Stability and Robustness of RBF Interpolation

Jean-Luc Bouchot*
Chair for Mathematics C (Analysis), RWTH Aachen University, Pontdriesch 10
52062 Aachen, Germany
bouchot@mathc.rwth-aachen.de

Keaton Hamm †
Department of Mathematics, Vanderbilt University, 1326 Stevenson Center
Nashville, TX 37240, USA
keaton.hamm@vanderbilt.edu

Abstract

We consider how some methods of uniform and nonuniform interpolation by translates of radial basis functions – specifically the so-called general multiquadrics – perform in the presence of certain types of noise. These techniques provide some new avenues for interpolation on bounded domains that is different from the existing literature by using fast Fourier transform methods to approximate cardinal functions associated with the RBF.

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1 Introduction

The classical sampling problem may be asked in two parts: first, for a given class of signals, does it suffice to know the samples, or values, of a signal at a given discrete set of points in order to recover the signal in some manner? Second, how might the signals be recovered, and moreover, how might it be done in a computationally efficient way? There are many theoretical and practical answers to this problem in various settings, and perhaps the most fundamental result is the classical Whittaker–Kotelnikov–Shannon sampling theorem [26], which

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states that $L_2(\mathbb{R})$ functions whose Fourier transform is supported on $[-\pi, \pi]$, for example, may be recovered in L_2 and uniformly via

$$f(x) = \sum_{j\in\mathbb{Z}} f(j)\frac{\sin(\pi(x-j))}{\pi(x-j)}.$$

While Whittaker [30] saw the series above as a *cardinal interpolation series*, i.e. evaluating the right-hand side at $k \in \mathbb{Z}$ produces $f(k)$, it was later shown that the convergence was uniform for bandlimited signals.

The drawback of this sampling formula for practical considerations is that the series is difficult to approximate well by truncation since the cardinal sine function $\operatorname{sinc}(x) := \sin(\pi x)/(\pi x)$ decays slowly (like $|x|^{-1}$). There is an abundance of literature tracing back to this fundamental theorem, and correspondingly many techniques to get around the slow decay of sinc. One such method is oversampling, however this can be costly in practice. Another method intimately related to the analysis here is what I. J. Schoenberg, the father of spline theory, terms *summability methods*. Specifically, one attempts to replace sinc in the series above by another function which decays more rapidly, nonetheless requiring that the new series is close to the original signal in whatever way one wants to measure (e.g. in L_2 or uniformly).

Some study of summability methods using cardinal functions formed from translates of a single *radial basis function* (RBF) – one which satisfies $\phi(x) = \phi(|x|)$– has been made [3, 4, 5, 11, 12, 13, 17, 24, 27]. Cardinal functions are those which satisfy the interpolatory condition $L(k) = \delta_{0,k}$, $k \in \mathbb{Z}$, where of course sinc is the canonical example. Such cardinal functions fashioned from radial basis functions have a special form in the Fourier transform domain as discussed in the sequel.

Building on these results, there are many techniques for sampling at nonuniform sets in $\mathbb{R}^d$. Of course, the analysis is typically much simpler in one dimension, whereas many of the techniques that are currently known in higher dimensions rely on the geometry of the points in $\mathbb{R}^d$ in a nontrivial way. Even the first part of the classical sampling question leaves some deep open questions in this area and has seen links with many interesting realms of mathematics including space-tiling, convex geometry, basis theory, and abstract harmonic analysis. Of interest to this work are those nonuniform sampling methods which use RBFs [8, 10, 16, 25]. For a survey of some of these themes using multiquadrics, of which this article is a continuation, consult [9].

The primary contribution here is to analyze what happens to various interpolation schemes involving RBFs for either bandlimited or time-limited signals in the presence of noise. We consider two kinds of noise and the effect they have on the sampling and reconstruction of certain classes of signals, and we also give some indication of the computational feasibility and methodology for performing the sampling scheme. The primary method will be that of sampling via interpolation, and consequently, this allows us to compare our method with the traditional literature on RBF interpolation on compact domains.

The rest of the paper is laid out as follows: we list some basic definitions and facts in Section 2, followed by the definition of the interpolation method in Section 3. Section 4 recalls the recovery results in this setup for both uniform and nonuniform sampling. Section 5 uses the combination of uniform and nonuniform results from the previous section to determine what happens under two distinct types of noise. Then, in Section 6, approximation rates in terms of the spacing of the samples in one dimension and the effect of noise on them are considered. Section 7 discusses how our method compares with classical RBF interpolation theory for compactly supported functions, and we end with a brief discussion of extensions in Section 8.

2 Definitions and Basic Notions

If $\Omega \subset \mathbb{R}^d$ is a set of positive Lebesgue measure, then let $L_p(\Omega)$ be the typical Banach space of p–integrable (or essentially bounded in the case $p=\infty$) Lebesgue measurable functions on Ω with its usual norm. If no set is specified, it is to be assumed that $\Omega = \mathbb{R}^d$. Similarly, $\ell_p(I)$ are the usual sequence spaces of p–summable sequences indexed by a set I, and if no set is specified, we mean $\ell_p(\mathbb{Z}^d)$. Let $C_0(\mathbb{R}^d)$ be the space of continuous functions on $\mathbb{R}^d$ which vanish at infinity.

For $k \in \mathbb{N}$, let $W_p^k(\Omega)$ be the Sobolev space of $L_p(\Omega)$ functions whose derivatives of order at most k are all in $L_p(\Omega)$. If $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index, then let D^α be the derivative operator given by $D^\alpha g = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} g$. Then the seminorm on the Sobolev space may be defined by

$$|g|_{W_p^k(\Omega)} := \max_{|\alpha|=k} \left(\int_\Omega |D^\alpha g(x)|^p dx \right)^{\frac{1}{p}} = \max_{|\alpha|=k} \|D^\alpha g\|_{L_p(\Omega)},$$

and the following is a norm on W_p^k: $\|g\|_{W_p^k(\Omega)} := \|g\|_{L_p(\Omega)} + |g|_{W_p^k(\Omega)}$. Again, if no set is specified, we refer to $W_p^k(\mathbb{R}^d)$.

For a function $f \in L_1$, define its Fourier transform via

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \xi \rangle} dx,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^n$. Thus under suitable conditions (for example, if f is continuous and $\widehat{f} \in L_1$) the following inversion formula holds: $f(x) = (\widehat{f})^\vee(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{i\langle \xi, x \rangle} d\xi$. The Fourier transform can be uniquely extended to a linear isometry of L_2 onto inself, and under this normalization, the Parseval/Plancherel Identity states that $2\pi\|f\|_{L_2} = \|\widehat{f}\|_{L_2}$.

Next, given a set $S \subset \mathbb{R}^d$ of positive Lebesgue measure, define the Paley–Wiener space of S–bandlimited functions via

$$PW_S := \{f \in L_2(\mathbb{R}^d) : \widehat{f} = 0 \text{ a.e. outside } S\}.$$

To reduce notational encumbrance, in the univariate case, we denote the space as follows: $PW_\sigma := \{f \in L_2(\mathbb{R}) : \widehat{f} = 0 \text{ a.e. outside } [-\sigma, \sigma]\}$. The Paley–Wiener Theorem states that an equivalent definition of the latter is the space of entire functions of exponential type $\sigma > 0$ whose restriction to $\mathbb{R}$ is in L_2. As all Paley–Wiener spaces are isometrically isomorphic, we typically restrict ourselves to the canonical space PW_π; however all of the results mentioned here may be dilated to a space with different band-size.

The interpolation scheme considered in the sequel will use the following ideas for point distributions in $\mathbb{R}^d$.

Definition 2.1.

(1) A sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ is a *complete interpolating sequence* (CIS) for PW_S provided for every $a \in \ell_2(\mathbb{N})$, there exists a unique $f \in PW_S$ such that $f(x_n) = a_n$, $n \in \mathbb{N}$.

(2) A sequence $(f_n)_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is a *Riesz basis* for $\mathcal{H}$ provided (f_n) is complete and the following inequality holds for all $a \in \ell_2(\mathbb{N})$:

$$A\|a\|_{\ell_2}^2 \leq \left\| \sum_{n\in\mathbb{N}} a_n f_n \right\|_{\mathcal{H}}^2 \leq B\|a\|_{\ell_2}^2. \tag{1}$$

For Paley–Wiener spaces, complete interpolating sequences are equivalent to Riesz bases of exponentials in the corresponding L_2 space in the Fourier domain via the following theorem.

Theorem 2.2 ([31], Theorem 9, p. 143)**.** *Let $S \subset \mathbb{R}^d$ be a set of positive Lebesgue measure. Then $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ is a CIS for PW_S if and only if $\left(e^{-i\langle x_n, \cdot\rangle}\right)_{n\in\mathbb{N}}$ is a Riesz basis for $L_2(S)$.*

It should be noted that the matter of existence of a CIS for the Paley–Wiener space over a given set S is somewhat delicate whenever $d > 1$. For example, it is unknown if such a sequence exists when $S = B_2^d$, the Euclidean ball in $\mathbb{R}^d$. However, there are some examples of interest, namely finite unions of disjoint intervals in $\mathbb{R}$ [20] or disjoint cubes with parallel axes in $\mathbb{R}^d$ [21]. There are some specific types of polytopes that admit Riesz bases of exponentials as well, such as zonotopes with vertices having rational coordinates [15], or centrally symmetric polytopes whose faces of co-dimension 1 are also centrally symmetric and whose vertices lie on a lattice [7].

For subsequent use, we catalog here some facts related to Riesz bases of exponentials. First, it bears noting that in dimension 1, such bases are abundant by the following classical result.

Theorem 2.3 (Kadec's 1/4–Theorem, [14])**.** *If $(x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$ satisfies*

$$\sup_{j\in\mathbb{Z}} |x_j - j| < \frac{1}{4},$$

then (x_j) is a CIS or PW_π. Moreover, the bound is sharp, as $(e^{\pm i(n-1/4)\cdot})_{n\in\mathbb{N}}$ is not a Riesz basis for $L_2[-\pi,\pi]$.

There are higher dimensional analogues of Kadec's theorem, for example, see [1, 2, 28]. Having a sufficient condition, we also note that a necessary condition for (x_j) to be a CIS is that it is separated, i.e. $\inf_{j\neq k}|x_k - x_j| > 0$. For a complete characterization by Pavlov using Muckenhoupt's A_p condition in terms of zeros of so-called sine-type entire functions, see [22].

There are also some important notions of stability of complete interpolating sequences which will be required.

Proposition 2.4. *If $(x_j)_{j\in\mathbb{N}}$ is a CIS for PW_S, and $(y_j)_{j\in\mathbb{N}}$ is such that $y_j \neq x_j$ for only finitely many $j \in \mathbb{N}$, then (y_j) is also a CIS for PW_S.*

Theorem 2.5. *If $(x_j)_{j\in\mathbb{N}}$ is a CIS for PW_S, then there exists a positive constant L such that if $|y_j - x_j| \leq L$ for every $j \in \mathbb{N}$, then (y_j) is a CIS for PW_S.*

3 The Interpolation Scheme

The primary concern of this paper is to analyze a scheme which samples a smooth function via interpolation from a shift-invariant space of certain radial basis functions. To wit, consider the following general problem: given a function f with a certain order of smoothness (e.g. in PW_S or $W_p^k(\mathbb{R}^d)$), a separated sequence $X := (x_j)_{j\in\mathbb{N}} \subset \mathbb{R}^d$, and a radial basis function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $\phi(x) = \phi(|x|)$, we aim to find an interpolating function of the form

$$\mathscr{I}_\phi f(x) = \sum_{j\in\mathbb{N}} a_j \phi(x - x_j), \quad x \in \mathbb{R}^d, \tag{2}$$

which satisfies

$$\mathscr{I}_\phi f(x_k) = f(x_k), \quad k \in \mathbb{N}.$$

When we need to make clear the reliance on the sequence X, we will use the notation $\mathscr{I}_\phi^X f$.

The sequel will primarily emphasize interpolation using the so-called *general multiquadrics* as kernels. These are defined using two parameters via

$$\phi_{\alpha,c}(x) := (|x|^2 + c^2)^\alpha.$$

To avoid notational encumbrance, we adopt the convention $L_{\alpha,c} := L_{\phi_{\alpha,c}}$, and $\mathscr{I}_{\alpha,c} := \mathscr{I}_{\phi_{\alpha,c}}$. It should be noted that all of the techniques and results in what follows have analogues for many different kernels; however, for ease of exposition, we focus on the general multiquadrics and note the extensions in Section 8.

Let us first note that if $c > 0$, $\alpha < -1/2$ and X is a CIS for PW_π, then for any $f \in PW_\pi$, a multiquadric interpolant $\mathscr{I}_{\alpha,c} f$ as in (2) exists [16]. Furthermore, the interpolant is unique (i.e. the sequence (a_j) is uniquely determined as the solution to the equation $\mathcal{M}a = y$, where $y_j = f(x_j)$, and $\mathcal{M} := (\phi(x_j - x_k))_{j,k\in\mathbb{N}}$), and $\mathscr{I}_{\alpha,c}$ is a bounded linear operator from $PW_\pi \to C_0 \cap L_2(\mathbb{R})$.

4 Recovery Results

Here we recall some of the recovery results for bandlimited functions using the interpolation method set out in the previous section.

Theorem 4.1 (cf. [9], Theorem IV.1)**.** *Let $\alpha < -1/2$ and let X be a complete interpolating sequence for PW_π. If $f \in PW_\pi$, then $\mathscr{I}^X_{\alpha,c} f \in L_2(\mathbb{R})$ and*

$$\lim_{c\to\infty} \|\mathscr{I}^X_{\alpha,c} f - f\|_{L_2(\mathbb{R})} = 0,$$

and

$$\lim_{c\to\infty} |\mathscr{I}^X_{\alpha,c} f(x) - f(x)| = 0$$

uniformly on $\mathbb{R}$.

Moreover, if $f \in PW_\sigma$ for some $\sigma < \pi$,

$$\|\mathscr{I}^X_{\alpha,c} f - f\|_{L_2(\mathbb{R})} \leq C e^{-c(\pi-\sigma)} \|f\|_{L_2(\mathbb{R})}, \tag{3}$$

where the constant C depends on α and X, but not on c.

While the first part of this theorem only says something about the asymptotic behaviour of the interpolants for functions whose Fourier transform is fully supported in the band of the Paley-Wiener space, we nonetheless obtain exponential convergence in terms of the shape parameter, c, of the multiquadric when oversampling, corresponding to the same notions in classical sampling theory. Currently, no good approximation rates in terms of c are known when $\widehat{f}$ has support on the full interval $[-\pi, \pi]$.

The following shows similar convergence phenomena in higher dimensions. We eliminate some of the details, but the main idea is that since it is unknown whether or not Riesz bases of exponentials for the Euclidean ball exist, one approximates the ball with convex bodies (zonotopes) which do have such bases in certain cases. Additionally, the interpolation is of functions in the Paley-Wiener space over a smaller ball, and thus exponential decay rates are achieved because the method is one of oversampling.

Theorem 4.2 ([10], Theorem 4.7)**.** *Let $\alpha < -d/2$. Suppose that $\delta \in (2/3, 1)$ and $\beta \in (0, 3\delta - 2)$. Suppose that S is a symmetric convex body such that $\delta B_2^d \subset S \subset B_2^d$ and that $(e^{-i\langle x_k, \cdot \rangle})_{k\in\mathbb{Z}}$ is a Riesz basis for $L_2(S)$. Then for every $f \in PW_{\beta B_2}$,*

$$\lim_{c\to\infty} \|\mathscr{I}_{\alpha,c}^X f - f\|_{L_2(\mathbb{R}^d)} = 0,$$

and

$$\lim_{c\to\infty} |\mathscr{I}_{\alpha,c}^X f(x) - f(x)| = 0$$

uniformly on $\mathbb{R}^d$. Moreover, there exists a constant $C > 0$ such that for every $f \in PW_{\beta B_2}$,

$$\max\{\|\mathscr{I}_{\alpha,c}^X f - f\|_{L_2(\mathbb{R}^d)}, \|\mathscr{I}_{\alpha,c}^X f - f\|_{L_\infty(\mathbb{R}^d)}\} \leq C \left(\frac{c\beta}{\delta}\right)^{\alpha + \frac{d+1}{2}} e^{-c(3\delta - 2 - \beta)} \|f\|_{L_2(\mathbb{R}^d)}, \tag{4}$$

where C is independent of c.

5 Interpolation in the Presence of Noise

Given the preliminaries above, we turn our attention to considering how the interpolation scheme behaves in the presence of noise. There are two main kinds of noise that will be considered: noisy data, and so-called *jitter error.*

5.1 Stability under perturbation of sample points

Jitter error corresponds to the case when the sample points X are perturbed. That is, instead of sampling at $X := (x_j)_{j\in\mathbb{Z}}$, we sample at $\tilde{X} := (\tilde{x_j})_{j\in\mathbb{Z}}$ with $\tilde{x_j} = x_j + \varepsilon_j$ for some bounded perturbation $(\varepsilon_j) \in \ell_\infty(\mathbb{Z})$. Physically, this may correspond to non-ideal sensors which have some error in the timing of the sampling.

Notice that it follows from Theorems 4.1 and 4.2 that the recovery results therein are independent under perturbations of the sample points at least as long as the perturbed points still form a complete interpolating sequence. So if $\tilde{X} = X + \varepsilon$ is a complete interpolating sequence for the Paley Wiener space, we have

$$\lim_{c\to\infty} \|I_{\alpha,c}^{\tilde{X}} f - f\|_{L_2(\mathbb{R}^d)} = \lim_{c\to\infty} \|I_{\alpha,c}^X f - f\|_{L_2(\mathbb{R}^d)} = 0.$$

Of course, the rate of convergence may differ, though it is difficult to relate how.

Proposition 5.1. *Suppose that X satisfies Kadec's 1/4–Theorem, and $\sup_{j\in\mathbb{Z}} |x_j - j| = L < 1/4$. Then if $\|(\varepsilon_j)_j\|_\infty < 1/4 - L$, $\tilde{X}$ given by $\tilde{x_j} = x_j + \varepsilon_j$ is a CIS for PW_π.*

Proof. Notice that $\tilde{X}$ still satisfies the condition of Kadec's Theorem. $\square$

Similarly, if $\varepsilon_j = 0$ for all but finitely many j, Proposition 2.4 implies that Y is again a CIS. Theorem 2.5 also implies that for any CIS X, there exists a constant L such that if $\|\varepsilon\|_{\ell_\infty} \leq L$, then $\tilde{X}$ is again a CIS.

However, one drawback is that this L may be very small. One can see this, for example, because the 1/4-Theorem is sharp, so if X was perturbed from the integer lattice arbitrarily close to 1/4, a small perturbation might fail. Using the example in Theorem 2.3, one can take $x_n = n - 1/4 - \varepsilon$ for $n > 0$ and $x_n = -n + 1/4 + \varepsilon$ for $n < 0$, where $\varepsilon > 0$ is fixed. Then there is a perturbation of X of norm ε which fails to be a CIS.

Nonetheless, we may make some estimate on L based not on the magnitude of $\|\varepsilon\|_{\ell_\infty}$, but on the so-called *frame bounds* of the basis $(e^{-ix_j\cdot})_j$. Note that it follows from (1) that with the same constants $A, B > 0$ (the frame bounds), for any $f \in PW_\pi$,

$$A\|f\|_{L_2(\mathbb{R})}^2 \leq \sum_{j\in\mathbb{Z}} |f(x_j)|^2 \leq B\|f\|_{L_2(\mathbb{R})}^2. \tag{5}$$

The following can be found in [6]:

Proposition 5.2. *Suppose that $(e^{-ix_j\cdot})_{j\in\mathbb{Z}}$ is a Riesz basis for $L_2[-\pi,\pi]$ with frame bounds $A, B > 0$. Then if $0 < L < \pi^{-1}\ln\left(\sqrt{\frac{A}{B}}+1\right)$, and $\tilde{x_j} = x_j + \varepsilon_j$, with $\|\varepsilon\|_{\ell_\infty} \leq L$, $(e^{-i\tilde{x_j}\cdot})_{j\in\mathbb{Z}}$ is a Riesz basis for $L_2[-\pi,\pi]$ with frame bounds $A(1-\sqrt{C})^2$ and $B(1+\sqrt{C})^2$, where $C = \frac{B}{A}(e^{\pi L}-1)^2$.*

5.2 Robustness to noisy samples

Consider what happens if, instead of sampling $f(x_j)$ exactly, we actually measure $y_j = f(x_j) + \delta_j$. For now, assume that $(\delta_j) \in \ell_2$, and $\|(\delta_j)_j\|_{\ell_2} \leq \delta$. In this case, the noise is added to the signal, and could appear as random background noise, or in some cases deterministic (or adversarial) noise. There are many ways to model such noise, but our focus here will be on that which is square-summable.

Given noisy samples, let $\widetilde{I_{\alpha,c}^X f}$ be the interpolant of the data y_j. Note that $(y_j) \in \ell_2$ by the condition on the noise sequence (δ_j). Consequently, on account of Definition 2.1, there is a unique $g \in PW_S$ such that $g(x_j) = y_j$. Thus, by uniqueness of the interpolant, there is a unique $g \in PW_S$ such that $\widetilde{I_{\alpha,c}^X f} = I_{\alpha,c}^X g$, and the following holds.

Theorem 5.3. *Let S and X be as in Theorem 4.1 or Theorem 4.2, and let $y_j = f(x_j) + \delta_j$ with $\|(\delta_j)\|_{\ell_2} \leq \delta$. Then for every $f \in PW_S$,*

$$\|\widetilde{I_{\alpha,c}^X f} - f\|_{L_2} \leq \frac{\delta}{\sqrt{A}} + o(1), \quad c \to \infty,$$

where A is as in (5).

Proof. Let $g \in PW_S$ be the function described above. Then we have

$$\|\widetilde{I^X_{\alpha,c}}f - f\|_{L_2} = \|I^X_{\alpha,c}g - f\|_{L_2} \leq \|I^X_{\alpha,c}g - g\|_{L_2} + \|g - f\|_{L_2} =: N_1 + N_2.$$

It follows from Theorem 4.2 that $N_1 = o(1)$, $c \to \infty$ (and in fact we may take the upper bound of N_1 to be the right hand side of (3) or (4) if one applies). Applying (5), we estimate

$$\|g - f\|_{L_2} \leq A^{-\frac{1}{2}}\|(g(x_j) - f(x_j))_j\|_{\ell_2} = A^{-\frac{1}{2}}\|(\delta_j)_j\|_{\ell_2} \leq A^{-\frac{1}{2}}\delta.$$

□

6 Approximation Rates Based on Spacing

As discussed in the previous section, the approximation rates in terms of the shape parameter, c, of the multiquadric are maintained in the presence of noise (hence the error of approximation is dominated by the ℓ_2 norm of the noise as in Theorem 5.3). But another type of approximation rate has been considered. Specifically, we fix a CIS, X, and consider interpolation at hX for $0 < h \leq 1$, and we tune the shape parameter of the multiquadric to reflect the dilation. That is, we interpolate from the space

$$\left\{\sum_{j\in\mathbb{Z}} a_j\phi_{\alpha,1}(\cdot - hx_j) = \sum_{j\in\mathbb{Z}} a_j\left(|\cdot - hx_j|^2 + 1\right)^\alpha : (a_j)_{j\in\mathbb{Z}} \subset \mathbb{R}\right\}.$$

In the uniform interpolation setting ($X = \mathbb{Z}$) these interpolants have a special structure, which will be discussed later. To emphasize the distinction (and the reliance of the shape parameter on h) we write this new interpolant in a different manner as $I^{hX}_\alpha f$. One may show that the relation to the original interpolant is $I^{hX}_\alpha f(x) = \frac{1}{h}\mathscr{I}_{\alpha,h^{-1}}f^h\left(\frac{x}{h}\right)$, where $f^h(x) := hf(hx)$. When α is fixed, we drop the subscript to ease the notation.

To begin our analysis of the effect of noise on this process, we recall the following.

Theorem 6.1 ([8], Theorem 3.4)**.** *Suppose that $\alpha < -1/2$, $k \in \mathbb{N}$, $0 < h \leq 1$, and X is a CIS for PW_π. Then there exists a constant C independent of h such that for every $f \in W^k_2(\mathbb{R})$,*

$$\|I^{hX}f - f\|_{L_2(\mathbb{R})} \leq Ch^k|f|_{W^k_2(\mathbb{R})}. \tag{6}$$

Remark 6.2. Note again that the estimate in Theorem 6.1 is invariant under perturbing the CIS in a certain manner. Specifically, if X is a CIS for PW_π, and so is Y, then (6) holds for both interpolants albeit with a different constant C. Moreover, one finds via the triangle inequality that

$$\|I^{hX}f - I^{hY}f\|_{L_2} \leq (C_X + C_Y)h^k|f|_{W^k_2}. \tag{7}$$

6.1 Noise in nonuniform interpolation

To discuss the question of reconstruction from noisy data in the setting we have described in this section, it is pertinent to recall a theorem on the stability of interpolating a given Sobolev function via a bandlimited one as an intermediate step to analyzing the interpolant.

Theorem 6.3 ([8], Theorem 3.1)**.** *Let $k\in\mathbb{N}$, $h>0$, and let X be a fixed CIS for PW_π. There exists a constant $C=C_{k,X}$, independent of h, such that for every $f\in W_2^k(\mathbb{R})$, there exists a unique $F\in PW_{\frac{\pi}{h}}$ such that*

$$F(hx_j)=f(hx_j),\quad j\in\mathbb{Z},$$

$$|F|_{W_2^k}\leq C|f|_{W_2^k},$$

and

$$\|F-f\|_{L_2}\leq Ch^k|f|_{W_2^k}.$$

We also have the following uniform bound on the interpolants.

Theorem 6.4 ([8], Theorem 3.3)**.** *For each $k\geq 0$, there is a constant C, independent of h, such that*

$$|I^{hX}f|_{W_2^k}\leq C|f|_{W_2^k},\quad f\in W_2^k(\mathbb{R}).$$

Suppose $h>0$ is fixed. Again, suppose that we measure $y_j(h)=f(hx_j)+\delta_j(h)$ with $\sup\limits_h\|(\delta_j(h))_j\|_{\ell_2}\leq\delta$. Let $\widetilde{I^{hX}}f$ be the interpolant of $y_j(h)$. Then we have the following rate of approximation.

Theorem 6.5. *Under the assumptions of Theorem 6.3, there is a constant C such that, for every $f\in W_2^k(\mathbb{R})$,*

$$\|\widetilde{I^{hX}}f-f\|_{L_2}\leq Ch^k|f|_{W_2^k}+\frac{\delta}{\sqrt{A}},$$

where A is the lower frame bound for X given by (5)*.*

Proof. The first key observation is that hX is a CIS for $PW_{\frac{\pi}{h}}$, which can be seen because Riesz bases are preserved under bounded, invertible linear transformations. Consequently, let $g\in PW_{\frac{\pi}{h}}$ be the unique function such that $g(hx_j)=y_j(h)$. Then we have

$$\|\widetilde{I^{hX}}f-f\|_{L_2}=\|I^{hX}g-f\|_{L_2}\leq\|I^{hX}g-I^{hX}f\|_{L_2}+\|I^{hX}f-f\|_{L_2}=:N_1+N_2.$$

Theorem 6.1 implies that $N_2\leq Ch^k|f|_{W_2^k}$. Secondly, Theorem 6.4 with $k=0$ implies that

$$N_1\leq\|I^{hX}(g-f)\|_{L_2}\leq C\|g-f\|_{L_2}.$$

Let $F \in PW_{\frac{\pi}{h}}$ be the function given by Theorem 6.3 such that $F(hx_j) = f(hx_j)$. Then we have

$$\|g - f\|_2 \leq \|g - F\|_2 + \|F - f\|_2 \leq \|g - F\|_2 + Ch^k |f|_{W_2^k}.$$

From (5),

$$\|g - F\|_2 \leq A^{-\frac{1}{2}} \|(g(hx_j) - f(hx_j))_j\|_{\ell_2} = A^{-\frac{1}{2}} \|(\delta_j(h))_j\|_{\ell_2} \leq A^{-\frac{1}{2}} \delta.$$

Thus $N_1 \leq Ch^k |f|_{W_2^k} + A^{-\frac{1}{2}} \delta$.

Combining the estimates on N_1 and N_2 completes the proof. □

6.2 Uniform Sampling

In the uniform case (when X is a lattice), the interpolation scheme we have discussed has some special properties, including the possibility of using growing kernels such as multiquadrics with positive α. Additionally, the interpolants themselves have a simpler form as they lie in the span of shifts of a single *cardinal function* which behaves like the classical cardinal sine function.

Given a multiquadric, we form a cardinal function $L_{\alpha,c}$ satisfying $L_{\alpha,c}(j) = \delta_{0,j},\ j \in \mathbb{Z}$, which lies in the closure of the linear span of $(\phi_{\alpha,c}(\cdot - j))_{j \in \mathbb{Z}}$. Then the multiquadric interpolant can be written as

$$\mathscr{I}_{\alpha,c}^{\mathbb{Z}} f(x) := \sum_{j \in \mathbb{Z}} f(j) L_{\alpha,c}(x - j). \tag{8}$$

The cardinal function $L_{\alpha,c}$ can be defined by its Fourier transform:

$$\widehat{L_{\alpha,c}}(\xi) := \frac{\widehat{\phi_{\alpha,c}}(\xi)}{\sum\limits_{k \in \mathbb{Z}} \widehat{\phi_{\alpha,c}}(\xi + 2\pi k)}, \quad \xi \in \mathbb{R} \setminus \{0\}. \tag{9}$$

Note that the series in (8) is rather similar to the series in the classical Whittaker-Kotelnikov-Shannon sampling theorem if $f \in PW_\pi$, but with the sinc function replaced by the cardinal function associated with the general multiquadric. This indeed was the observation made by Schoenberg, who instigated the study of cardinal spline interpolation. The point was to apply a sort of summability method to the sinc series in the sampling theorem because the fact that $\operatorname{sinc}(x) = O(|x|^{-1})$ implies that it takes a rather large number of terms to well-approximate the series via truncation. If the cardinal functions $L_{\alpha,c}$ decay faster than the sinc function, then the series in (8) will be easier to approximate by truncation; of course, the question then is whether or not the cardinal interpolant $\mathscr{I}_{\alpha,c}^{\mathbb{Z}} f$ is close to f (either in L_2 or uniformly depending on the kind of guarantees one desires). For an in-depth study of the decay rates of the cardinal functions associated with general multiquadrics, see [11], especially Section

4 therein. For almost all values of α, $L_{\alpha,c}$ decays faster than $|x|^{-1}$. We note also that for the cases $\alpha = \pm 1/2$, such considerations were already made by Buhmann [4] and Riemenschneider and Sivakumar [23]. Additionally, L_p approximation rates of the same order as in Theorem 6.1 for interpolation at $h\mathbb{Z}$ can be found in [12].

6.3 Interpolation of Compactly Supported Functions

Let us now turn our attentions to some more practical considerations which may prove useful in applications. Of course, everything from Theorem 5.3 to Theorem 6.1 holds whenever we take $X = \mathbb{Z}$, which is clearly a CIS for PW_π. Let us consider for the moment what happens whenever we consider interpolation of univariate compactly supported Sobolev functions. To wit, consider $g \in W_2^k(\mathbb{R})$ with $\mathrm{supp}(g) \subset [-1, 1]$ (this choice of interval is arbitrary for ease of presentation, and can easily be dilated). Then for $N \in \mathbb{N}$, the interpolant $I^{N^{-1}\mathbb{Z}} f$ is actually interpolating f at the sequence $\{\frac{j}{N}\}_{j=-N}^{N}$, and has the following form via the relation established at the beginning of this section:

$$I^{N^{-1}\mathbb{Z}} f(x) = \sum_{j=-N}^{N} f\left(\frac{j}{N}\right) L_{\alpha,c}(Nx - j). \tag{10}$$

Consequently, Theorem 6.1 shows that the approximation rate in this case has an upper bound in terms of the number of sample points. Namely, if $X = \{\frac{j}{N}\}_{j=-N}^{N}$, then

$$\|I^{N^{-1}\mathbb{Z}} f - f\|_{L_2} \leq C N^{-k} |f|_{W_2^k} = C|X|^{-k} |f|_{W_2^k}. \tag{11}$$

Also, Theorem 5.3 still holds with h replaced by $|X|^{-1}$ as well.

But now, consider $N^{-1}X = \{\frac{x_j}{N}\}_{J=-N}^{N}$ to be an arbitrary set of distinct points in $[-1, 1]$ (i.e. x_j are arbitrary in $[-N, N]$). Then let

$$\widetilde{x_k} := \begin{cases} k, & |k| > N \\ x_k, & |k| \leq N. \end{cases}$$

Evidently, $\widetilde{X}$ is a CIS for PW_π because it was formed by perturbing only finitely many integers. It follows from (7) that

$$\begin{aligned} \|I^{N^{-1}\widetilde{X}} f - f\|_{L_2} &\leq \|I^{N^{-1}\widetilde{X}} f - I^{N^{-1}\mathbb{Z}} f\|_{L_2} + \|I^{N^{-1}\mathbb{Z}} f - f\|_{L_2} \\ &\leq (C_{\widetilde{X}} + 2C_{\mathbb{Z}}) N^{-k} |f|_{W_2^k}. \end{aligned} \tag{12}$$

The constant $C_{\widetilde{X}}$ depends on a few things: the minimum spacing of the sequence (i.e. $\inf_{j \neq k} |x_j - x_k|$), and the frame bounds for the basis as in (1). However, by Proposition 5.2, we have the following.

Theorem 6.6. *Let $L < \ln(2)/\pi$. There is a constant C such that for any $\widetilde{X}$ with $\sup_{k\in\mathbb{Z}}|\widetilde{x_k} - k| < L$,*

$$\|I^{N^{-1}\widetilde{X}}f - f\|_{L_2} \leq CN^{-k}|f|_{W_2^k}, \quad f \in W_2^k[-1,1].$$

This theorem implies that if one wishes to approximate a compactly supported f by its interpolant $I^{N^{-1}\widetilde{X}}f$, it suffices to consider the more simple uniform interpolant $I^{N^{-1}\mathbb{Z}}f$. The usefulness of this will be discussed further in the next section.

7 Computational Feasibility

In this section, we investigate the computational feasibility, peculiarities, and potential advantages of the interpolation based on cardinal functions compared to traditional radial basis function (RBF) theory. For ease of presentation we limit our discussion to problems in one dimension. As discussed in Section 6, consider a function $f \in W_2^k(\mathbb{R})$ whose support lies inside $[-1,1]$, and we interpolate at a sequence of points $(x_j)_{j=-N}^N \subset [-1,1]$. The classical method using RBFs is to do interpolate from the linear span of $\{\phi_{\alpha,c}(\cdot - x_j) : j = -N, \cdots, N\}$. The drawback in this case if ϕ is a multiquadric or the Gaussian kernel is that forming the interpolant can be computationally quite expensive as it is formed by inverting the matrix $\mathcal{M}_N := (\phi_{\alpha,c}(x_k - x_j))_{k,j=-N}^N$ and applying it to the data vector $y_j = f(x_j)$. Part of the problem is that if the minimum spacing of the points is h, then the condition number for this matrix can be as bad as e^{1/h^2} ([19]), which is very bad. The other disadvantage of this framework is that it is not robust to noise. RBF interpolation is very good at recovering smooth functions, but is sensitive to noise unless other smoothing techniques are applied.

On the other hand, given a sequence $(x_j)_{j=-N}^N \subset [-1,1]$, by (12) and Theorem 6.6, we may simply use the uniform interpolant $I^{N^{-1}\mathbb{Z}}f$ to approximate f. The benefit of this, is that $I^{N^{-1}\mathbb{Z}}f$ is less difficult to compute. Indeed, one must first estimate the function $\widehat{L_{\alpha,c}}$ by truncating the series in the denominator of (9), then evaluate $L_{\alpha,c}$ via the Fast Fourier Transform (FFT). Then one directly forms the series in (10) from the already known sample values $f(j/N)$. Moreover, as discussed in both of the previous two sections, this method enjoys the advantage of being robust to noise. Notice however, that this interpolation scheme is different in the sense that $I^{N^{-1}\mathbb{Z}}f$ is in the span of $(L_{\alpha,c}(\cdot - j))_{j=-N}^N$, which in turn is in the span of $(\phi_{\alpha,c}(\cdot - j))_{j\in\mathbb{Z}}$, as opposed to only the span of $2N+1$ translates of the multiquadric.

7.1 Approximation of the Fourier transform of the cardinal function

As stated above, one first needs to truncate the series in the denominator of the Fourier transform of the cardinal function. It is known (see for instance [29, Theorem 8.15]) that the Fourier transform of a multiquadric is given (in one dimension) by

$$\widehat{\phi_{\alpha,c}}(\xi) = \frac{2^{1+\alpha}}{\Gamma(-\alpha)} \left(\frac{c}{|\xi|}\right)^{\alpha+1/2} K_{\alpha+1/2}(c|\xi|), \quad \xi \in \mathbb{R}\backslash\{0\}, \tag{13}$$

where $K_\nu(r) := \int_0^\infty e^{-r\cosh(t)} \cosh(\nu t)\mathrm{d}t, \quad r > 0, \nu \in \mathbb{R}$ is the modified Bessel function of the second kind. Note that these functions have a pole at the origin and decay exponentially.

It follows that the truncation of the series in the Fourier transform of the cardinal function associated with the general multiquadrics is possible thanks to the fast decay of the Bessel function. In particular, we have the following.

Theorem 7.1. *Let $\varepsilon > 0$. Let $\alpha \in \mathbb{R}$ and $\alpha < 0$. For any $c > 0$, there exists a natural number $\tau := \tau_{c,\alpha,\varepsilon} \in \mathbb{N}$, such that for all $\xi \in \mathbb{R}$, there exists a $k_\xi \in \mathbb{Z}$ with*

$$\left|\sum_{k\in\mathbb{Z}} \widehat{\phi_{\alpha,c}}(\xi + 2\pi k) - \sum_{k=k_\xi-\tau}^{k_\xi+\tau} \widehat{\phi_{\alpha,c}}(\xi + 2\pi k)\right| \le \varepsilon \left|\sum_{k\in\mathbb{Z}} \widehat{\phi_{\alpha,c}}(\xi + 2\pi k)\right|. \tag{14}$$

Proof. First note that $S_{\alpha,c}(\xi) := \sum_{k\in\mathbb{Z}} \widehat{\phi_{\alpha,c}}(\xi + 2\pi k)$ is 2π-periodic. It is straightforward to see that (14) is equivalent to finding τ such that for any $\xi^* \in (-\pi, \pi)$

$$\left|\sum_{k\in\mathbb{Z}} \widehat{\phi_{\alpha,c}}(\xi^* + 2\pi k) - \sum_{k=-\tau}^{\tau} \widehat{\phi_{\alpha,c}}(\xi^* + 2\pi k)\right| \le \varepsilon \left|\sum_{k\in\mathbb{Z}} \widehat{\phi_{\alpha,c}}(\xi^* + 2\pi k)\right|.$$

From [29, Lemma 5.13] it follows that for $\nu \in \mathbb{R}$, $0 \le K_\nu(r) \le \sqrt{2\pi} r^{-1/2} e^{-r} e^{\nu^2/(2r)}$. In particular, let $r_k := \xi^* + 2\pi k > 0$, for some $k > \tau$; it follows, with $\nu = \alpha + 1/2$ that

$$\widehat{\phi_{\alpha,c}}(r_k) \le \frac{2^{1+\alpha}}{\Gamma(-\alpha)} c^\alpha \sqrt{2\pi} r_k^{-\alpha-1} e^{-cr_k} e^{\frac{(\alpha+1/2)^2}{2cr_k}}.$$

With $\lambda := \frac{2^{1+\alpha}}{\Gamma(-\alpha)} c^\alpha \sqrt{2\pi}$, this expression simplifies to $\widehat{\phi_{\alpha,c}}(r_k) \le \lambda r_k^{-\alpha-1} e^{-cr_k} e^{\frac{(\alpha+1/2)^2}{2cr_k}}$. For k large enough, there exists a constant $\gamma > 0$ such that

$$\widehat{\phi_{\alpha,c}}(r_k) \le \gamma e^{-cr_k}. \tag{15}$$

Plugging back in the definition of r_k yields

$$\widehat{\phi_{\alpha,c}}(r_k) \le \gamma e^{-c\xi^*} e^{-2\pi ck}, \quad \text{for any } k > \tau.$$

Similarly, given $k < -\tau$, we arrive at the following estimate:

$$\widehat{\phi_{\alpha,c}}(r_k) \leq \gamma e^{c\xi^*} e^{2\pi ck}, \quad \text{for any } k < -\tau.$$

Summing for all k outside of $\{-\tau, \cdots, \tau\}$ finally yields, for $\xi \in \mathbb{R}$

$$\begin{aligned}\left|\sum_{k\in\mathbb{Z}} \widehat{\phi_{\alpha,c}}(\xi + 2\pi k) - \sum_{k=k_\xi-\tau}^{k_\xi+\tau} \widehat{\phi_{\alpha,c}}(\xi + 2\pi k)\right| &= \sum_{k<-\tau} \widehat{\phi_{\alpha,c}}(\xi^* + 2\pi k) + \sum_{k>\tau} \widehat{\phi_{\alpha,c}}(\xi^* + 2\pi k) \\ &\leq \sum_{k<-\tau} \gamma e^{2\pi c} e^{-c\xi^*} e^{2\pi ck} + \sum_{k>\tau} \gamma e^{-c\xi^*} e^{-2\pi ck} \\ &\leq \frac{2\gamma e^{-c\xi^*}}{1 - e^{-2\pi c}} e^{-2\pi c\tau}\end{aligned}$$

The sum for $k \in \mathbb{Z}$ can be estimated from below (see for instance [11, Proof of Prop.3.2]) by

$$\sum_{k\in\mathbb{Z}} \widehat{\phi_{\alpha,c}}(\xi) \geq D^{-4\pi c},$$

for a certain constant $D := D_{\alpha,c} > 0$. Therefore, for (14) to be valid, it suffices to find τ such that

$$\frac{2\gamma e^{-c\xi^*}}{1 - e^{-2\pi c}} e^{-2\pi cN} \leq \varepsilon D e^{-4\pi c},$$

which is achieved whenever (noticing that $\xi^* > 0$)

$$\tau \geq 2 + \frac{\ln(\varepsilon^{-1})}{2\pi c} + \frac{\ln\left(\frac{2\gamma}{(1-e^{-2\pi c})D}\right)}{2\pi c}.$$

□

Remark 7.2. A careful analysis of the proof of Proposition 3.2 from [11] gives insight on how to pick D. For instance, for $0 > \alpha \geq -1$, one can choose $D \leq \beta \frac{2^{1+\alpha}}{\Gamma(-\alpha)} c^\alpha (2\pi)^{-\alpha-1} e^{-2\pi c}$, where $\beta := \beta_\alpha$ is given in [29, Corollary 5.12].

Remark 7.3. The constant γ appearing in (15) can be easily picked in some particular cases. For instance, for $\alpha = -1$, then $\gamma := \frac{\sqrt{2\pi}}{c} e^{\frac{1}{16c\pi}}$. In this case, Theorem 7.1 is satisfied for $\tau \geq 2 + \frac{\ln\left(\frac{2\sqrt{2\pi}}{c}\right)}{2\pi c} + \frac{1}{32\pi^2 c^2} - \frac{\ln\left((1-e^{-2\pi c})D\right)}{2c\pi}$.

7.2 Particular cases of Theorem 7.1

Poisson kernel: case $\alpha = -1$ This case is associated to the approximation using a Poisson kernel as the basis function. The Bessel function involved can be simplified to $K_{-1/2}(r) = \sqrt{\frac{\pi}{2r}}e^{-r}$. Carrying out a similar analysis as in the proof of Theorem 7.1 with $r_k = \xi^* + 2\pi k \neq 0$ and $k > \tau \geq 1$, yields

$$\begin{aligned}\widehat{\phi_{-1,c}}(r_k) &= c^{-1}\sqrt{\frac{\pi}{2}}|r_k|^{-1/2}e^{-c|r_k|},\\ &\leq \frac{e^{-c\xi^*}}{2c}e^{-2\pi ck}.\end{aligned}$$

When $k < -\tau \leq -1$, $\widehat{\phi_{-1,c}}(r_k) \leq \frac{e^{c\xi^*}}{2c}e^{2\pi ck}$. Hence,

$$\begin{aligned}\sum_{k>\tau}\widehat{\phi_{-1,c}}(r_k) &\leq \frac{e^{-c(\xi^*+2\pi)}}{2c}\frac{e^{-2\pi c\tau}}{1-e^{-2\pi c}}, \qquad \text{and}\\ \sum_{k<-\tau}\widehat{\phi_{-1,c}}(r_k) &\leq \frac{e^{c(\xi^*-2\pi)}}{2c}\frac{e^{-2\pi c\tau}}{1-e^{-2\pi c}},\end{aligned}$$

whereby,

$$\sum_{|k|>\tau}\widehat{\phi_{1,c}}(\xi^*+2\pi k) \leq \frac{e^{c(\xi^*-2\pi)}}{c(1-e^{-2\pi c})}e^{-2\pi c\tau}.$$

We want now to ensure the condition $\frac{e^{c(\xi^*-2\pi)}}{c(1-e^{-2\pi c})}e^{-2\pi c\tau} \leq \varepsilon De^{-4\pi c}$ where D is given by Remark 7.2 as follows:

$$D_{-1} = \frac{1}{2c\sqrt{2}}. \tag{16}$$

Finally, putting everything together and using the fact that $\xi^* - 2\pi < 0$, we get that

$$\tau \geq \frac{\ln(\varepsilon^{-1})}{2\pi c} + \frac{1}{2} + \frac{3\ln(2)}{4\pi c} - \frac{\ln(1-e^{-2\pi c})}{2\pi c} \tag{17}$$

ensures a relative error of the truncated series within ε, for any $\varepsilon > 0$. As an example, let us consider the accuracy of a single precision machine $\varepsilon = 10^{-16}$ and a shape coefficient $c = 1$. In this case, $\tau \geq 6.53$ is sufficient. In other words, only 15 evaluations are required for an accurate estimation of the Fourier transform of the cardinal function. Similarly, for a double precision machine with $\varepsilon = 10^{-32}$, only 27 coefficients in the expansion are sufficient for the accurate estimation of the Fourier transform.

Gaussian case As mentioned before, every result stated here holds when the Gaussian kernel, $g_\lambda := e^{-\lambda|\cdot|^2}$, is used. Typically, one considers $0 < \lambda \leq 1$, and the limiting results above hold for $\lambda \to 0^+$. For simplicity, we omit the calculations for the Gaussian as they are rather similar to the Poisson kernel. One finds that to obtain relative error ε, one needs $\tau \geq \frac{2}{\pi^2}|\ln(\varepsilon/4)| + 4$ terms, whence for machine precision, 12 terms are sufficient.

Remark 7.4. In the end, starting with (10), we are left with the numerical approximation of the cardinal function at a given point $x \in [-N, N]$. This can be done by first evaluating its (approximate) Fourier transform $\widehat{L_{\alpha,c}}(\xi)$ at the sampling points $\xi_k = k/(2N+1)$, for $-N \leq k \leq N$. Following Theorem 7.1, it suffices to evaluate Bessel functions at only few sampling points and sum them together. Finally, a direct application of a Discrete Fourier Transform allows for the computation of the cardinal function $L_{\alpha,c}$ at some (uniform) sampling points $(x_j)_{j\in\mathcal{J}}$ for some uniform finite set $\mathcal{J} \subset [-N, N]$ via

$$x(k) = \sum_{n=-N}^{N} \widehat{L_{\alpha,c}}(\xi_n) e^{2\pi ikn/|\mathcal{J}|}.$$

Note that with the use of a Fast Fourier Transform algorithms, this can be achieved with a total complexity of $\mathcal{O}(N\log(N))$. For a sampling set $\mathcal{J}$ large enough – in other words, which samples the Fourier transform of the cardinal function $\widehat{L_{\alpha,c}}$ with enough details – we are able to build an accurate estimation of $L_{\alpha,c}$ at uniform points. Then one can interpolate to approximate the points in the expansion (10) that have not been calculated exactly via the FFT.

8 Extensions

In what follows, assume that S is a symmetric, convex body in $\mathbb{R}^d$, i.e. a compact, convex set which is symmetric about the origin. Under some restrictions on the sequence X, the kernel ϕ, and the smoothness of the space under consideration, such interpolants exist. To wit, consider the following notion developed by Ledford in [16] and in higher dimensions in [10].

Definition 8.1. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is a *d-dimensional interpolator for PW_S* if the following hold:

(I1) $\phi(x) = \int_{\mathbb{R}^d} \psi(\xi)e^{i\langle x,\xi\rangle}d\xi = \psi^\vee(x)$, for some $\psi \in L_1 \cap L_2$.

(I2) $\widehat{\phi} \geq 0$ and there exists an $\varepsilon > 0$ such that $\widehat{\phi} \geq \varepsilon$ on S.

(I3) If $M_j := \sup_{\xi\in S\setminus\frac{1}{2}S} |\widehat{\phi}(2^j\xi)|$, $j \in \mathbb{N}$, then $\left(2^{\frac{jd}{2}}M_j\right) \in \ell_1$.

With this definition in hand, the following holds.

Theorem 8.2 ([10], Theorem 3.3)**.** *Let $S \subset \mathbb{R}^d$ a symmetric convex body. Suppose that X is a CIS for PW_S and that ϕ is a d-dimensional interpolator for PW_S.*

(i) For every $f \in PW_S$, there exists a unique sequence $(a_j) \in \ell_2$ such that

$$\sum_{j\in\mathbb{N}} a_j \phi(x_k - x_j) = f(x_k), \quad k \in \mathbb{N}.$$

(ii) The Interpolation Operator $\mathscr{I}_\phi : PW_S \to L_2(\mathbb{R}^d)$ defined by

$$\mathscr{I}_\phi f(\cdot) = \sum_{j\in\mathbb{N}} a_j \phi(\cdot - x_j),$$

where (a_j) is as in (i), is a well-defined, bounded linear operator from PW_S to $L_2(\mathbb{R}^d)$. Moreover, $\mathscr{I}_\phi f$ belongs to $C_0(\mathbb{R}^d)$.

We note as in [10, Section 4] that the following are examples of d-dimensional interpolators for PW_S where S is as above: the Gaussian kernel $g_\lambda := e^{-\lambda|\cdot|^2}$, for any parameter $\lambda > 0$, the general inverse multiquadrics $\phi_{\alpha,c} := (|\cdot|^2 + c^2)^\alpha$ for $\alpha < -d/2, c > 0$, and the functions $g_\alpha := (e^{-\alpha|\cdot|^p})^\vee$ for $\alpha > 0, p > 0$.

Every theorem stated here holds for such kernels. In particular, more details may be found in [8, 10, 16, 17, 25]

References

[1] B. A. Bailey, An asymptotic equivalence between two frame perturbation theorems, in *Proceedings of Approximation Theory XIII: San Antonio 2010*, Eds: M. Neamtu, L. Schumaker, Springer New York (2012), 1-7.

[2] B. A. Bailey, Sampling and recovery of multidimensional bandlimited functions via frames, *J. Math. Anal. Appl.* **370** (2010), 374-388.

[3] B. J. C. Baxter, The asymptotic cardinal function of the multiquadratic $\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$ as $c \to \infty$, *Comput. Math. Appl.*, **24**(12), (1992), 1-6.

[4] M. Buhmann, Multivariate cardinal interpolation with radial-basis functions, *Constr. Approx.* **6.3** (1990), 225-255.

[5] M. D. Buhmann, *Radial basis functions: theory and implementations*, Vol. 12. Cambridge University Press, 2003.

[6] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.* **72**(2) (1952), 341-366.

[7] S. Grepstad and N. Lev, Multi-tiling and Riesz bases, *Adv. Math.* **252** (2014), 1-6.

[8] K. Hamm, Approximation rates for interpolation of Sobolev functions via Gaussians and allied functions, *J. Approx. Theory*, **189** (2015), 101-122.

[9] K. Hamm, Sampling and Recovery Using Multiquadrics, 11th International Conference on Sampling Theory and Applications (SampTA 2015), Washington D.C.

[10] K. Hamm, Nonuniform sampling and recovery of bandlimited functions in higher dimensions, To appear, arXiv: 1411.5610.

[11] K. Hamm and J. Ledford, Cardinal interpolation with general multiquadrics, to appear in *Adv. Comput. Math.*

[12] K. Hamm and J. Ledford, Cardinal interpolation with general multiquadrics: convergence rates, Submitted. arXiv: 1506.07387.

[13] T. Hangelbroek, W. Madych, F. Narcowich and J. Ward, Cardinal interpolation with Gaussian kernels, *J. Fourier Anal. Appl.* **18** (2012), 67-86.

[14] M. I. Kadec, The exact value of the Paley-Wiener constant, *Dokl. Adad. Nauk SSSR* **155** (1964), 1243-1254.

[15] M. N. Kolountzakis, Multiple lattice tiles and Riesz bases of exponentials, *Proc. Amer. Math. Soc.* **143** (2015), 741-747.

[16] J. Ledford, Recovery of Paley-Wiener functions using scattered translates of regular interpolators, *J. Approx. Theory* **173** (2013), 1-13.

[17] J. Ledford, On the convergence of regular families of cardinal interpolators, *Adv. Comput. Math.* **41** (2015), no. 2, 357-371.

[18] J. Ledford, Convergence properties of spline-like cardinal interpolation operators acting on ℓ^p data, (submitted) arXiv:1312.4062, 2013.

[19] F. J. Narcowich, N. Sivakumar, and J. D. Ward, On condition numbers associated with radial-function interpolation, *J. Math. Anal. Appl.* **186**(2) (1994), 457-485.

[20] G. Kozma and S. Nitzan, Combining Riesz bases, *Invent. Math.*, **199**(1) (2015), 267-285.

[21] G. Kozma and S. Nitzan, Combining Riesz bases in $\mathbb{R}^d$, preprint, arXiv: 1501.05257.

[22] B. S. Pavlov, The basis property of a system of exponentials and the condition of Muckenhoupt, *Dokl. Akad. Nauk SSSR* **247** (1979), 37-40.

[23] S. D. Riemenschneider and N. Sivakumar, On the cardinal-interpolation operator associated with the one-dimensional multiquadric, *East J. Approx.* **7**, no. 4 (1999), 485-514.

[24] S. D. Riemenschneider and N. Sivakumar, Cardinal interpolation by Gaussian functions: A survey, *J. Analysis* **8** (2000), 157-178.

[25] Th. Schlumprecht and N. Sivakumar, On the sampling and recovery of bandlimited functions via scattered translates of the Gaussian, *J. Approx. Theory* **159** (2009), 128-153.

[26] C. E. Shannon, Communication in the presence of noise, *Proc. IRE*, **37** (1949), 10-21.

[27] N. Sivakumar, A note on the Gaussian cardinal-interpolation operator, *Proc. Edinb. Math. Soc. (2)* **40** (1997), 137-150.

[28] W. Sun, X. Zhou, On the stability of multivariate trigonometric systems, *J. Math. Anal. Appl.* **235** (1999), 159-167.

[29] H. Wendland, *Scattered Data Approximation*, Cambridge University Press (2004).

[30] E. T. Whittaker, On the functions which are represented by the expansions of the interpolation theory, *Proc. Royal Soc. Edinburgh*, Sec. A, Vol. 35 (1915), 181-194.

[31] R. M. Young, *An Introduction to Nonharmonic Fourier Series, Revised Edition*, Academic Press (2001).