If $\rho_1 > 0$, while the other correlations are zero, then there is a positive risk premium. For high productivity assets, (set $\underline{q}_t = -\infty$ in equation (44)) the risk premium is proportional to σ_c, and so this is the traditional case. However, when $\underline{q}_t$ is finite then a change in aggregate volatility does not affect the risk premium proportionately, because it changes β. As $\left(q-\underline{q}\right) \searrow 0$, the asset is becoming riskier, and the volatility of its returns will be rising. This will alter the covariance of the returns with consumption, and thereby change the risk premium. A similar effect would prevail if $\rho_2 < 0$.

In this model, it is not clear that σ_c is the sole or obvious measure of aggregate volatility. One might consider σ_Z to be an alternative feature that reflects volatility, and in the benchmark model of section (4), it is the case that $\sigma_Z = \sigma_c$. So what is the impact of an increase in σ_Z? Through equation (26) one can see that an increase in σ_Z can have a complicated effect on the asset price. It has a direct impact by changing the value of B, but also changes the parameter β. Whether an increase in σ_Z raises or lowers prices depends on a variety of parameters. A change in the value of σ_Z has the obvious effect of raising the volatility of $\underline{q}_t$, which raises the riskiness of the asset. This also changes the growth rate of relative productivity $\left(q_t - \underline{q_t}\right)$, which influences the discounted value of dividends foregone when the asset ceases payoffs. Next, it can influence how the capital gain portion of the return will be correlated with consumption (if $\rho_2 \neq 0$) or with the dividend-price ratio (if $\rho_3 \neq 0$).

It may seem counter-intuitive that an increase in some measure of aggregate risk would not raise the risk premium of assets. However, the analysis presented here shows that this simple,

[38] In equation (26) you want to think of $e^{\beta\left(q-\underline{q}\right)}\left(e^{-\underline{q}}\right)$ as the discounted value of the foregone dividends when the asset ceases being productive, and $e^{\beta\left(q-\underline{q}\right)}$ can be thought of as the expected value of the random discount factor, over possible dates at which the asset may become terminal. Therefore, a change in β alters this discount factor.

straightforward view may not be correct because the relationship between aggregate volatility and excess returns can be complicated. Furthermore, the effect of a change in aggregate risk may have a different impact on two otherwise identical assets (in terms of parameters), whose sole difference is the current level of productivity.

8 Comparison to the Existing Literature

There has been a number of recent studies that have sought to investigate the impact that growth through creative destruction can have on financial market outcomes. Some of these papers have incorporated an exit decision for firms into the analysis. However, the contribution here is to show that having this exit decision be endogenous and related to firm productivity, can give rise to important financial market behavior. Furthermore, this analysis explores how factors that are external to the firm might influence the exit decision, and therefore be reflected in asset prices and returns. These findings add additional support to the findings of Fama and French [8] who state that "... our results suggest that the stock risks are multidimensional."

It has been documented that the US equity or risk premia is countercyclical, or higher at business cycle troughs than at peaks (see Harvey [15], Schwert [27], Chou, Engle and Kane [5], or Li [23]). Some research has attempted to explain this using countercyclical risk aversion (Campbell and Cochrane [4]) or changing volatility to the consumption process (Bansal and Yaron [2]). Guvenen [13] employs a model of limited stock market participation and heterogeneity in the elasticity of intertemporal substitution in consumption to try to explain this observation, as well as many others.

The analysis of this paper, specifically in equation (44), is also useful for aiding in our understanding of other features of the risk premium. This formula highlights how different factors could contribute to excess returns, and not all of these factors are necessarily related to the behavior of consumption.

To understand why the risk premium might be countercyclical, consider that during periods of slow or negative growth, the value of $q_t - \underline{q}_t$ were to fall for most firms, which would imply the survival probabilities for these firms would fall. Equation (44) then suggests that the risk premium for these firms would increase at that time. The behavior of the risk premium is then dependent on the behavior of both q_t and $\underline{q}_t$, and there can be many potential factors that contribute to the behavior of these variables.

9 Final Remarks

In a vibrant economy, there are new firms and products born while older ones may exit. This birth and mortality of firms and products, while usually taken for granted, does not seem to have manifested itself in the study of its implications for the pricing of assets issued by these firms. It was shown that introducing these issues into an otherwise conventional model of asset pricing can produce novel implications for asset prices and returns that are not present in the existing literature. Viewed from this perspective, it may be that this mortality or survivorship of firms may be a fundamental explanation for movements in asset prices and rates of return.

It is natural to think that changes in firms-specific productivity can affect the mortality of these firms, and this in turn should be closely reflected directly in the asset prices of those firms. The analysis presented here shows that there is an inverse, and non-linear relationship between the volatility of asset prices, and the level of a firm's relative productivity. It is also conceivable that there are factors external to the firm that can influence the rates of return in a manner that does not seem to be present in the existing literature. What is not as easily understood and appreciated is that these asset prices should also be influenced by the expected mortality or lifetime of the firm and that this feature may be influenced by arcane external phenomena, such as the R&D efforts of people who may not yet have even started a firm. This makes it difficult to even assess the magnitude of the risks inherent in some asset prices.

The existing literature frequently characterizes the trade-off between expected excess returns, and the standard deviation of these returns as a rather simple frontier. That is the implication of the consumption-based CAPM. But the model studied in this paper suggests that this relationship is anything but simple, and it is a frontier that needs considerable exploration. There is a multitude of factors that could affect both the expected returns and the riskiness of these returns, in a non-trivial or non-linear manner. The analysis presented here focused on only one "outside" or external factor ($\underline{q}$) that could influence returns. However one can also introduce many such factors. The risk-return trade-off then would become a multi-dimensional relationship. The standard deviation of returns to the asset may contain a multitude of risk factors not directly related to the current or future payoff of the asset.

In conclusion, it is appropriate to return to where this exploration began, which is to think about the implications that this analysis can have for the interaction between growth, or creative destruction, and related financial phenomena. From the perspective of the model studied here, it is apparent why an increase in general asset prices may not necessarily be a rational prelude to a period of increased innovation or growth. To the extent that such an increase in prices results from an increase in the survival functions of existing assets (produced by a fall in $\underline{q}_t$), this may imply a *reduction* in future growth and innovation that would benefit incumbent firms at the expense of potential entrants. Asset prices do indeed contain information, but deciphering it can be challenging.

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Figure 1

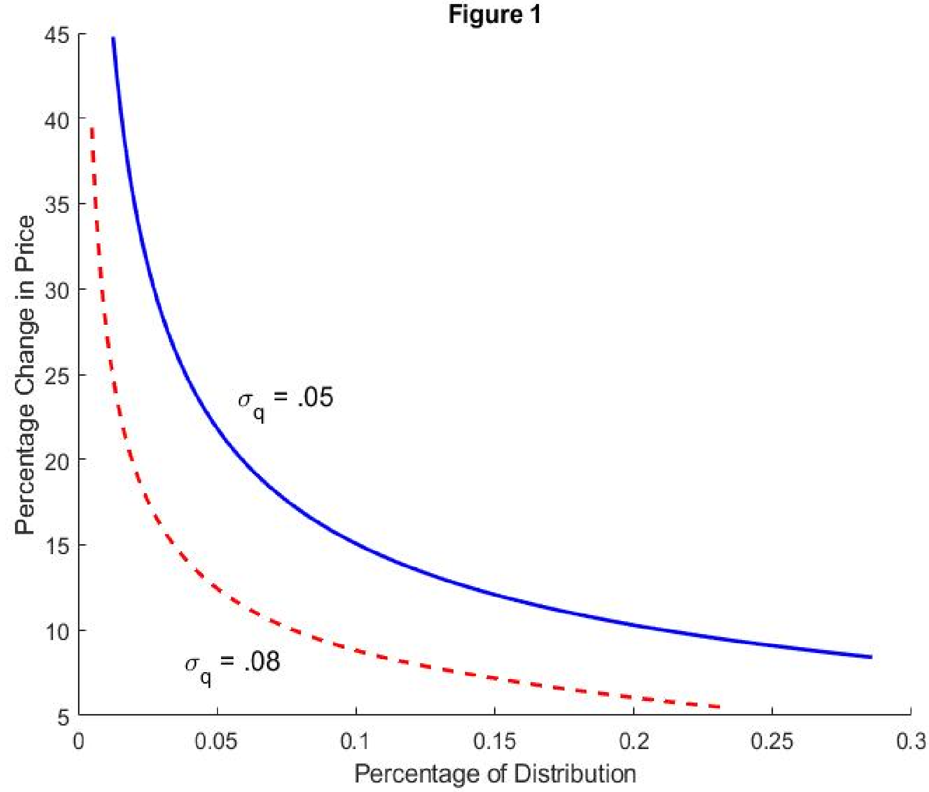
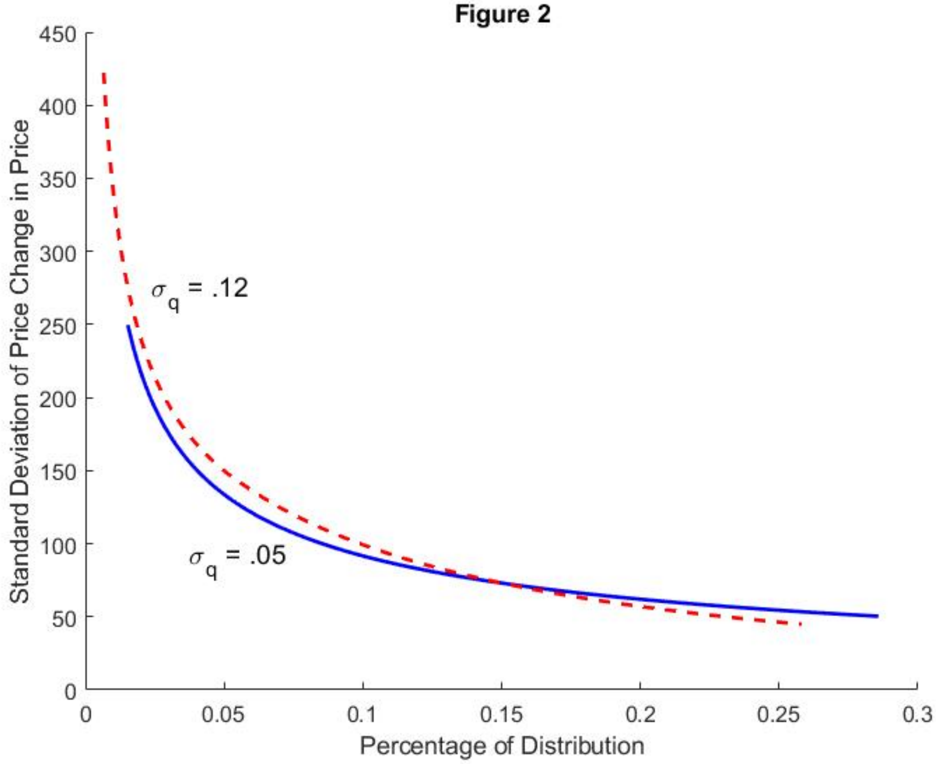


Figure 2



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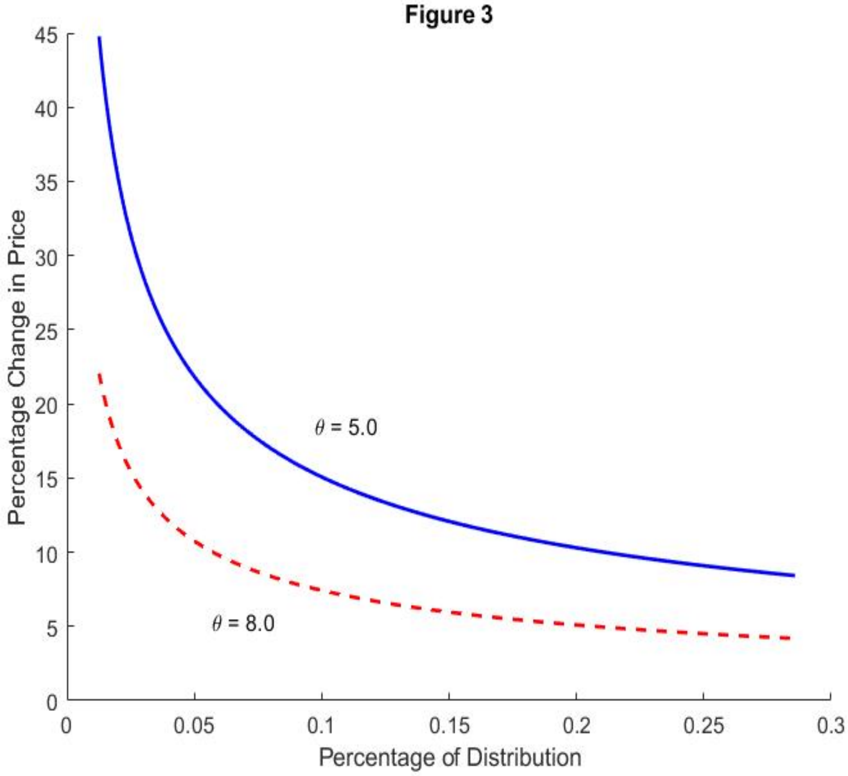

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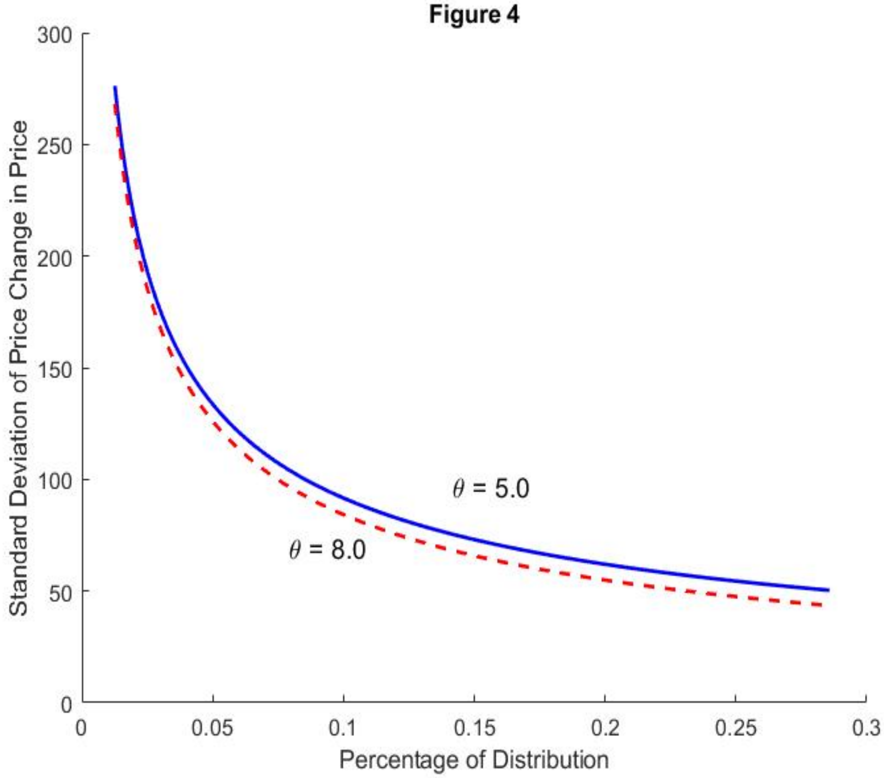

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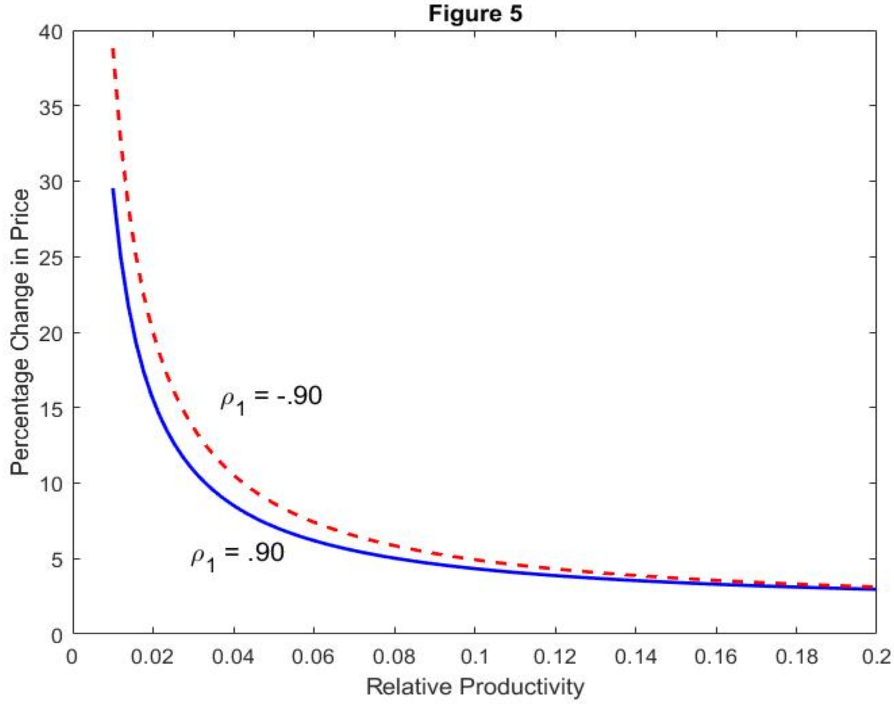

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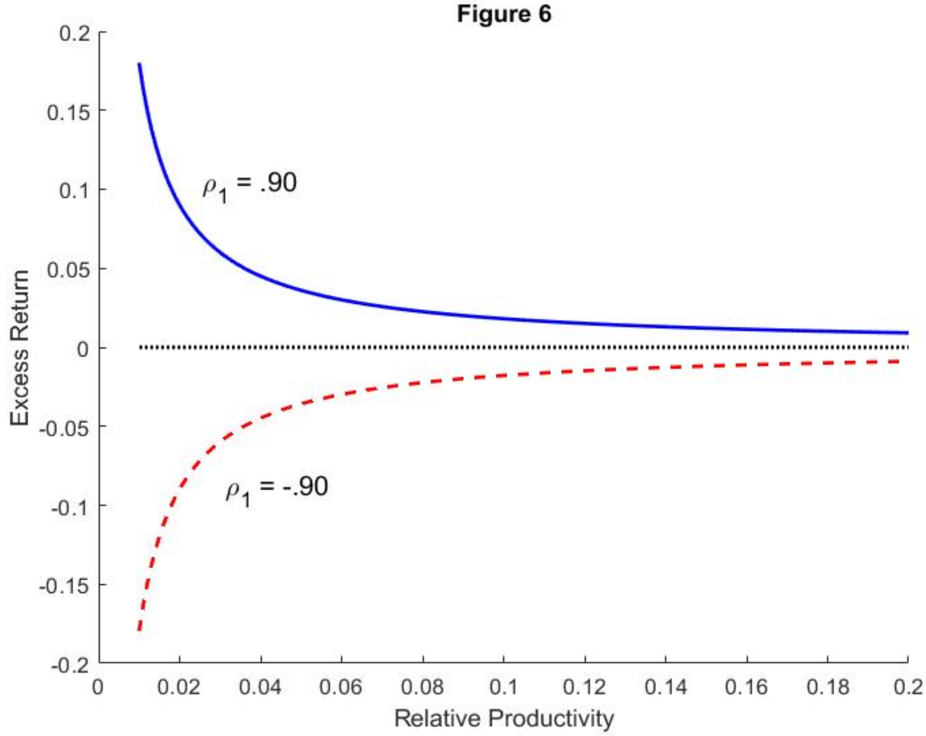

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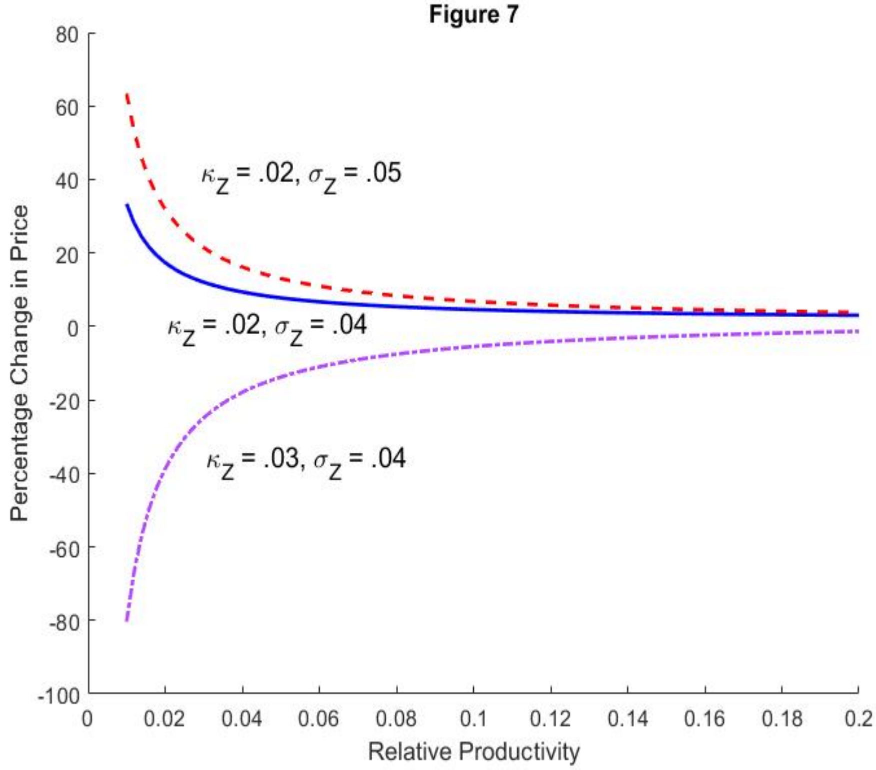


Figure 8

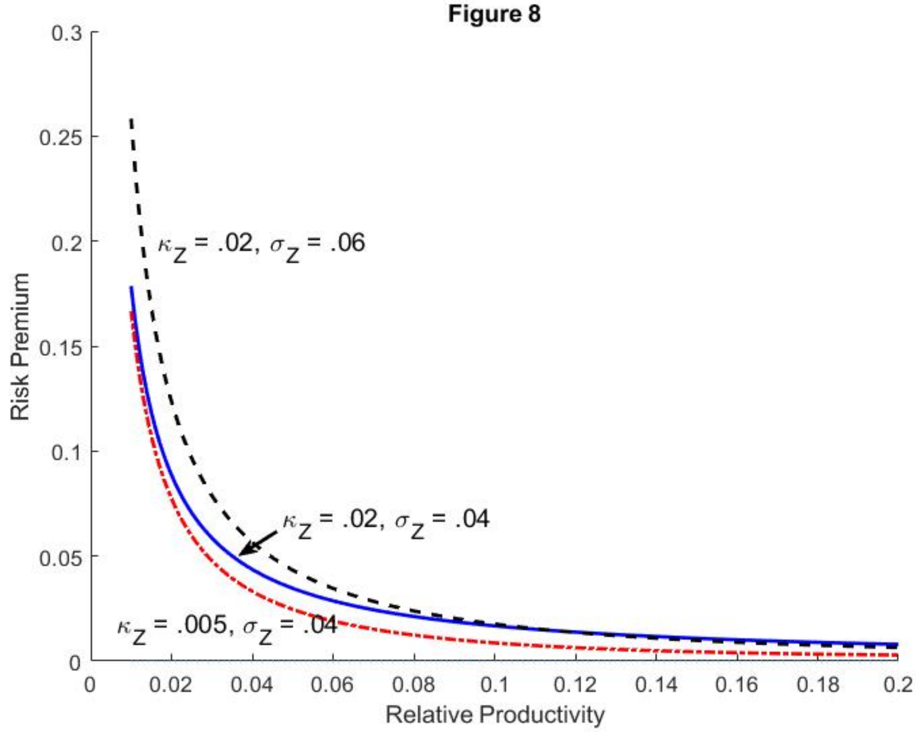


Figure 9

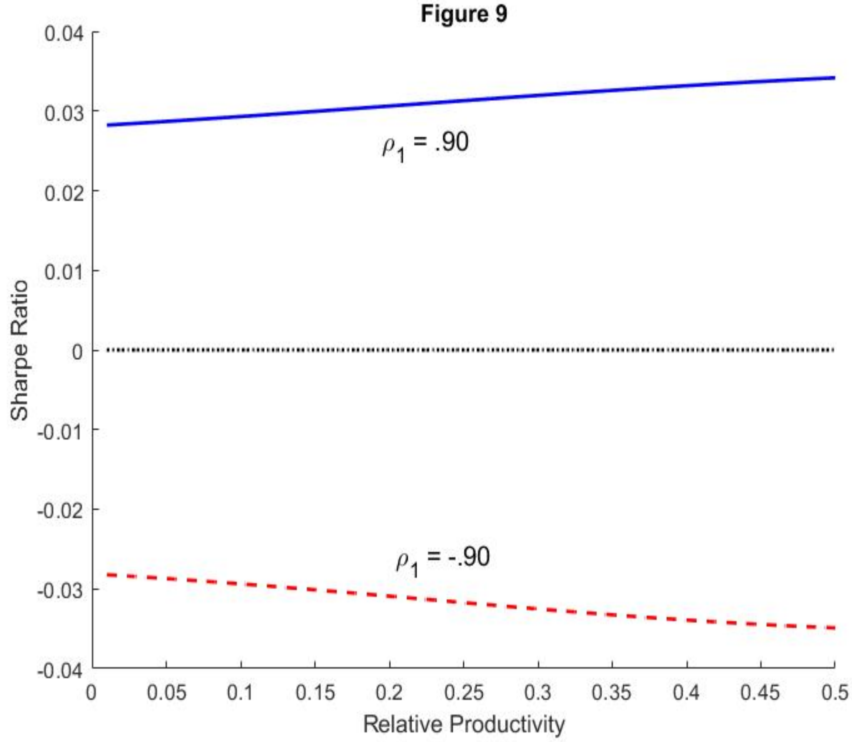


Figure 10

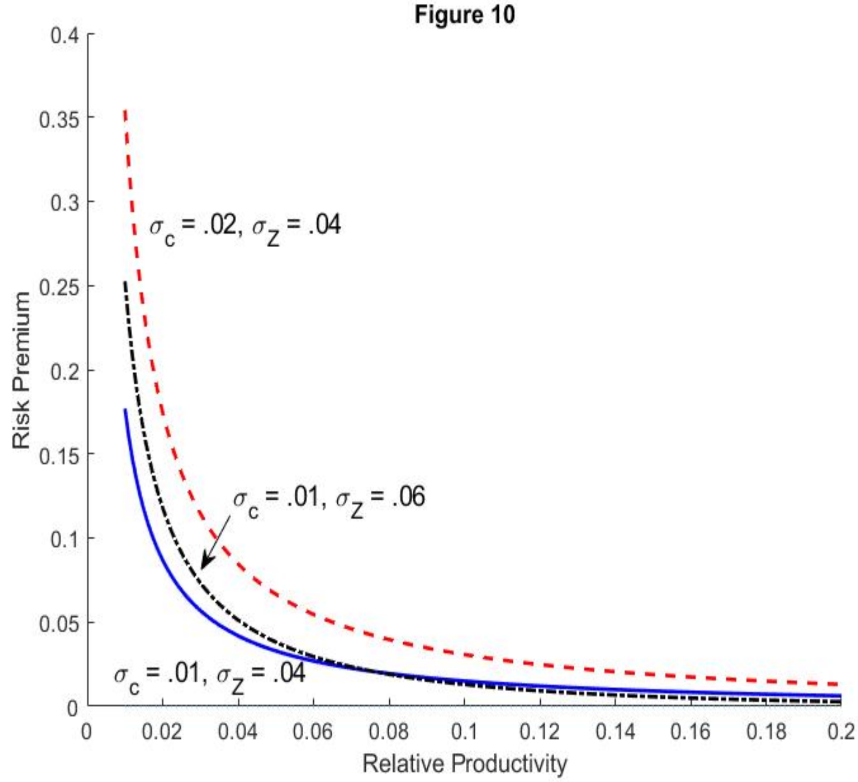


Figure 11

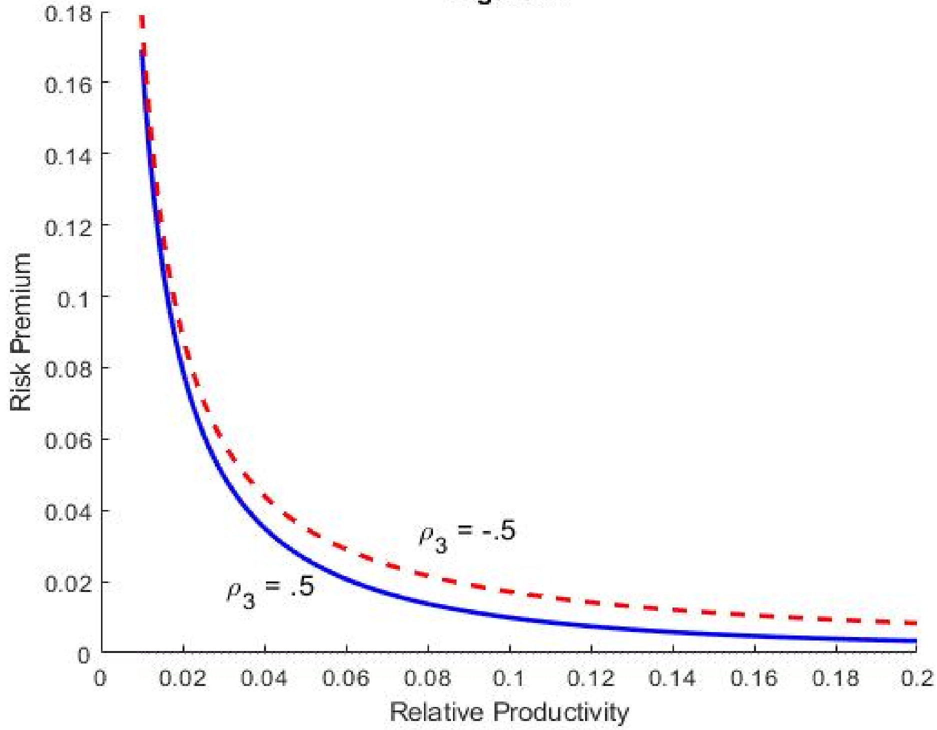


Figure 12

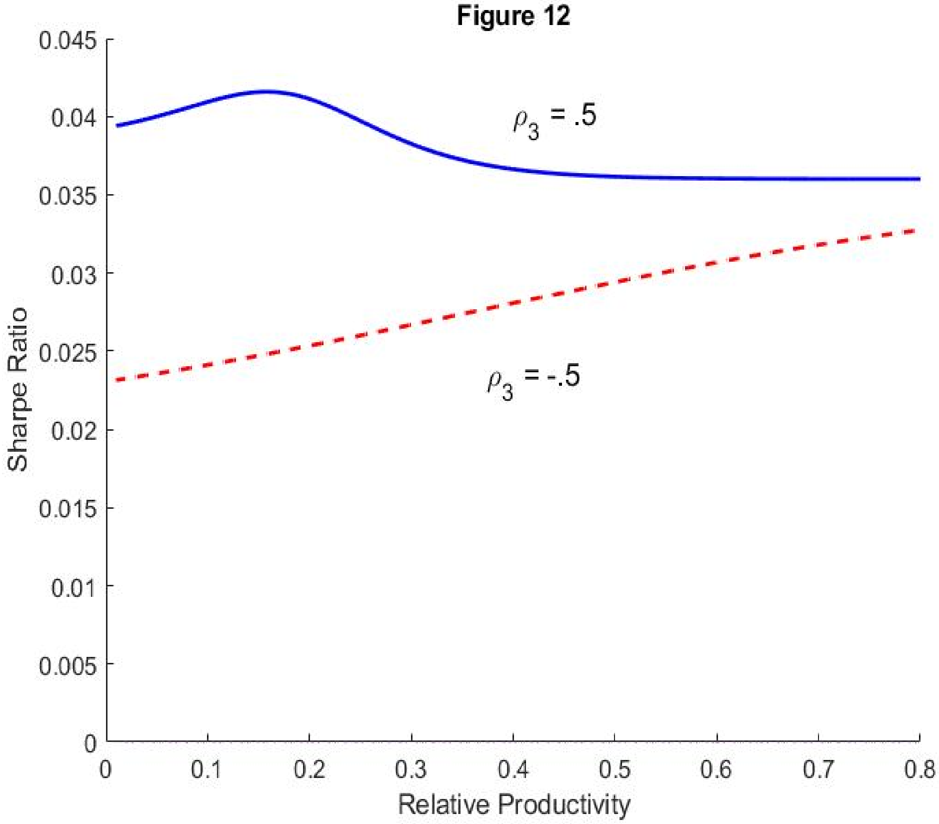
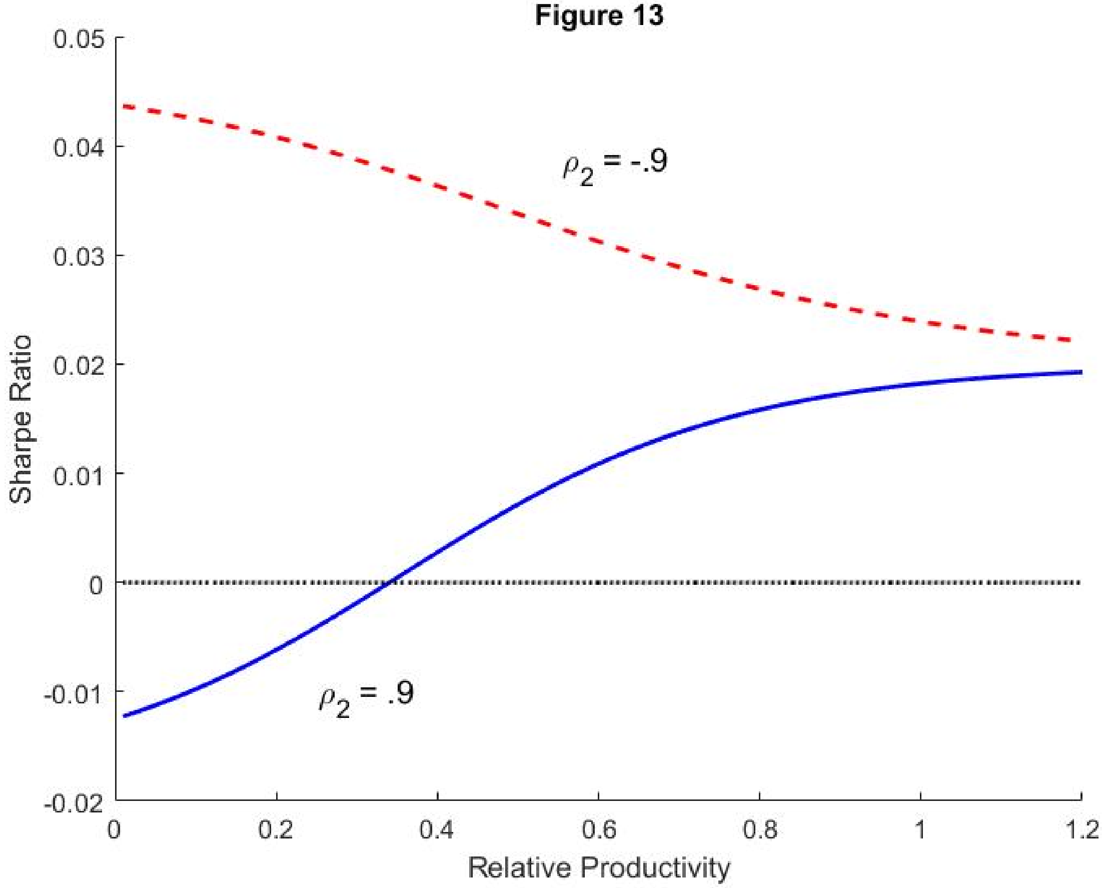


Figure 13


0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
Sharpe Ratio
ρ_2 = -.9
ρ_2 = .9
0
0.2
0.4
0.6
0.8
1
1.2
Relative Productivity