

JSJ-Decompositions of Coxeter Groups over Virtually Abelian Splittings

M. Mihalik

July 13, 2010

Abstract

The idea of “JSJ-decompositions” for 3-manifolds began with work of Waldhausen and was developed later through work of Jaco, Shalen and Johansen. It was shown that there is a finite collection of 2-sided, incompressible tori that separate a closed irreducible 3-manifold into pieces with strong topological structure.

Sela introduced JSJ-decompositions for groups, an idea that has flourished in a variety of directions. The general idea is to consider a certain class \mathcal{G} of groups and splittings of groups in \mathcal{G} by groups in another class \mathcal{C} . E.g. Rips and Sela considered splittings of finitely presented groups by infinite cyclic groups. For an arbitrary group G in \mathcal{G} , the goal is to produce a “unique” graph of groups decomposition Ψ of G with edge groups in \mathcal{C} so that Ψ reveals all graph of groups decompositions of G with edge groups in \mathcal{C} . More specifically, if H is a vertex group of Ψ then either there is no \mathcal{C} -group that splits both G and H , or H has a special “surface group-like” structure. Vertex groups of the second type are standardly called orbifold groups.

For a finitely generated Coxeter system (W, S) , we produce a reduced JSJ-decomposition Ψ for splittings of W over virtually abelian subgroups. We show Ψ is unique (up to conjugate vertex groups) and each vertex and edge group is generated by a subset of S (and so Ψ is “visual”). The construction of Ψ is algorithmic. If $V \subset S$ generates an orbifold (vertex) group then $V = T \cup M$, where $\langle M \rangle$ is virtually abelian, $\langle T \rangle$ is virtually a closed surface group or virtually free and $\langle V \rangle = \langle M \rangle \times \langle T \rangle$.

email:michael.l.mihalik@vanderbilt.edu

Subject Classifications: 20F65, 20F55, 20E08, 57M07, 57M60

1 Introduction

The theory of JSJ-decompositions has its origins in the work of Waldhausen [19] on characteristic submanifolds of a 3-manifold and later work of Jaco-Shalen [10] and Johansen [11]. For a closed, irreducible, oriented 3-manifold there is a finite collection of embedded 2-sided incompressible tori that separate the manifold into pieces, each of which is a Seifert fibered space or an atoroidal and acylindrical space. This gives a graph of groups decomposition of the fundamental group with edge groups free abelian of rank 2.

Sela [17] introduced the notion of JSJ-decomposition for a general class of groups and showed that word hyperbolic groups have JSJ-decompositions over infinite cyclic splittings. Rips-Sela [15] generalize this to finitely presented groups. Scott-Swarup [16] consider splittings corresponding to virtually polycyclic groups with restrictions on Hirsch length and extend these results to virtually abelian groups of bounded rank. Dunwoody-Sageev [6] and then Fujiwara-Papasoglu [8] gave JSJ-decompositions for finitely presented groups over slender splittings.

A group is *slender* if all of its subgroups are finitely generated. The class of slender groups is contained in the class of *small* groups which are defined in terms of actions on trees. If a group contains a non-abelian free group it is not small. Coxeter groups containing no non-abelian free group are in fact virtually abelian and decompose in a special way amenable to our results (see theorem 13).

In analogy with the 1-ended assumptions of Rips-Sela [15], and following Dunwoody-Sageev [6] directly, we define the class of minimal virtually abelian splitting subgroups of W . If a Coxeter group splits over a minimal subgroup that contains no non-abelian free group, then the splitting subgroup is in fact virtually abelian. Hence for our purposes, there is no difference between (minimal) splittings over small, slender or virtually abelian groups and we only consider splittings of Coxeter groups over virtually abelian subgroups.

If (W, S) is a finitely generated Coxeter system, a graph of groups decomposition Ψ of W is *visual* if each edge and vertex group is generated by some subset of S and the bonding maps are inclusions. The main theorem of [13] states that for any graph of groups decomposition Λ of W there is a visual decomposition Ψ such that each vertex (respectively edge) group of Ψ is conjugate to a subgroup of a vertex (respectively edge) group of Λ . This result is used extensively in this paper and the basics of visual decompositions are reviewed in section 2.

Our construction of a JSJ-decomposition of a finitely generated Coxeter group W , with virtually abelian edge groups begins with a graph of groups decomposition Ψ_1 of W with edge groups that are minimal virtually abelian splitting subgroups of W and such that Ψ_1 is a maximal such decomposition without edge groups that are “crossing splitters”. We also show that Ψ_1 is unique (up to conjugate vertex groups) and visual. We call Ψ_1 a *level 1 JSJ-decomposition of W with virtually abelian edge groups*. If $V \subset S$ and $\langle V \rangle$ is a vertex group of Ψ_1 , then $\langle V \rangle$ may contain virtually abelian subgroups that split $\langle V \rangle$ and W non-trivially. Minimal splitting subgroups of this type are not necessarily minimal virtually abelian splitting subgroups of W . It is shown that any graph of groups decomposition of $\langle V \rangle$ with such edge groups is compatible with Ψ_1 and a maximal such decomposition that avoids crossing splitters, is unique (up to conjugate vertex groups) and visual. Replacing all such vertex groups of Ψ_1 by such graph of groups decompositions gives Ψ_2 , a *level 2 JSJ-decomposition of W with virtually abelian edge groups*. Continuing, we eventually have a visual graph of groups decomposition Ψ such that if $\langle V \rangle$ is a vertex group of Ψ , then the only virtually abelian splitting subgroups of $\langle V \rangle$ (that also split W) are crossing. We call Ψ a *JSJ-decomposition of W with respect to virtually abelian splittings* and show that Ψ is unique (up to conjugate vertex groups). If Λ and Φ are graph of groups decompositions of a group W , then say *the decomposition of Λ induced by Φ is compatible with Λ* if for each vertex group V of Λ , the decomposition of V induced by the action of V on the Bass-Serre tree for Φ is compatible with Λ . In sections 7 and 8, we prove results that imply our main theorem:

Theorem 1 *Suppose (W, S) is a finitely generated Coxeter system and Ψ the reduced JSJ-decomposition of W with virtually abelian edge groups. Then:*

1. Ψ is visual, unique up to conjugate vertex groups, and algorithmically defined.
2. If Φ is a graph of groups decomposition of W with virtually abelian edge groups then the decomposition of Ψ induced by Φ is compatible with Ψ .
3. If both W and a vertex group $\langle V \rangle$ of Ψ ($V \subset S$) split nontrivially over a virtually abelian subgroup of $\langle V \rangle$, then $\langle V \rangle$ decomposes as $\langle T \rangle \times \langle M \rangle$ where $T \cup M = V$, M generates a virtually abelian group and the presentation diagram of T is either a loop of length ≥ 4 (in which case T generates a group that is virtually a closed surface group) or the

presentation diagram of T is a disjoint union of vertices and simple paths (in which case T generates a virtually free group with graph of groups decomposition such that each vertex group is either \mathbb{Z}_2 or finite dihedral and each edge group is either trivial or \mathbb{Z}_2).

Vertex groups of the type described by part 3 of theorem 1 are called *orbifold* vertex groups. If H is a non-orbifold vertex group of our JSJ-decomposition Ψ , then H does not split non-trivially over a virtually abelian subgroup that also splits W non-trivially. In particular, if Φ is another graph of groups decomposition of W with virtually abelian edge groups, then H is a subgroup of a conjugate of a vertex group of Φ .

The construction of our JSJ-decompositions with virtually abelian edge groups is algorithmic. Given the presentation diagram Γ of a Coxeter system (W, S) , theorem 13 allows us to determine the subsets of S that generate virtually abelian subgroups. Those that separate Γ , algebraically split W . A result in section 4 allows us to easily decide which of these are the visual minimal virtually abelian splitting subgroups of (W, S) at all stages of the construction of the i^{th} -level JSJ-decompositions. It is equally easy to “visually” determine which of these splitting subgroups are crossing and hence build JSJ-decompositions. Our main result distinguishes the two types of vertex groups of our JSJ-decompositions. Those with no crossing subgroups are indecomposable with respect to virtually abelian splittings of W and those with crossing subgroups, which are traditionally called orbifold vertex groups.

If Φ is a graph of groups decomposition for a group G , V is a vertex of Φ with incident edge E , and the groups of V and E agree, then one can collapse the decomposition Φ across the edge E to obtain a smaller (more reduced) decomposition of G . As a simple example consider the splitting $A *_C E *_E D$ that collapses to $A *_C D$. While edges such as E seem to contribute somewhat artificial splittings to Φ , this type of edge is important for the JSJ decompositions produced by Fujiwara and Papasoglu. The advantage of the unreduced splitting $A *_C E *_E D$ is that it exhibits the decomposition $\langle A \cup E \rangle *_E D$ whereas $A *_C D$ does not. Our decompositions are reduced. The connection between our decompositions and the Fujiwara/Papasoglu decompositions is that their decompositions collapse to ours.

In [9], Guirardel-Levitt, develop the idea that a JSJ decomposition should be a deformation space satisfying a universal property. A deformation space (introduced by Forester in [7]) is a collection of G -trees, which in fact is

a contractible complex. In the correct setting, the deformation spaces of Guirardel-Levitt contain the trees constructed in [6], [8] and [15], as well as the ones constructed in this paper. For Coxeter groups the trees for the Fujiwara/Papasoglu decomposition and our decomposition are in the same deformation space.

Finally, we point out several aspects of this paper that separate it from the other JSJ results in the literature. We produce the first explicit constructions of JSJ decompositions for a general class of groups (Coxeter) over a specific set of splittings (virtually abelian). While the theme of minimal splittings is somewhat fundamental to JSJ decompositions, the phenomena of having only mutually crossing minimal splitters, highlighted in lemma 22, is not. (In the general theory, mutually crossing splitters would be “hyperbolic-hyperbolic”.) Lemma 22 basically says that under the right circumstances, if A and B generate minimal visual virtually abelian splitting subgroups and A separates B in the presentation diagram, then B must separate A . Hence the notion of “crossing” splittings subgroups arises. A second combinatorial highlight appears in proposition 23. This result basically states that minimal visual virtually abelian splitters separate vertex groups into exactly two parts. Perhaps the most remarkable aspect of the paper is the explicit structure taken on by our orbifold groups. Theorem 37 confirms this structure, which is significantly more rigid than what is known in general settings.

2 Basic Facts and Background Results

A thorough discussion of graphs of groups decompositions of Coxeter groups is given in [13]. We briefly discuss the aspects of this theory necessary to this paper. Every Coxeter group has a set of order 2 generators and so there is no non-trivial map of a Coxeter group to \mathbb{Z} . In particular, no Coxeter group is an HNN extension of any sort and any graph of groups decomposition of a Coxeter group has graph a tree. Hence the decompositions of Coxeter groups are a straightforward generalization of amalgamated product decompositions. For a graph of groups decomposition Λ of a group G , the Bass-Serre tree T for Λ has vertices (respectively edges) the cosets wV where $w \in G$ and V is a vertex (respectively edge) group of Λ . There is a left action of G on T and an element g of G stabilizes the coset wV iff $g \in wVw^{-1}$. If V is a vertex of Λ with vertex group $\Lambda(V)$, and Φ a graph of groups decomposition of $\Lambda(V)$, then Φ is *compatible* with Λ if for each edge E of Λ incident to V ,

$\Lambda(E)$ is contained in a $\Lambda(V)$ -conjugate of a vertex group of Φ . In this case V can be replaced by Φ to produce a finer graph of groups decomposition of G . A graph of groups decomposition Λ is *reduced* if no edge between distinct vertices has edge group the same as an end point vertex group. If a graph of groups is not reduced, we may collapse a vertex group across an edge, where the edge group is the same as the endpoint vertex group, giving a smaller graph of groups decomposition of the original group.

Lemma 2 (See [13]) *Suppose Λ is a reduced graph of groups decomposition of a group G , the underlying graph for Λ is a tree, U and V are vertices of Λ , and $g\Lambda(U)g^{-1} \subset \Lambda(V)$ for some $g \in G$ then $U = V$ and $g \in \Lambda(U)$. \square*

The following result is straightforward.

Lemma 3 *Suppose Ψ is a graph of groups decomposition of a group G , V is a finitely generated vertex group of Ψ , and \mathcal{E} is a collection of subgroups of V such that for any $K \in \mathcal{E}$, V splits non-trivially and Ψ -compatibly over K . If Λ is a graph of groups decomposition of V with edge groups in \mathcal{E} , then Λ is compatible with Ψ . \square*

The next result easily follows from the combinatorics of group actions on trees or more practically from the exactness of the Mayer-Vietoris sequence for a pair of groups.

Lemma 4 *Suppose a group G splits as $A *_C B$. If there is no non-trivial homomorphism from G or C to \mathbb{Z} , then there is no non-trivial homomorphism from A or B to \mathbb{Z} . In particular, if Λ is a graph of groups decomposition of a Coxeter group and no edge group of Λ maps non-trivially to \mathbb{Z} , then no vertex group of Λ maps non-trivially to \mathbb{Z} . \square*

We take a *Coxeter presentation* to be given as

$$P = \langle S : (st)^{m(s,t)} \ (s, t \in S, m(s, t) < \infty) \rangle$$

where $m : S^2 \rightarrow \{1, 2, \dots, \infty\}$ is such that $m(s, t) = 1$ iff $s = t$, and $m(s, t) = m(t, s)$. In the group with this presentation, the elements of S represent distinct elements of order 2 and a product st of generators has order $m(s, t)$. A *Coxeter group* W is a group having a Coxeter presentation and a *Coxeter system* (W, S) is a Coxeter group W with generating subset S corresponding to the generators in a Coxeter presentation of W . When the order of the

product of a pair of generators is infinite there will be no defining relator for that pair of generators and we will say that the generators are *unrelated*. Our basic reference for Coxeter groups is Bourbaki [2]. A *special* or *visual* subgroup for a Coxeter system (W, S) , is a subgroup of W generated by a subset of S . If W' is the visual subgroup generated by $S' \subseteq S$ in a Coxeter system (W, S) , then (W', S') is also a Coxeter system. More specifically the following result (see [2]) is fundamental to the study of Coxeter groups.

Proposition 5 *Suppose (W, S) is a Coxeter system and $P = \langle S : (st)^{m(s,t)}$ for $m(s, t) < \infty \rangle$ (where $m : S^2 \rightarrow \{1, 2, \dots, \infty\}$) is a Coxeter presentation for W . If $A \subset S$, then $(\langle A \rangle, A)$ is a Coxeter system with Coxeter presentation $\langle A : (st)^{m'(s,t)}$ for $m'(s, t) < \infty \rangle$ (where $m' = m|_{A^2}$). \square*

Given a group G and a generating set S , an S -*geodesic* for $g \in G$ is a shortest word in $S \cup S^{-1}$ such that the product of the letters of this word is g . The number of letters in an S -geodesic for g is the S -*length* of g . An important combinatorial fact about geodesics for a Coxeter system is called the “deletion condition”.

Proposition 6 (The Deletion Condition) *Suppose (W, S) is a Coxeter system and $w = a_1 \cdots a_n$ for $a_i \in S$. If $a_1 \cdots a_n$ is not geodesic then there are indices $i < j$ in $\{1, 2, \dots, n\}$ such that $w = a_1 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} a_{j+1} \cdots a_n$. I.e. a_i and a_j can be deleted. \square*

The information given by a Coxeter presentation may be conveniently expressed in the form of a labeled graph. We define the *presentation diagram* of the system (W, S) to be the labeled graph $\Gamma(W, S)$ with vertex set S , and an (undirected) edge labeled $m(s, t)$ between distinct vertices s and t when $m(s, t) < \infty$. The connected components of the presentation diagram $\Gamma(W, S)$ correspond to visual subgroups which are the factors in a free product decomposition of W . In contrast, a *Coxeter graph* has vertex set S and labeled edges when $m(s, t) \neq 2$. The components of a Coxeter graph corresponding to direct product factors of W . By proposition 5, a presentation diagram of a visual subgroup of W generated by a subset $S' \subseteq S$ is the induced subgraph of $\Gamma(W, S)$ with vertex set S' .

Suppose $\Gamma(W, S) = \Gamma_1 \cup \Gamma_2$ is a union of induced subgraphs and let $\Gamma_0 = \Gamma_1 \cap \Gamma_2$ (so vertices and edges of $\Gamma(W, S)$ are in Γ_1 or Γ_2 or both, and Γ_0 is the induced subgraph consisting of the vertices and edges in both). Equivalently,

suppose Γ_0 is an induced subgraph with $\Gamma(W, S) - \Gamma_0$ having at least two components, Γ_1 is Γ_0 together with some of these components and Γ_2 is Γ_0 together with the other components. We say in this case that Γ_0 *separates* $\Gamma(W, S)$ (separates it into at least two components). Then it is evident from the Coxeter presentation that W is an amalgamated product of visual subgroups corresponding to Γ_1 and Γ_2 over the visual subgroup corresponding to Γ_0 . Amalgamated product decompositions with visual factors and visual amalgamated subgroup are easily seen in the presentation diagram and we call such an amalgamated product a *visual splitting* of W .

We say that Ψ is a *visual graph of groups decomposition* of W (for a given Coxeter system (W, S)), if each vertex and edge group of Ψ is visual for (W, S) , the injections of each edge group into its endpoint vertex groups are given simply by inclusion, and the fundamental group of Ψ is isomorphic to W by the homomorphism induced by the inclusion map of vertex groups into W . A sequence of compatible visual splittings of W will result in such a decomposition. In [13], we study general graph of groups decompositions of Coxeter groups and how these are related to visual graph of groups decompositions. The main result of [13] shows that an arbitrary graph of groups decomposition of a Coxeter group can be refined (in a certain sense) to a visual graph of groups decomposition.

Theorem 7 *Suppose (W, S) is a Coxeter system and W is a subgroup of the fundamental group of a graph of groups Λ . Then W has a visual graph of groups decomposition Ψ where each vertex group of Ψ is a subgroup of a conjugate of a vertex group of Λ , and each edge group of Ψ is a subgroup of a conjugate of an edge group of Λ . Moreover, Ψ can be taken so that each visual subgroup of W that is a subgroup of a conjugate of a vertex group of Λ is a subgroup of a vertex group of Ψ . \square*

The following three results are important technical facts proved in [13].

Lemma 8 *Suppose (W, S) is a Coxeter system. A graph of groups Ψ with graph a tree, where each vertex group and edge group is a visual subgroup for (W, S) and each edge map is given by inclusion, is a visual graph of groups decomposition of W iff each edge in the presentation diagram of W is an edge in the presentation diagram of a vertex group and, for each generator $s \in S$, the set of vertices and edges with groups containing s is a nonempty subtree in Ψ . \square*

If Ψ is a visual graph of groups decomposition for the Coxeter system (W, S) , it is convenient to label the vertices of Ψ by the subsets of S that generate the corresponding vertex groups. So if $Q \subset S$ is a vertex of Ψ , then $\Psi(Q) = \langle Q \rangle$. The same convention is not used for edge groups as distinct edges may have the same edge group.

Separation properties of various subsets of a presentation diagram $\Gamma(W, S)$ for a Coxeter system (W, S) are frequently analyzed in this paper. If A and B are subsets of Γ we say A separates B in Γ if there are points b_1 and b_2 of $B - A$ such that any path in Γ from b_1 to b_2 intersects A .

Lemma 9 *Suppose (W, S) is a Coxeter system, S is finite, Ψ is a visual graph of groups decomposition of W , E' is an edge of Ψ and $\Psi(E') = \langle E \rangle$ for $E \subset S$. If $\{x, y\} \subset S - E$, and $x \in X$ and $y \in Y$ for X and Y vertices of Ψ on opposite sides of E' , then E separates x and y in $\Gamma(W, S)$. \square*

Lemma 10 *Suppose (W, S) is a finitely generated Coxeter system, and Ψ is a visual graph of groups decomposition of W . If C is a complete subset of the presentation diagram $\Gamma(W, S)$, then there is a vertex V of Ψ such that $C \subset V$. \square*

In this paper we require more than just the statement of theorem 7. The technique to produce a visual decomposition is easy to describe and useful to the constructions in this paper. Under the hypothesis of theorem 7 let T be the Bass-Serre tree for Λ . In the proof of theorem 7, it is shown that W has a visual graph of groups decomposition with graph T and vertex (respectively edge) group at gV generated by the subset of S that stabilizes gV . Since S is finite, this visual graph of groups reduces to a finite reduced graph of groups decomposition of W satisfying the conclusion of theorem 7. In this paper we make repeated use of this construction and refer to it as *the visual graph of groups given by the construction for theorem 7*.

The next lemma follows from a result of Kilmoyer (see section 4 of [13]).

Lemma 11 *Suppose (W, S) is a Coxeter system, $I, J \subset S$, and d is a minimal length double coset representative in $\langle I \rangle w \langle J \rangle$. Then $\langle I \rangle \cap d \langle J \rangle d^{-1} = \langle K \rangle$ for $K = I \cap (dJd^{-1})$ and, $d^{-1} \langle K \rangle d = \langle J \rangle \cap (d^{-1} \langle I \rangle d) = \langle K' \rangle$ for $K' = J \cap d^{-1} I d = d^{-1} K d$. In particular, if $w = idj$ for $i \in \langle I \rangle$ and $j \in \langle J \rangle$ then $\langle I \rangle \cap w \langle J \rangle w^{-1} = i \langle K \rangle i^{-1}$ and $\langle J \rangle \cap w^{-1} \langle I \rangle w = j^{-1} \langle K' \rangle j$. \square*

3 Decomposing Coxeter Groups With No Non-Abelian Free Subgroup

The Euclidean simplex reflection groups and irreducible finite Coxeter groups are catalogued in Coxeter's book [3]. Euclidean simplex groups are infinite and virtually abelian. In his thesis [12], D. Krammer classifies free abelian subgroups of Coxeter groups. If a Coxeter group contains no non-abelian free group, the Coxeter group is virtually free abelian and decomposes as a direct product of finite Coxeter groups and Euclidean simplex groups in a special way - a result well-known to experts. We include an elementary proof of this result, using the following lemma (which is a direct consequence of theorems 7.2.2, 7.3.1, 12.1.19 and exercise 12.1.14 of J. Ratcliffe's book [14]).

Lemma 12 *Consider the collection of Coxeter systems (W, S) such that*

1. $\Gamma(W, S)$ is complete,
2. W does not decompose as $\langle A \rangle \times \langle B \rangle$ for A and B non-trivial subsets of S , and
3. for each $s \in S$, $\langle S - \{s\} \rangle$ is either finite or Euclidean.

Then W is finite, Euclidean or contains a free subgroup of rank 2. \square

Theorem 13 *Suppose (W, S) is a finitely generated Coxeter system and W does not contain a non-abelian free group. Then S is the disjoint union of sets which commute with one another and each generates a finite group or an Euclidean simplex group.*

Proof: If $a, b \in S$ such that $m(a, b) = \infty$, let $C = S - \{a, b\}$. Then $W = \langle \{a\} \cup C \rangle *_{\langle C \rangle} \langle \{b\} \cup C \rangle$. The index of $\langle C \rangle$ in both $\langle \{a\} \cup C \rangle$ and $\langle \{b\} \cup C \rangle$ is 2, since, W contains no free group of rank 2. Hence $\langle C \rangle$ is normal in $\langle \{a\} \cup C \rangle$. For each $c \in C$, $aca \in \langle C \rangle$. Any geodesic in $\langle C \rangle$ uses only letters in C , and by the deletion condition, $aca = c$. Thus a commutes with C as does b . The group $\langle a, b \rangle$ splits off as a direct factor of W . Hence we may assume $\Gamma(W, S)$ is complete (and infinite). Note that condition 1) of the lemma is satisfied.

Suppose (W, S) is a counterexample to the theorem, with $|S|$ small as possible. Note that $|S| \geq 3$, and W does not visually decompose as a non-trivial direct product, so that condition 2) of the lemma is satisfied. If

$s \in S$, we have $\langle S - \{s\} \rangle$ is a visual product of a finite group and Euclidean simplex groups. Choose a and b distinct elements of S . Write $\langle S - \{a\} \rangle = \langle F_a \rangle \times \langle E_1 \rangle \times \cdots \times \langle E_p \rangle$ (so $F_a \cup (\cup_{i=1}^p E_i) = S - \{a\}$) where $\langle F_a \rangle$ is finite and each $\langle E_i \rangle$ is Euclidean. Similarly write $\langle S - \{b\} \rangle = \langle F_b \rangle \times \langle K_1 \rangle \times \cdots \times \langle K_q \rangle$. If $b \in E_i$ for some i , then assume $i = 1$. If $p > 1$, then $E_2 \subset K_j$ for some j . For any proper subset K of K_j , $\langle K \rangle$ is finite, and so $E_2 = K_j$. But then E_2 commutes with $S - E_2$ which is impossible. Hence $\{p, q\} \subset \{0, 1\}$.

If $F_a \neq \emptyset$ and $E_1 \neq \emptyset$, we may choose $b \in F_a$, so that $\{a, b\} \cap E_1 = \emptyset$. This implies $E_1 = K_1$ and E_1 commutes with $S - E_1 (= F_a \cup \{a\})$, which is impossible. Hence either F_a or E_1 is empty. Since a is arbitrary in S , condition 3) of the previous lemma is satisfied. \square

4 Minimal Virtually Abelian Splitting Subgroups

If Λ is a graph of groups decomposition of a group W and G is a vertex group of Λ , then a virtually abelian subgroup A of G is a *minimal virtually abelian splitting subgroup* for (Λ, G) if G splits non-trivially and compatibly with Λ over A , and there is no virtually abelian subgroup B of W such that G splits non-trivially and compatibly with Λ over B , and $B \cap A$ has infinite index in A and finite index in B .

For a Coxeter system (W, S) , Ψ a visual graph of groups decomposition for (W, S) and G a vertex group of Ψ , let $\mathcal{C}(\Psi, G)$ be the set of virtually abelian subgroups of G that split G non-trivially and Ψ -compatibly. Let $M(\Psi, G)$ be the set of minimal virtually abelian splitting subgroups for (Ψ, G) .

If Ψ is the trivial graph of groups decomposition for (W, S) (with one vertex), then define $\mathcal{C}(W, S) \equiv \mathcal{C}(\Psi, W)$ and $M(W, S) \equiv M(\Psi, W)$. Observe:

1. If a vertex group G of Ψ has more than 1-end, then each member of $M(\Psi, G)$ is a finite group.
2. For a given finitely generated Coxeter group W , the ranks of the virtually abelian subgroups of W are bounded.
3. If $A \subset S$ and $\langle A \rangle \in M(\Psi, G)$ then A satisfies the conclusion of theorem 13. Hence A is the disjoint union of sets that commute with one another such that (at most) one generates a finite group and each other generates an Euclidean simplex group.

If (W, S) is a Coxeter system and $A \subset S$ is such that $\langle A \rangle$ is virtually abelian, define $E(A)$ to be the set of generators of the Euclidean factors of the visual direct product decomposition of $\langle A \rangle$ given by theorem 13.

If A is a minimal virtually abelian splitting subgroup of a Coxeter group, then by theorem 7, A contains a subgroup of finite index which is isomorphic to a Coxeter group. Hence there is no non-trivial homomorphism of A to \mathbb{Z} . Lemma 4 implies the next result.

Lemma 14 *Suppose W is a finitely generated Coxeter group and Λ is a graph of groups decomposition of W with minimal virtually abelian splitting subgroups as edge groups. Then Λ is a tree and no vertex group of Λ maps non-trivially to \mathbb{Z} . \square*

If Ψ is a visual graph of groups decomposition for the Coxeter system (W, S) , $V(\subset S)$ is a vertex of Ψ and Ψ_1 is a visual decomposition for $(\langle V \rangle, V)$, then Ψ_1 is *visually compatible* with Ψ if for each edge group $\langle E \rangle$ ($E \subset S$) of an edge of Ψ incident to V , E is a subset of K for some vertex $K(\subset V)$ of Ψ_1 . In particular, if $A \subset V$ is such that $\langle A \rangle$ is an edge group of Ψ_1 , then we say $\langle A \rangle$ *splits V visually and compatibly with Ψ* .

Lemma 15 *Suppose (W, S) is a finitely generated Coxeter system, Ψ is a reduced visual graph of groups decomposition for (W, S) , $V \subset S$ is a vertex of Ψ , and Λ is a reduced graph of groups decomposition of $\langle V \rangle$ such that each edge group of Λ is in $M(\Psi, \langle V \rangle)$. Let Ψ' be the reduced visual decomposition for Λ given by the construction for theorem 7, then Ψ' is visually compatible with Ψ , each edge group of Ψ' is in $M(\Psi, \langle V \rangle)$ and if $U \subset V$ such that $\langle U \rangle$ is an edge group of Ψ' , then U separates V in $\Gamma(W, S)$.*

Proof: By lemma 3, Λ is compatible with Ψ . So if E' is an edge of Ψ incident to V and $E \subset S$ is such that $\langle E \rangle = \Psi(E')$, then $\langle E \rangle$ is a subgroup of a $\langle V \rangle$ -conjugate of a vertex group of Λ . Hence E stabilizes a vertex of the Bass-Serre tree for Λ . But then by the construction for theorem 7, E is a subset of a vertex group of Ψ' and Ψ' is visually compatible with Ψ . Each edge group of Ψ' is conjugate to a subgroup of an edge group of Λ and so must be in $M(\Psi, \langle V \rangle)$. Suppose $A, B \subset V$ are distinct vertices of Ψ' incident to the edge U of Ψ' . There exists $a \in A - U$ and $b \in B - U$. If Ψ'' is the visual graph of groups decomposition obtained from Ψ by replacing the vertex V by Ψ' , then applying lemma 9 to Ψ'' shows U separates a and b in Γ . \square

The next lemma is a direct consequence of V. Deodhar's results in [4].

Lemma 16 Suppose (W, S) is a Coxeter system, $A \subset S$, $\langle A \rangle$ is infinite and there is no non-trivial $F \subset A$ such that $\langle F \rangle$ is finite and $\langle A \rangle = \langle A - F \rangle \times \langle F \rangle$. If $w \in W$ such that $w\langle A \rangle w^{-1} \subset \langle B \rangle$ for $B \subset S$, then $uAu^{-1} = A \subset B$ for u the minimal length double coset representative of $\langle B \rangle w \langle A \rangle$. In particular, if $w\langle A \rangle w^{-1} = \langle B \rangle$, then $A = B$. \square

Observe that if (W, S) is a Coxeter system, and $W = \langle F \rangle \times \langle G \rangle = \langle H \rangle \times \langle I \rangle$ for $F \cup G = S = H \cup I$. Then $W = \langle F \rangle \times \langle H - F \rangle \times \langle G \cap I \rangle$.

Proposition 17 Suppose (W, S) is a Coxeter system, $A \subset S$ and $\langle A \rangle = \langle B \rangle \times \langle C \rangle$ where $B \cup C = A$, and C is the (unique) largest such subset of A such that $\langle C \rangle$ is finite. If $w\langle A \rangle w^{-1} \subset \langle H \rangle$ for $w \in W$ and $H \subset S$, then $B \subset H$. In particular, if $\langle A \rangle$ and $\langle H \rangle$ are virtually abelian then $E(A) \subset E(H)$.

Proof: Let d be a minimal length double coset representative of $\langle H \rangle w \langle A \rangle$. So $d\langle A \rangle d^{-1} \subset \langle H \rangle$ and $d\langle A \rangle d^{-1} = d\langle A \rangle d^{-1} \cap \langle H \rangle = \langle K \rangle$ for $K = dAd^{-1} \cap H$. Hence $K = dAd^{-1}$ ($d^{-1}Kd \subset A$ and generates $\langle A \rangle$). Apply lemma 16. \square

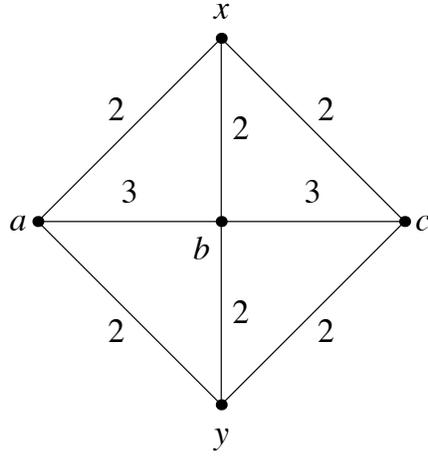


Figure 1: $\Gamma(W, S)$

Example 1. Consider the Coxeter system (W, S) with presentation diagram shown in figure 1. Clearly, $\langle x, y, b \rangle \in M(W, S)$ and the element bc conjugates $\{x, y, b\}$ to $\{x, y, c\}$. So, $\langle x, y, c \rangle \in M(W, S)$. Hence a visual subgroup in $M(W, S)$ need not separate the presentation diagram $\Gamma(W, S)$.

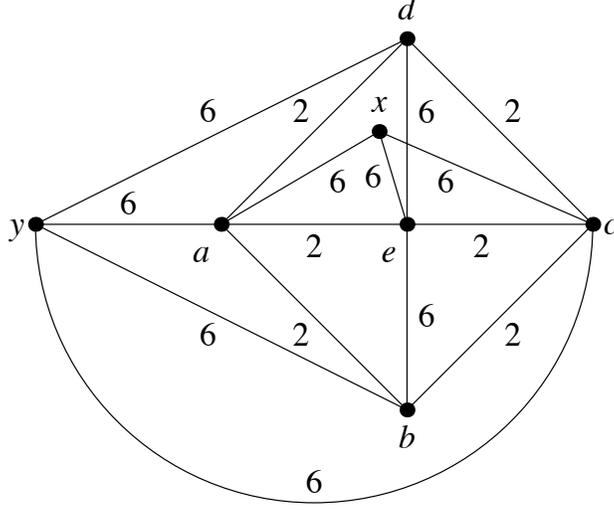


Figure 2: $\Gamma(W, S)$

Example 2. The group with presentation diagram shown in figure 2, splits non-trivially and visually over both $\langle a, b, c, d \rangle \in \mathcal{C}(W, S)$ and $\langle a, c, e \rangle$. As W is 1-ended and $\langle a, c, e \rangle$ is 2-ended, $\langle a, c, e \rangle \in M(W, S)$. Hence $\langle a, b, c, d \rangle \notin M(W, S)$. Furthermore, no $A \subset \{a, b, c, d\}$ is such that $\langle A \rangle \in M(W, S)$.

Proposition 18 *Suppose (W, S) is a Coxeter system, Ψ is a visual graph of groups decomposition for (W, S) and $V \subset S$ a vertex of Ψ . If $A \subset V$ and $\langle A \rangle \in \mathcal{C}(\Psi, \langle V \rangle)$, then there exists $B \subset V$ such that B separates V in $\Gamma(W, S)$, $\langle B \rangle \in M(\Psi, \langle V \rangle)$ and $E(B) \subset E(A)$.*

Proof: If $\langle A \rangle \in M(\Psi, \langle V \rangle)$, let $G = \langle A \rangle$. Otherwise, let G be a minimum rank element of $\mathcal{C}(\Psi, \langle V \rangle)$ such that $G \cap \langle A \rangle$ has infinite index in $\langle A \rangle$ and finite index in G . In any case, $G \in M(\Psi, \langle V \rangle)$. By Lemma 15, there exists $B \subset V$ and $v \in \langle V \rangle$ such that B separates V in Γ , $v\langle B \rangle v^{-1} \subset G$ and $\langle B \rangle \in M(\Psi, \langle V \rangle)$. Hence $\text{Rank}(\langle B \rangle) = \text{Rank}(G) \leq \text{Rank}(\langle A \rangle)$ and $\langle A \rangle \cap v\langle B \rangle v^{-1}$ has finite index in $v\langle B \rangle v^{-1}$.

By lemma 11, $\langle A \rangle \cap v\langle B \rangle v^{-1} = a\langle K \rangle a^{-1}$, for some $a \in \langle A \rangle$ and $K \subset A$. Hence $E(K) \subset E(A)$. By lemma 11, $\langle B \rangle \cap v^{-1}\langle A \rangle v = b\langle K' \rangle b^{-1}$ for some $b \in \langle B \rangle$ and $K' \subset B$ where K and K' are conjugate. Hence by lemma 16

$E(K) = E(K')$. As $b\langle K'\rangle b^{-1}$ has finite index in $\langle B\rangle$, $E(K') = E(B)$. Hence $E(B) \subset E(A)$. \square

We can now easily recognize V -separating visual subgroups in either $\mathcal{C}(\Psi, \langle V\rangle)$ or $M(\Psi, \langle V\rangle)$.

Corollary 19 *Suppose (W, S) is a Coxeter system, Ψ is a reduced visual graph of groups decomposition for (W, S) , $V \subset S$ is a vertex group of Ψ , $A \subset V$ such that $\langle A\rangle$ splits $\langle V\rangle$ non-trivially and visually compatible with Ψ . Then $\langle A\rangle \in M(\Psi, \langle V\rangle)$ iff $\langle A\rangle$ is virtually abelian, and there is no $B \subset V$ such that $\langle B\rangle$ is virtually abelian, $\langle B\rangle$ splits $\langle V\rangle$ visually compatible with Ψ , and $E(B)$ is a proper subset of $E(A)$. \square*

For (W, S) a Coxeter system, Ψ a reduced visual graph of groups decomposition for (W, S) , $V \subset S$ such that V is a vertex of Ψ , and $A \subset V$ such that A separates V in $\Gamma(W, S)$ and $\langle A\rangle \in M(\Psi, \langle V\rangle)$, we have not ruled out the possibility that there is $x \in E(A)$ such that $A - \{x\}$ separates V in Γ . It may be that $x \in E$ for $\langle E\rangle$ the group of an edge incident to V in Ψ and that in Γ , $A - \{x\}$ separates x from some other point of E , so that the visual splitting of W over $\langle A - \{x}\rangle$ is not compatible with Ψ (and so the minimality of $\langle A\rangle$ is not violated). In our main applications, Ψ will be an n^{th} -stage JSJ-decomposition and we will show there is no $B \subset V$ such that $\langle B\rangle$ is virtually abelian and B separates E in Γ for $\langle E\rangle$ the group of an edge of Ψ incident to V . But, until that point of the paper is reached, we add a restriction to the statements of some of our results in order to deal with this contingency.

Lemma 20 *Suppose (W, S) is a Coxeter system, Ψ is a reduced visual graph of groups decomposition for (W, S) , $V \subset S$ is a vertex of Ψ , $A \subset V$ separates V in $\Gamma(W, S)$, $\langle A\rangle$ is virtually abelian, and there is no $x \in E(A)$ such that $A - \{x\}$ separates V in $\Gamma(W, S)$. If K is a component of $\Gamma - A$ which intersects V non-trivially, then for each $a \in E(A)$ there is an edge from K to a .*

Proof: Otherwise, $A - \{a\}$ separates V in Γ . \square

Lemma 21 *Suppose Γ is a graph with vertex set S , and $A \subset S$ separates $\{b_1, b_2\}$ in Γ . If $b \in S$ is adjacent to both b_1 and b_2 in Γ , then $b \in A$. \square*

Lemma 22 *Suppose (W, S) is a Coxeter system, Ψ is a reduced visual graph of groups decomposition for (W, S) , $V \subset S$ is a vertex of Ψ , $A \subset S$ such that $\langle A \rangle$ is virtually abelian, and $B \subset V$ such that $\langle B \rangle$ is virtually abelian, B separates V in $\Gamma(W, S)$ and there is no $x \in E(B)$ such that $B - \{x\}$ separates V in $\Gamma(W, S)$. If A separates B in Γ then B separates A in Γ . In particular, (by lemma 21) $A - \{a_1, a_2\} = B - \{b_1, b_2\}$ for a_1, a_2 unrelated elements of A and b_1, b_2 unrelated elements of B . (So $\text{Rank}(A) = \text{Rank}(B)$.) Furthermore, if $A \subset V$, $\langle A \rangle \in \mathcal{C}(\Psi, \langle V \rangle)$, $\langle B \rangle \in M(\Psi, \langle V \rangle)$, and for each $D \subset V$ such that $\langle D \rangle \in M(\Psi, \langle V \rangle)$ there is no $x \in E(D)$ such that $D - \{x\}$ separates V in $\Gamma(W, S)$ then $\langle A \rangle \in M(\Psi, \langle V \rangle)$.*

Proof: Write $B = \{b_1, b_2\} \cup M$, where A separates b_1 and b_2 in Γ and $\langle M \rangle$ commutes with $\{b_1, b_2\}$. By lemma 21, $M \subset A$ and so $A \cap B = M$. If B does not separate A , then $A - M \subset K$ for K a component of $\Gamma - B$. For $i \in \{1, 2\}$, let K_i be distinct components of $\Gamma - B$ containing $t_i \in V$. Assume $K_1 \neq K_2$. Then $t_1 \notin A$. We may assume t_1 is in a component of $\Gamma - A$ not containing b_2 . By hypothesis, $\{b_1\} \cup M$ does not separate V in Γ . Choose a shortest path from t_1 to t_2 avoiding $\{b_1\} \cup M$. Then this path passes through b_2 (exactly once). As A separates b_2 and t_1 , we let s be the first vertex of A in our path. Then $s \in A - M \subset K$ and we have connected $t_1 \in K_1$ to $A - M \subset K$ in $\Gamma - B$, which is nonsense. By lemma 21, $A - \{a_1, a_2\} = B - \{b_1, b_2\} \equiv M$.

It remains to show that if $A \subset V$, $\langle V \rangle \in \mathcal{C}(\Psi, \langle V \rangle)$ and $\langle B \rangle \in M(\Psi, \langle V \rangle)$ then $\langle A \rangle \in M(\Psi, \langle V \rangle)$. Otherwise, proposition 18 implies there is $D \subset V$ such that D separates V in Γ , $\langle D \rangle \in M(\Psi, \langle V \rangle)$, $E(D) \subset E(A) (= \{a_1, a_2\} \cup E(M))$ and $\text{Rank}(\langle D \rangle) < \text{Rank}(\langle A \rangle)$. If $\{a_1, a_2\} \subset D$, then as B separates a_1 and a_2 , the argument above shows $D - \{a_1, a_2\} = B - \{b_1, b_2\}$. But then $\text{Rank}(\langle D \rangle) = \text{Rank}(\langle B \rangle) = \text{Rank}(\langle A \rangle)$ which is nonsense. If $a_1 \notin D$, then $E(D) \subset M$. This is impossible as $\langle B \rangle \in M(\Psi, \langle V \rangle)$ and $\langle B \rangle = \langle b_1, b_2 \rangle \times \langle M \rangle$. Similarly for a_2 . \square

Remark 1. A splitting of a group G over a subgroup U is called *elliptic* with respect to a splitting of G over V if U stabilizes a point in the Bass-Serre tree for the splitting of G over V . Otherwise, the splitting over U is called *hyperbolic* with respect to the splitting over V . Hence a pair of splittings can be elliptic-elliptic, hyperbolic-elliptic, elliptic-hyperbolic or hyperbolic-hyperbolic. Lemma 22 basically says the pairs of splittings we are interested in are either elliptic-elliptic or hyperbolic-hyperbolic. Because of the more specific combinatorial nature of our splittings we use a different terminology.

If $\langle A \rangle$ and $\langle B \rangle$ are elements of $M(\Psi, \langle V \rangle)$ and A and B separate one another in Γ , we say $\langle A \rangle$ and $\langle B \rangle$ *cross* or are *crossing* in $M(\Psi, \langle V \rangle)$.

Proposition 23 *Suppose (W, S) is a Coxeter system, Ψ is a reduced visual graph of groups decomposition for (W, S) , $V \subset S$ is a vertex of Ψ , $A \subset S$ such that $\langle A \rangle$ is virtually abelian, and $B \subset V$ such that $\langle B \rangle$ is virtually abelian, B separates V in $\Gamma(W, S)$ and there is no $x \in E(B)$ such that $B - \{x\}$ separates V in $\Gamma(W, S)$. If A separates B in $\Gamma(W, S)$ then $\Gamma(W, S) - B$ has exactly 2-components which intersect V non-trivially.*

Proof: By lemma 22, we may assume $A = \{a_1, a_2\} \cup M$ and $B = \{b_1, b_2\} \cup M$, where a_1 and a_2 are unrelated and separated by B , b_1 and b_2 are unrelated and separated by A , and M commutes with $\{a_1, a_2, b_1, b_2\}$. For $i \in \{1, 2\}$, let K_i be the component of $\Gamma - B$ containing a_i .

If K is a component of $\Gamma - B$ which intersects V non-trivially, then by lemma 20, there is an edge connecting b_i to K for $i \in \{1, 2\}$. If additionally, $K \neq K_i$ for $i \in \{1, 2\}$, then $K \cap A = \emptyset$, and so $K \cup \{b_1, b_2\}$ is a connected subset of $\Gamma - A$. This is impossible since A separates b_1 and b_2 in Γ . \square

Lemma 24 *Suppose (W, S) is a Coxeter system and Λ is a graph of groups decomposition of W . If $A \subset S$, $\langle A \rangle$ is virtually abelian and for any unrelated x, y in A , $\{x, y\}$ stabilizes a vertex of T_Λ (the Bass-Serre tree for Λ), then A stabilizes a vertex of T_Λ . In particular, if Ψ is the visual graph of groups for Λ given by the construction for theorem 7, then $A \subset V$ for $\langle V \rangle$ a vertex group of Ψ .*

Proof: The group $\langle A \rangle$ decomposes as $\langle a_1, b_1 \rangle \times \cdots \times \langle a_n, b_n \rangle \times \langle F \rangle$ where $A = \{a_1, b_1, \dots, a_n, b_n\} \cup F$, $m(a_i, b_i) = \infty$ and F generates a complete subdiagram of $\Gamma(W, S)$. If the statement of the lemma fails, assume n is minimal among all counterexamples. Note that $n > 0$ since F is FA (see [13]). By the minimality of n , $\langle a_1, \dots, a_n, b_2, \dots, b_n, F \rangle$ stabilizes a vertex V_1 of T_Λ , $\langle a_2, \dots, a_n, b_1, \dots, b_n, F \rangle$ stabilizes V_2 and $\langle a_1, b_1 \rangle$ stabilizes V_3 . As T_Λ is a tree, there is a vertex V of T_Λ common to the three geodesics connecting pairs in $\{V_1, V_2, V_3\}$, and A stabilizes V . \square

5 Weak $M(\Psi, \langle V \rangle)$ -JSD Decompositions

If (W, S) is a Coxeter system, Ψ is a reduced visual graph of groups decomposition for (W, S) and $V \subset S$ is a vertex of Ψ , then a reduced Ψ -compatible graph of groups decomposition, Λ , of $\langle V \rangle$ is *weakly $M(\Psi, \langle V \rangle)$ -JSJ* if

1. each edge group of Λ is in $M(\Psi, \langle V \rangle)$ and
2. each element of $M(\Psi, \langle V \rangle)$ is a subgroup of a conjugate of a vertex group of Λ .

A reduced Ψ -compatible visual graph of groups decomposition Ψ_1 of $\langle V \rangle$ looks weakly $M(\Psi, \langle V \rangle)$ -JSJ if

1. each edge group of Ψ_1 is in $M(\Psi, \langle V \rangle)$ and
2. no edge group $\langle E \rangle$ ($E \subset V$) of Ψ_1 is crossing.

At this point we consider decompositions that are precursors to the n^{th} -stage JSJ-decompositions. For a finitely generated Coxeter system (W, S) we say a visual graph of groups decomposition Ψ is *JSJ-amenable* if Ψ is reduced and for any vertex $V \subset S$ of Ψ and any $E \subset S$ such that $\langle E \rangle$ is the group of an edge incident to V , there is no $A \subset V$ such that $\langle A \rangle$ is virtually abelian and A separates E in Γ . In particular, if Ψ is JSJ-amenable, $V \subset S$ is a vertex of Ψ , $A \subset V$ separates V in Γ and $\langle A \rangle \in M(\Psi, \langle V \rangle)$ then there is no $x \in E(A)$ such that $A - \{x\}$ separates V in Γ . (This remark should be compared with the one following corollary 19). In section 7 (proposition 34) we show that n^{th} -stage JSJ-decompositions are JSJ-amenable.

Proposition 25 *Suppose (W, S) is a Coxeter system, Ψ is a visual graph of groups decomposition for (W, S) that is JSJ-amenable and $V \subset S$ is a vertex of Ψ . A reduced visual Ψ -compatible graph of groups decomposition Ψ_1 of $\langle V \rangle$ looks weakly $M(\Psi, \langle V \rangle)$ -JSJ if and only if it is weakly $M(\Psi, \langle V \rangle)$ -JSJ.*

Proof: Assume Ψ_1 is weakly $M(\Psi, \langle V \rangle)$ -JSJ. If Ψ_1 does not look weakly $M(\Psi, \langle V \rangle)$ -JSJ, then there is $E \subset V$ such that $\langle E \rangle$ is the edge group of an edge E' of Ψ_1 and $T \subset V$ such that $\langle T \rangle \in M(\Psi, \langle V \rangle)$ and $\langle E \rangle$ crosses $\langle T \rangle$ in $M(\Psi, \langle V \rangle)$. Assume E separates elements t_1 and t_2 of T in Γ . By lemma 22, $E = \{e_1, e_2\} \cup N$ and $T = \{t_1, t_2\} \cup N$ where e_1 and e_2 are unrelated, t_1 and t_2 are unrelated and N commutes with $\{e_1, e_2, t_1, t_2\}$. Since Ψ_1 is weakly $M(\Psi, \langle V \rangle)$ -JSJ, there is a vertex $U \subset V$ of Ψ_1 such that $\langle T \rangle$ is conjugate to a subgroup of $\langle U \rangle$. By proposition 17, $\{t_1, t_2\} \subset U$. As Ψ_1 is a tree, we may assume E' is an edge of Ψ_1 incident to U . Let Q be the vertex of E' opposite U . Let Ψ' be the graph of groups decomposition obtained from Ψ by replacing $\langle V \rangle$ by Ψ_1 . Since Ψ_1 is reduced, there exists $x \in Q - E$.

By proposition 23, $\Gamma - E$ has exactly two components which intersect V non-trivially, one containing t_1 and the other containing t_2 . Hence x can be connected to t_1 or t_2 by a path in $\Gamma - E$. This is impossible as E' separates U and Q in Ψ' and so by lemma 9, E separates $U - E$ and $Q - E$ in Γ .

Suppose Ψ_1 looks weakly $M(\Psi, \langle V \rangle)$ -JSJ and $B \in M(\Psi, \langle V \rangle)$. By proposition 18, there is $A \subset V$ such that $\langle A \rangle \in M(\Psi, \langle V \rangle)$ and some conjugate of $\langle A \rangle$ is a subgroup of (finite index in) B .

Suppose x and y are elements of A and there is no vertex group of Ψ_1 containing $\{x, y\}$. Then x and y are separated by E in $\Gamma(W, S)$ for E an edge of Ψ_1 . This is impossible as Ψ_1 looks weakly $M(\Psi, \langle V \rangle)$ -JSJ. We conclude that $\{x, y\} \subset U$ for some $U \subset V$ a vertex of Ψ_1 .

By lemma 24, A is a subgroup of a conjugate of a vertex group of Ψ_1 . Let T be the Bass-Serre tree for Ψ_1 and U a vertex of T stabilized by $\langle A \rangle$. Let B' be a conjugate of B such that $\langle A \rangle$ has finite index in B' . If the cosets of $\langle A \rangle$ in B' are $\langle A \rangle, b_1 \langle A \rangle, \dots, b_n \langle A \rangle$, then the orbit of $B'U$ in T is $U, b_1 U, \dots, b_n U$. By corollary 4.8 of [5], B' stabilizes some vertex of T . Equivalently, B is a subgroup of a conjugate of a vertex group of Ψ_1 , and hence Ψ_1 is weakly $M(\Psi, \langle V \rangle)$ -JSJ. \square

Lemma 26 *Suppose (W, S) is a Coxeter system, Ψ a reduced visual graph of groups decomposition for (W, S) , $V \subset S$ a vertex of Ψ , Λ a reduced Ψ -compatible graph of groups decomposition of $\langle V \rangle$, Ψ_1 the reduced visual decomposition for Λ from the construction for theorem 7, $E \subset V$ such that $\langle E \rangle$ is an edge group of Ψ_1 , T the Bass-Serre tree for Λ and E' the edge of T such that $E = \{v \in V : v \text{ stabilizes } E'\}$. If $\{x, y\} \subset V - E$ and x (respectively y) stabilizes the vertex X (respectively Y) of T where X and Y are on different sides of E' , then x and y are in different components of $\Gamma - E$.*

Proof: If Φ is obtained from Ψ by replacing V by Λ , then T is a subtree of the Bass-Serre tree for Φ . Hence, E is an edge of Ψ_Φ the visual decomposition for Φ given by the construction for theorem 7. Now apply lemma 9. \square

Proposition 27 *Suppose (W, S) is a Coxeter system, Ψ is a visual graph of groups decomposition for (W, S) that is JSJ-amenable, $V \subset S$ is a vertex of Ψ , Λ is a reduced graph of groups decomposition of $\langle V \rangle$ with edge groups in $M(\Psi, \langle V \rangle)$, and Ψ_1 is the reduced visual decomposition for Λ given by the construction for theorem 7. Then Ψ_1 is weakly $M(\Psi, \langle V \rangle)$ -JSJ if and only if Λ is weakly $M(\Psi, \langle V \rangle)$ -JSJ.*

Proof: By lemmas 3 and 15, Ψ_1 and Λ are Ψ -compatible, and for each $E \subset V$ such that $\langle E \rangle$ is an edge group of Ψ_1 , $\langle E \rangle$ is in $M(\Psi, \langle V \rangle)$ and E separates V in $\Gamma(W, S)$. Assume Λ is weakly $M(\Psi, \langle V \rangle)$ -JSJ. If Ψ_1 is not weakly $M(\Psi, \langle V \rangle)$ -JSJ, then proposition 25 implies Ψ_1 does not look weakly $M(\Psi, \langle V \rangle)$ -JSJ. I.e. there are E and B , subsets of V that separate $\Gamma(W, S)$ and generate crossing members of $M(\Psi, \langle V \rangle)$, such that E separates (in Γ) elements b_1 and b_2 of B , and $\langle E \rangle$ is an edge group of Ψ_1 . Let T be the Bass-Serre tree for Λ . The construction of visual decompositions for theorem 7 implies there is an edge E' of T such that $E = \{v \in V : v \text{ stabilizes } E'\}$ and since Ψ_1 is reduced, we may assume there are elements x and y of $V - E$ that stabilize vertices X and Y (respectively) of T on opposite sides of E' . Since Λ is weakly $M(\Psi, \langle V \rangle)$ -JSJ, there is a vertex U of T such that B stabilizes U . By proposition 23, $\Gamma - E$ has exactly two components which intersect V non-trivially, one containing b_1 and the other containing b_2 . There is an element z of $V - E$ that stabilizes a vertex of T on the side of E' opposite U . But by lemma 26, z cannot be in the same component of $\Gamma - E$ as b_1 or b_2 , which is nonsense. The converse is trivial. \square

6 $M(\Psi, \langle V \rangle)$ -JSJ Decompositions

Suppose (W, S) is a Coxeter system, Ψ is a reduced visual graph of groups decomposition for (W, S) , and $V \subset S$ is a vertex of Ψ . We define, a weakly $M(\Psi, \langle V \rangle)$ -JSJ decomposition Λ , to be $M(\Psi, \langle V \rangle)$ -JSJ if for any vertex group U of Λ and non-trivial Λ -compatible splitting of U over an $M(\Psi, \langle V \rangle)$ subgroup, the resulting (reduced) decomposition of Λ is not weakly $M(\Psi, \langle V \rangle)$ -JSJ.

Suppose Ψ_1 is a visual weak $M(\Psi, \langle V \rangle)$ -JSJ decomposition, $E \subset V$ such that $\langle E \rangle$ is a non-crossing element of $M(\Psi, \langle V \rangle)$, and E separates U in $\Gamma(W, S)$ for U a vertex of Ψ_1 . Since $\langle E \rangle$ is non-crossing, this splitting is Ψ_1 -compatible, visual and non-trivial. By lemma 3, the resulting decomposition of Ψ_1 is compatible with Ψ . We say a visual weakly $M(\Psi, \langle V \rangle)$ -JSJ decomposition Ψ_1 *looks* $M(\Psi, \langle V \rangle)$ -JSJ if for any $E \subset V$ such that $\langle E \rangle$ is a non-crossing member of $M(\Psi, \langle V \rangle)$ and vertex U of Ψ_1 such that $E \subset U \subset V$, E does not separate U in Γ .

The next proposition implies the existence of visual $M(\Psi, \langle V \rangle)$ -JSJ decompositions for a given JSJ-amenable graph of groups decomposition Ψ .

Proposition 28 *Suppose (W, S) is a Coxeter system, Ψ is a visual graph of groups decomposition of (W, S) that is JSJ-amenable, and $V \subset S$ is a vertex of Ψ . A visual weak $M(\Psi, \langle V \rangle)$ -JSJ decomposition Ψ_1 of $\langle V \rangle$, looks $M(\Psi, \langle V \rangle)$ -JSJ if and only if Ψ_1 is $M(\Psi, \langle V \rangle)$ -JSJ.*

Proof: Suppose Ψ_1 looks $M(\Psi, \langle V \rangle)$ -JSJ, but is not $M(\Psi, \langle V \rangle)$ -JSJ. Then there is $D \in M(\Psi, \langle V \rangle)$ and a vertex $U \subset V$ of Ψ_1 such that $\langle U \rangle$ splits non-trivially and Ψ_1 -compatibly as $A *_D B$, and the resulting reduced decomposition Λ (of $\langle V \rangle$), is weakly $M(\Psi, \langle V \rangle)$ -JSJ. Let Ψ' be the reduced visual decomposition for $A *_D B (= \langle U \rangle)$ given by the construction for theorem 7. As no conjugate of $\langle U \rangle$ is contained in A or B , Ψ' has more than one vertex. Any visual subgroup of $\langle U \rangle$ contained in a conjugate of A or B is contained in a vertex group of Ψ' by the construction for theorem 7. Hence Ψ' is compatible with Ψ_1 . Let Ψ'_1 be obtained from Ψ_1 by replacing $\langle U \rangle$ by Ψ' , and reducing. Then Ψ'_1 is a reduced visual graph of groups decomposition for Λ as given by the construction for theorem 7. By proposition 27, Ψ'_1 is weakly $M(\Psi, \langle V \rangle)$ -JSJ and so Ψ_1 did not look $M(\Psi, \langle V \rangle)$ -JSJ to begin with. The converse is trivial. \square

Theorem 29 *Suppose (W, S) is a Coxeter system, Ψ is a visual graph of groups decomposition of (W, S) that is JSJ-amenable, and $V \subset S$ is a vertex of Ψ . If Λ is a reduced $M(\Psi, \langle V \rangle)$ -JSJ graph of groups decomposition of $\langle V \rangle$, and Ψ_1 is the reduced visual decomposition derived from Λ by the construction for theorem 7, then*

1. Ψ_1 is an $M(\Psi, \langle V \rangle)$ -JSJ decomposition.
2. There is a (unique) bijection α of the vertices of Λ to the vertices of Ψ_1 such that for each vertex U of Λ , $\Lambda(U)$ is conjugate to $\Psi(\alpha(U))$.
3. Each edge group of Λ is conjugate to a special subgroup of $\langle V \rangle$.

Proof: The decomposition Ψ_1 is weakly $M(\Psi, \langle V \rangle)$ -JSJ by Proposition 27. Suppose U is a vertex of Λ with vertex group $A = \Lambda(U)$. We wish to show that A is a subgroup of a conjugate of a vertex group of Ψ_1 . Otherwise, the action of A on T , the Bass-Serre tree for Ψ_1 , defines a non-trivial reduced graph of groups decomposition Φ of A such that each edge group of Φ is a subgroup of a conjugate of an $(M(\Psi, \langle V \rangle))$ edge group of Ψ_1 . If A does not stabilize a vertex of T , then Φ is nontrivial. If C is an edge group of

Λ incident with A , then C contains a conjugate of an edge group Q of Ψ_1 . Since C and Q are in $M(\Psi, \langle V \rangle)$, a conjugate of Q is of finite index in C and at the same time stabilizes an edge of T . But then the orbit of this edge in T under the action of C is finite. By Corollary 4.8 of [5], C stabilizes some vertex of T and so is contained in a conjugate of a vertex group of Φ . Hence Φ is compatible with Λ . We also require Φ to be compatible with Ψ . Suppose $D \subset V$ is an edge of Ψ incident to V , such that some conjugate of $\langle D \rangle$ is a subgroup of A . Then $D \subset Q$ for Q a vertex of Ψ_1 . Hence $\langle D \rangle$ stabilizes a vertex group of T and so is a subgroup of a conjugate of a vertex group of Φ , as required. Replacing A in Λ by this decomposition and reducing gives a Ψ -compatible reduced graph of groups decomposition Λ' of $\langle V \rangle$ with $M(\Psi, \langle V \rangle)$ edge groups. By hypothesis, Λ' is not weakly $M(\Psi, \langle V \rangle)$ -JSJ. Hence there exists B , an $M(\Psi, \langle V \rangle)$ subgroup of $\langle V \rangle$ such that $\langle V \rangle$ splits non-trivially over B and B is not a subgroup of a conjugate of a vertex group of Λ' . Since Λ is weakly $M(\Psi, \langle V \rangle)$ -JSJ, B is a subgroup of a conjugate of A . But, since Ψ_1 is weakly $M(\Psi, \langle V \rangle)$ -JSJ, B is a subgroup of a conjugate of a vertex group of Ψ_1 and hence of a vertex group of Φ and of Λ' . We conclude that A is a subgroup of a conjugate of a vertex group of Ψ_1 , as desired.

If $A (= \Lambda(U))$ is a subgroup of a conjugate of $\Psi_1(U') (\equiv \langle U' \rangle)$ for U' a vertex of Ψ_1 , then, since $\Psi_1(U')$ is a subgroup of a conjugate of a vertex group of Λ , A is a subgroup of a conjugate of a vertex group of Λ . By lemma 2 the vertex group A at U is a subgroup of a conjugate of a vertex group at U'' only if $U = U''$ and (since Λ is a tree) the conjugate is by an element of A . But then A is conjugate to $\Psi_1(U')$. Since no vertex group of Ψ_1 is contained in a conjugate of another, U' is uniquely determined, and we set $\alpha(U) = U'$. Since each vertex group $\Psi_1(U')$ is contained in a conjugate of some $\Lambda(U)$ which is in turn conjugate to $\Psi_1(\alpha(U))$ we must have $U' = \alpha(U)$ and each U' is in the image of α .

If Ψ_1 is not $M(\Psi, \langle V \rangle)$ -JSJ, then it does not look $M(\Psi, \langle V \rangle)$ -JSJ and some vertex group W_1 of Ψ_1 visually splits nontrivially and Ψ_1 -compatibly over an $M(\Psi, \langle V \rangle)$ visual subgroup U_1 to give a Ψ -compatible visual graph of groups decomposition Φ of $\langle V \rangle$ with U_1 an edge group. Now W_1 is conjugate to a vertex group A of Λ . As a subgroup of $\langle V \rangle$, A acts on the Bass-Serre tree T' for Φ , but A cannot stabilize a vertex of T' , otherwise W_1 stabilizes a vertex of T' and we assumed W_1 split nontrivially. As above this contradicts the assumption that Λ is $M(\Psi, \langle V \rangle)$ -JSJ, implying instead that Ψ_1 is $M(\Psi, \langle V \rangle)$ -JSJ.

Since Λ is a tree, we can take each edge group of Λ as contained in its

endpoint vertex groups taken as subgroups of $\langle V \rangle$. Hence each edge group is simply the intersection of its incident vertex groups (up to conjugation). Since vertex groups of Λ correspond to conjugates of vertex groups in Ψ_1 , their intersection is conjugate to a visual subgroup by Lemma 11. \square

Theorem 29 shows that all $M(\Psi, \langle V \rangle)$ -JSJ decompositions for Coxeter groups are basically visual. The next collection of lemmas lead to a proof of the uniqueness of $M(\Psi, \langle V \rangle)$ -JSJ decompositions for Coxeter groups.

Lemma 30 *Suppose (W, S) is a Coxeter system, Ψ is a visual graph of groups decomposition of (W, S) that is JSJ-amenable, $V \subset S$ is a vertex of Ψ and Ψ_1 is a visual $M(\Psi, \langle V \rangle)$ -JSJ decomposition for $\langle V \rangle$. If $D \subset V$ is such that D separates V in $\Gamma(W, S)$ and $\langle D \rangle \in M(\Psi, \langle V \rangle)$, then there is $Q \subset V$ a vertex of Ψ_1 such that $D \subset Q$. Furthermore, if $\langle D \rangle$ is non-crossing in $M(\Psi, \langle V \rangle)$, then $E(D)$ is a subset of an edge group of Ψ_1 .*

Proof: By definition some conjugate of D is a subset of $\langle Q \rangle$ for Q a vertex of Ψ_1 , but we require more. By proposition 17, $E(D) \subset Q$. We have $D = E(D) \cup N$ where $N \subset V$ generates a finite subgroup of $\langle V \rangle$ which commutes with $E(D)$. As N determines a complete subdiagram of $\Gamma(\langle V \rangle, V)$, N is a subset of some vertex of Ψ_1 . Let D' be a largest subset of D such that $N \subset D'$, $D' \subset Q_1 \subset V$ and Q_1 is a vertex of Ψ_1 . If $D' \neq D$, let $a \in D - D'$. Observe that $\{a\} \cup N$ generates a finite group and so there exists $Q_2 (\subset V)$ a vertex of Ψ_1 such that $\{a\} \cup N \subset Q_2$. If U is a vertex common to the three Ψ_1 -geodesics connecting pairs in $\{Q, Q_1, Q_2\}$, then $\{a\} \cup D' \subset U$ contrary to the definition of D' . We conclude $D' = D$ as required.

The second part of the lemma is not used in the rest of the paper and is tedious to prove. The reader may wish to skip this argument on a first reading.

Assume $\langle D \rangle$ is non-crossing in $M(\Psi, \langle V \rangle)$ and Q is a vertex of Ψ_1 with $D \subset Q \subset V$. Assume $E(D) \neq \emptyset$. If Q' is a vertex of Ψ_1 , distinct from Q and $E(D) \subset Q'$ then for any $F \subset V$ such that $\langle F \rangle$ is the edge group of an edge on the Ψ_1 -geodesic connecting Q and Q' , $E(D) \subset F$. Hence we may assume:

(*) Q is the only vertex of Ψ_1 such that $E(D) \subset Q$.

Even as D separates V in Γ , D does not separate Q in Γ (otherwise, since D is non-crossing and does not separate E (for $\langle E \rangle$ an edge group of Ψ) in Γ , $\langle Q \rangle$ splits non-trivially and compatibly with Ψ_1 and Ψ over D - contrary to our JSJ assumption on Ψ_1). Hence $Q - D \subset C$ for C a component of $\Gamma - D$.

Next we show that if C' is a component of $\Gamma - D$ other than C such that C' intersects V non-trivially, and $x \in E(D)$ then there is a vertex $V_{x,C'}$ of Ψ_1 such that $x \in V_{x,C'}$ and $V_{x,C'} \cap C' \neq \emptyset$. First observe that $C' \cap Q = \emptyset$. By lemma 20, for each $x \in E(D)$ there is an edge $[xa]$ of Γ such that $a \in C'$. Let $x \equiv x_0, x_1, \dots, x_n$ be the consecutive vertices of a path in Γ from x to $x_n \in V \cap C'$ such that for $i > 0$, $x_i \in C'$. We may assume n is the smallest integer such that $x_n \in V \cap C'$. Since $x \notin C'$, $n \neq 0$. Let Ψ' be the visual graph of groups decomposition of W obtained by replacing the vertex V of Ψ by Ψ_1 . By lemma 8, for each $i \in \{1, 2, \dots, n\}$ there is a vertex $V_i \subset S$ of Ψ' such that $\{x_{i-1}, x_i\} \subset V_i$. Define $V_0 \equiv Q$. Since $Q \cap C' = \emptyset$, $V_i \neq Q$ for $i > 0$. Let α_i be the Ψ' -geodesic from $Q \equiv V_0$ to V_i and β_i be the Ψ' -geodesic from V_i to V_{i+1} for $i \geq 1$. By lemma 8, if $B \subset S$ is a vertex of β_i , then $x_i \in C' \cap B$ and so β_i does not pass through Q . This implies that α_n can be written as a non-trivial subpath τ of α_1 followed by a path λ where each edge of λ is an edge of some β_i . Let $Q \equiv X_1, \dots, X_k \equiv V_n$ be the consecutive vertices of α_n . If X_i is the end point of τ then $x_0 \in X_i$ (lemma 8 implies x_0 is an element of every vertex of α). Since X_i is also the initial point of λ , $\{x_0, x_m\} \subset X_i$ for some $m \in \{1, \dots, n\}$. Since Ψ_1 is a subtree of Ψ' , there is $j \in \{1, \dots, k\}$ such that X_1, \dots, X_j are vertices Ψ_1 and X_{j+1}, \dots, X_k are not. If $j \geq i$, then as $\{x_0, x_m\} \subset X_i$, set $V_{x,C'} = X_i$ to finish. If $j < i$, then as Ψ_1 is a subtree of Ψ' , any Ψ' -geodesic from $V_n \equiv X_k$ to a vertex of Ψ_1 must have initial segment with consecutive vertices $V_n \equiv X_k, X_{k-1}, \dots, X_j$. Hence if $Q' \subset V$ is a vertex of Ψ_1 containing $x_n (\in V \cap C')$, then the Ψ' geodesic from V_n to Q' passes through X_j . As $x_n \in V_n \cap Q'$, lemma 8 implies $x_n \in X_j$. Since $j < i$, $x_0 \in X_j$ and set $V_{x,C'} = X_j$ to finish the claim.

Note that for all $x \in E(D)$ and for all components $C' \neq C$ of $\Gamma - D$, $V_{x,C'} \neq Q$ (as $C' \cap Q = \emptyset$). For $x \in E(D)$ and $C' \neq C$ a component of $\Gamma - D$, there may be more than one possible choice for $V_{x,C'}$. Assume that $E(D) \cap V_{x,C'}$ is maximal over all possible choices. I.e. if X is a vertex of Ψ_1 , $x \in X$, and $X \cap C' \neq \emptyset$ then the number of elements of $X \cap E(D)$ is less than or equal to the number of elements of $V_{x,C'} \cap E(D)$. By (*), $E(D) \not\subset V_{x,C'}$. Let $y \in E(D) - V_x$, then Q , $V_{y,C'}$ and $V_{x,C'}$ are distinct. There is a vertex U common to the three geodesics of Ψ_1 connecting pairs in $\{Q, V_{x,C'}, V_{y,C'}\}$ and so $[E(D) \cap V_{x,C'}] \cup \{y\} \subset U$. In particular, $x \in U$. By the maximality condition on $V_{x,C'} \cap E(D)$, $U \cap C' = \emptyset$. Hence $V_{x,C'} \neq U \neq V_{y,C'}$ and U separates $V_{x,C'}$ and $V_{y,C'}$ in Ψ_1 . If Ψ' is the graph of groups decomposition of W obtained from Ψ by replacing V by Ψ_1 , then Ψ_1 is a subtree of the tree Ψ' and U separates $V_{x,C'}$ and $V_{y,C'}$ in Ψ' . Suppose α is an edge path in C' (with

consecutive vertices c_0, \dots, c_n) connecting $c_0 \in C' \cap V_{x,C'}$ to $c_n \in C' \cap V_{y,C'}$. Let $C_0 = V_{x,C'}$, C_i be a vertex of Ψ' such that C_i contains $\{c_{i-1}, c_i\}$ and $C_{n+1} = V_{y,C'}$. If X is a vertex of the Ψ' -geodesic from C_i to C_{i+1} , then lemma 8 implies X contains $c_i \in C'$. Hence, α defines a path in Ψ' from $V_{x,C'}$ to $V_{y,C'}$ avoiding U (which is impossible). \square

For a Coxeter system (W, S) and vertex $V \subset S$ of a visual $M(W, S)$ -JSJ decomposition of (W, S) , V may not be a connected subset of $\Gamma(W, S)$.

Example 3. Let (W, S) be the Coxeter system given by $\Gamma(W, S)$ pictured in figure 3, where each edge has label 3. The only non-crossing visual virtually abelian splitting subgroups for this system are $\langle x, y \rangle$ and $\langle u, v \rangle$ ($\langle x, v \rangle$ and $\langle u, y \rangle$ are crossing). The JSJ-decomposition is given by:

$$\langle a, b, x, y \rangle *_{\langle x, y \rangle} \langle x, y, u, v \rangle *_{\langle u, v \rangle} \langle u, v, c, d \rangle$$

The set $\{x, y, u, v\}$ generates a vertex group of this decomposition and is not connected in Γ .

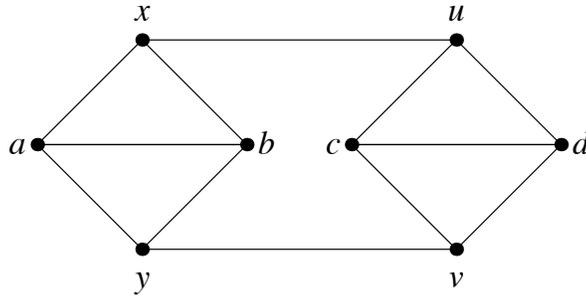


Figure 3: A disconnected JSJ vertex group

Still we have the following:

Lemma 31 *Suppose (W, S) is a Coxeter system, Ψ is a visual graph of groups decomposition of (W, S) that is JSJ-amenable, $V \subset S$ is a vertex of Ψ , and Ψ_1 is a visual $M(\Psi, \langle V \rangle)$ -JSJ decomposition of $\langle V \rangle$. If $D \subset V$, $\langle D \rangle \in M(\Psi, \langle V \rangle)$, D separates V in $\Gamma(W, S)$, and $\langle D \rangle$ is non-crossing in $M(\Psi, \langle V \rangle)$, then there is no $Q \subset V$ a vertex of Ψ_1 such that D separates Q in $\Gamma(W, S)$.*

Note that in the previous example, neither $\{x, y\}$ nor $\{u, v\}$ separates $\{x, y, u, v\}$ in Γ .

Proof: Suppose otherwise. By lemma 30, there is $Q' \subset V$ a vertex of Ψ_1 such that $D \subset Q'$. As Ψ_1 is $M(\Psi, \langle V \rangle)$ -JSJ, D does not separate Q' in Γ . In particular, $Q \neq Q'$. Suppose $F \subset V$ is such that $\langle F \rangle$ is the group of an edge of the Ψ_1 -geodesic connecting the vertices Q and Q' . As F separates Q and Q' in the tree Ψ_1 , (and since $D \subset Q'$ separates Q in Γ) F separates Q in Γ . But if $\langle F \rangle$ is the group of the edge of this geodesic incident to Q , there is an induced splitting of $\langle Q \rangle$ over $\langle F \rangle$ compatible with Ψ_1 (since $\langle F \rangle$ is non-crossing) and Ψ (since Ψ is JSJ-amenable). This is impossible as Ψ_1 is $M(\Psi, \langle V \rangle)$ -JSJ. \square

Theorem 32 *Suppose (W, S) is a Coxeter system, Ψ is a visual graph of groups decomposition of (W, S) that is JSJ-amenable, $V \subset S$ is a vertex of Ψ , and Ψ_1 and Ψ_2 are visual $M(\Psi, \langle V \rangle)$ -JSJ decomposition of $\langle V \rangle$. If $Q \subset V$ is a vertex of Ψ_1 , then Q is a vertex of Ψ_2 . I.e. The vertex groups of two visual $M(\Psi, \langle V \rangle)$ -JSJ decompositions of $\langle V \rangle$ are the same.*

Proof: It suffices to show that $Q \subset Q'$, for Q' a vertex of Ψ_2 . Otherwise, there exists $D \subset V$ an edge of Ψ_2 and D separates Q in Γ . This is contrary to lemma 31. \square

Theorems 29 and 32 imply:

Corollary 33 *Suppose (W, S) is a Coxeter system, Ψ is a visual graph of groups decomposition of (W, S) that is JSJ-amenable, $V \subset S$ a vertex of Ψ , Λ_1 and Λ_2 are $M(\Psi, \langle V \rangle)$ -JSJ decomposition of $\langle V \rangle$. Then there is a (unique) bijection α of the vertices of Λ_1 to the vertices of Λ_2 such that for each vertex Q of Λ_1 , $\Lambda_1(Q)$ is conjugate to $\Lambda_2(\alpha(Q))$. \square*

Remark 2. For a Coxeter system (W, S) , the vertex groups of two visual $M(W, S)$ -JSJ decompositions must be the same, but it is unreasonable to expect two such graph of groups decompositions to be exactly the same. As an example, a Coxeter diagram $\Gamma(W, S)$ may be such that $S = A \cup B \cup C$ where $A \cap B = A \cap C = B \cap C = E$ and a visual $M(W, S)$ -JSJ decomposition is given by $\langle A \rangle *_{\langle E \rangle} \langle B \rangle *_{\langle E \rangle} \langle C \rangle$ or $\langle A \rangle *_{\langle E \rangle} \langle C \rangle *_{\langle E \rangle} \langle B \rangle$. See for instance example 4 of section 8.

7 JSJ-Decompositions

Suppose (W, S) is a finitely generated Coxeter system. Let Ψ_0 be the trivial (single vertex) graph of groups decomposition of W . Then Ψ_0 is vacuously JSJ-amenable. If Ψ_1 is the (unique) visual $M(W, S)(= M(\Psi_0, \langle S \rangle))$ -JSJ decomposition of W and $E \subset S$ is such that $\langle E \rangle$ is the group of an edge incident to $V \subset S$ for V a vertex of Ψ_1 , then $\langle E \rangle \in M(\Psi_0, \langle S \rangle)$. If $A \subset V$ is such that $\langle A \rangle$ is virtually abelian and A separates E in $\Gamma(W, S)$ then lemma 22 implies $\langle A \rangle \in M(\Psi_0, \langle S \rangle)$. But then $\langle E \rangle$ and $\langle A \rangle$ are crossing in $M(\Psi_0, \langle S \rangle)$, contrary to the fact that $\langle E \rangle$ is an edge group of Ψ_1 (the (unique) visual $M(W, S)(= M(\Psi_0, \langle S \rangle))$ -JSJ decomposition of W). Instead, Ψ_1 is JSJ-amenable.

If $\langle V \rangle$ is a vertex group of Ψ_1 , lemma 3 implies the (visual) $M(\Psi_1, \langle V \rangle)$ -JSJ decomposition of $\langle V \rangle$ is compatible with Ψ_1 . Inductively assume Ψ_j is JSJ-amenable for all $j < i$. Let Ψ_i be obtained from Ψ_{i-1} by replacing each vertex group $\langle V \rangle$ of Ψ_{i-1} by the (unique) $M(\Psi_{i-1}, \langle V \rangle)$ -JSJ decomposition insured by theorem 32. By corollary 33, we may assume this decomposition is visual. The compatibility of this decomposition with Ψ_{i-1} is insured by repeated application of lemma 3. Our first result in this section will be to show Ψ_i is JSJ-amenable for all i , insuring our definitions are meaningful. We call Ψ_i the i^{th} -stage of the JSJ-decomposition for (W, S) . As all decompositions involved are visual, there is an integer n such that for every vertex group $\langle V \rangle$ of Ψ_n , and $\langle D \rangle \in M(\Psi_n, \langle V \rangle)$ if D separates $\langle V \rangle$ in $\Gamma(W, S)$ then $\langle D \rangle$ is crossing. In this case $\Psi_n = \Psi_{n+1}$ and we define Ψ_n to be a JSJ-decomposition of W . This decomposition is unique in the sense of corollary 33. In particular, if (W, S) and (W, S') are finitely generated Coxeter systems, then the JSJ-decompositions of W derived from (W, S) and (W, S') have conjugate vertex groups.

Proposition 34 *Suppose (W, S) is a finitely generated Coxeter system. Then Ψ_n , the n^{th} -stage of the JSJ-decomposition for (W, S) is JSJ-amenable for all n .*

Proof: We have shown Ψ_0 and Ψ_1 are JSJ-amenable. Assume Ψ_j is JSJ-amenable for all $j < n$. Then Ψ_n is a visual graph of groups and each edge group of Ψ_n is a member of $M(\Psi_j, \langle Q \rangle)$ for $j < n$. Suppose $V \subset S$ is a vertex of Ψ_n and $E \subset S$ is such that $\langle E \rangle$ is the group of an edge of Ψ_n incident to V . We must show there is no $A \subset V$ such that $\langle A \rangle$ is virtually abelian and A

separates E in Γ . Suppose otherwise. We may assume $\langle E \rangle$ is the group of an edge of the $M(\Psi_j, \langle U \rangle)$ -JSJ decomposition of $\langle U \rangle$ for U a vertex of Ψ_j , where $j < n$ and $V \subset U$. If $j < n - 1$, then A cannot separate E in Γ since Ψ_j is JSJ-amenable. If $j = n - 1$, and A separates E in Γ , then lemma 22 implies $A \in M(\Psi_{n-1}, \langle U \rangle)$ and that $\langle A \rangle$ and $\langle E \rangle$ are crossing in $M(\Psi_{n-1}, \langle U \rangle)$. But then $\langle E \rangle$ is not the group of the edge of the $M(\Psi_{n-1}, \langle U \rangle)$ decomposition of $\langle U \rangle$. Instead, Ψ_n is JSJ-amenable. \square

Proposition 35 *Suppose (W, S) is a Coxeter system, Ψ_i is the i^{th} -stage of the JSJ-decomposition for (W, S) , $V \subset S$ is a vertex of Ψ_i , $\langle A \rangle \in \mathcal{C}(W, S)$ and $B \subset V$ such that B separates V in $\Gamma(W, S)$ and $B \in M(\Psi_i, \langle V \rangle)$. If A separates B in $\Gamma(W, S)$ then $A \subset V$.*

Proof: This result is trivial for $i = 0$ as $V = S$ in this case. Suppose the proposition fails. Let $i \geq 1$ be the largest integer such that for all $j < i$, there is no $A \subset S$ and vertex V of Ψ_j such that $\langle A \rangle \in \mathcal{C}(W, S)$, A separates B in Γ for some $\langle B \rangle \in M(\Psi_j, \langle V \rangle)$, B separates V in Γ , and $A \not\subset V$. Then there is $\langle A \rangle \in \mathcal{C}(W, S)$, V' a vertex of Ψ_i , and $B \subset V'$ such that B separates V' in Γ , $\langle B \rangle \in M(\Psi_i, \langle V' \rangle)$, A separates B in Γ , and $A \not\subset V'$. Assume $\{b_1, b_2\} \subset B$ and A separates b_1 from b_2 in Γ . By lemma 22, $B - \{b_1, b_2\} = A - \{a_1, a_2\} \equiv M$ where B separates a_1 and a_2 in Γ . Let V be the vertex group of Ψ_{i-1} containing V' , and Ψ'_i the visual $M(\Psi_{i-1}, \langle V \rangle)$ -JSJ decomposition that splits $\langle V \rangle$ to give Ψ_i .

We show $B \notin M(\Psi_{i-1}, \langle V \rangle)$. Assume otherwise. As A separates B in Γ the definition of i implies that $A \subset V$. Lemma 22 implies $A \in M(\Psi_{i-1}, \langle V \rangle)$. As $M \cup \{a_1, a_2\} = A \not\subset V'$ and $M \subset B \subset V'$, we assume $a_1 \notin V'$. Then as Ψ'_i is a tree, there is $D \subset V$ such that $\langle D \rangle$ is the group of an edge of Ψ'_i incident to V' such that any path in Γ from a_1 to V' intersects D . Thus, $M \subset D$. Any path from a_1 to a_2 intersects $B \subset V'$, and so such a path intersects D . As $D \subset V'$, $a_1 \notin D$. If $a_2 \notin D$, then D separates a_1 and a_2 in Γ , so that D and A are crossing in $M(\Psi_{i-1}, \langle V \rangle)$. But no edge of Ψ'_i is crossing in $M(\Psi_{i-1}, \langle V \rangle)$. Instead, $a_2 \in D \subset V'$. As $\{b_1, b_2\} \subset V'$, and since any path from b_1 to b_2 contains a_1 or a_2 , any path from b_1 to b_2 intersects D . Hence, as $\langle D \rangle$ is non-crossing in $M(\Psi_{i-1}, \langle V \rangle)$, and since $B \in M(\Psi_{i-1}, \langle V \rangle)$, $\{b_1, b_2\} \cap D \neq \emptyset$. Furthermore, $\{b_1, b_2\} \not\subset D$, since otherwise, A would cross (the non-crossing) D in $M(\Psi_{i-1}, \langle V \rangle)$. We assume $b_1 \in D$ and $b_2 \in V' - D$. The set $\{a_2\} \cup M$ does not separate V in Γ as $A \in M(\Psi_{i-1}, \langle V \rangle)$. Let α be a shortest path in Γ from b_1 to b_2 , avoiding $\{a_2\} \cup M$. Then α contains

a_1 , and so the segment of α from a_1 to b_2 contains $d \in D$. The segment of α from d to b_2 avoids A . Now $\{b_1, d\} \subset D - (\{a_2\} \cup M)$. There is no edge connecting b_1 and d , since b_1 cannot be connected to b_2 avoiding A . As $\langle D \rangle$ is virtually abelian, theorem 13 implies a_2 is connected to b_1 and to d by an edge (labeled 2). The segment of α from a_1 to d followed by the edge from d to a_2 avoids B , which is nonsense. Instead, $\langle B \rangle \notin M(\Psi_{i-1}, \langle V \rangle)$ as desired.

Proposition 18 implies there is $K \subset V$ such that K separates V in Γ and $\langle K \rangle$ is a maximal rank element of $M(\Psi_{i-1}, \langle V \rangle)$ such that $E(K) \subset E(B)$. Then $\langle E(K) \rangle$ has infinite index in B . As above we assume $a_1 \notin V'$ and there is $D \subset V$ such that $\langle D \rangle$ is the group of an edge of Ψ'_i incident to V' such that any path in Γ from a_1 to V' intersects D . In particular $M \subset D$.

We show $E(K) = E(M) = E(D)$. If $\{b_1, b_2\} \subset K$, then A separates K and lemma 22 implies $Rank(K) = Rank(A) (= Rank(B))$ which is nonsense. Instead, $E(K) = E(K \cap B) \subset E(M)$. As $M \subset D$, $E(K) \subset E(M) \subset E(D)$. Since it is also true that $\{K, D\} \subset M(\Psi_{i-1}, \langle V \rangle)$, we see that $E(D) = E(K)$ and so $E(K) = E(M) = E(D)$.

Next we show $E(M)$ is not a subset of the group of an edge of Ψ_{i-1} (incident to V). Otherwise, there is a $j < i-1$, a vertex \hat{V} of Ψ_j containing V , and a non-crossing $F \in M(\Psi_j, \langle \hat{V} \rangle)$ such that $F \subset V$, and $E(M) \subset F$. But then K is non-crossing in $M(\Psi_j, \langle \hat{V} \rangle)$ and as K separates V in Γ , the visual $M(\Psi_j, \langle \hat{V} \rangle)$ -JSJ decomposition of $\langle \hat{V} \rangle$ was not maximal - which is nonsense.

As $E(M)$ is not a subset of the group of an edge of Ψ_{i-1} incident to V , $A \subset V$. If $a_2 \notin V'$, then there exists $D' \subset V$ such that $\langle D' \rangle$ is the group of an edge of Ψ'_i , D' separates a_2 from V' in Γ , and $E(M) = E(D')$. Note that $D - E(M)$ and $D' - E(M)$ generate finite groups. In particular, these sets define complete subgraphs of Γ . As $\langle B \rangle \in M(\Psi_i, V')$, $E(M)$ does not separate b_1 from b_2 in Γ . If α is a shortest path in Γ from b_1 to b_2 avoiding $E(M)$, then a_1 or a_2 is a vertex of α . If a_1 is a vertex of α , then some vertex t of α , between b_1 and a_1 belongs to $D - E(M)$ and some vertex s of α between a_1 and b_2 belongs to $D' - E(M)$. But there is an edge between t and s contradicting the minimality of α . Similarly if a_2 is a vertex of α . Instead, either a_1 or a_2 is an element of V' . If $a_2 \in V'$, then since $B \in M(\Psi_i, \langle V' \rangle)$, $\{a_2\} \cup M$ does not separate V' in Γ . Let α be a shortest path from b_1 to b_2 in Γ , avoiding $\{a_2\} \cup M$. Then a_1 is a vertex of α and as before, there exists t and s in $D - E(M)$ such that t is between b_1 and a_1 and s is between a_1 and b_2 . The edge joining t and s shortens α which is impossible. \square

Theorem 36 *Suppose Ψ is the visual JSJ-decomposition for (W, S) $\langle V \rangle$ is a vertex group of Ψ , Φ is an arbitrary graph of groups decomposition of W with virtually abelian edge groups, and T is the Bass-Serre tree for Φ . If $\langle A \rangle \in \mathcal{C}(W, S)$, then*

1. *A does not separate B in $\Gamma(W, S)$ if $\langle B \rangle$ is an edge group of Ψ ,*
2. *the decomposition of $\langle V \rangle$ induced by its action on T is compatible with Ψ , and*
3. *if $M(\Psi, \langle V \rangle)$ contains no crossing elements, then $\langle V \rangle$ stabilizes a vertex of Φ .*

Proof: If 1) fails, there is an integer i and a vertex group $\langle V \rangle$ of Ψ_i , the i^{th} -level visual JSJ-decomposition for (W, S) , such that $\langle B \rangle$ is an edge group of Ψ'_{i+1} the $M(\Psi_i, \langle V \rangle)$ visual JSJ-decomposition of $\langle V \rangle$. By proposition 35, $A \subset V$ and so by lemma 22, $\langle A \rangle$ and $\langle B \rangle$ are crossing in $M(\Psi_i, \langle V \rangle)$. But this is impossible as no edge of Ψ'_{i+1} is crossing in $M(\Psi_i, \langle V \rangle)$.

For the second part of the theorem, assume B is an edge of Ψ and $\langle B \rangle$ does not stabilize a vertex of T (equivalently, $\langle B \rangle$ is not a subgroup of a conjugate of a vertex group of Φ). Then the visual decomposition of (W, S) for Φ (given by the construction for theorem 7) has an edge group $\langle A \rangle$ such that A separates B in Γ - which is impossible.

For the third part of the theorem, let Λ be the decomposition of $\langle V \rangle$ determined by its action on Φ . Since Λ is compatible with Ψ , if Λ is non-trivial, then $M(\Psi, \langle V \rangle)$ is non-empty. But as Ψ is JSJ, $M(\Psi, \langle V \rangle)$ can only contain crossing elements. \square

8 Orbifold groups

If Ψ is the JSJ-decomposition of (W, S) and $\langle V \rangle$ is a vertex group of Ψ such that $M(\Psi, \langle V \rangle)$ contains no crossing members, then $\langle V \rangle$ is indecomposable with respect to its action on the Bass-Serre tree for any splitting of W over virtually abelian subgroups. If instead, $M(\Psi, \langle V \rangle)$ contains crossing members, then we say $\langle V \rangle$ is an *orbifold vertex group* of Ψ .

Theorem 37 *Suppose (W, S) is a finitely generated Coxeter system, Ψ is the (visual) JSJ-decomposition of (W, S) , and $\langle V \rangle$ is an orbifold vertex group of Ψ . Then $\langle V \rangle$ decomposes as $\langle T \rangle \times \langle M \rangle$ where $T \cup M = V$, M generates*

a virtually abelian group and the presentation diagram of T is either a loop of length ≥ 4 (in which case T generates a group that is virtually a closed surface group) or the presentation diagram of T is a disjoint union of vertices and simple paths (in which case T generates a virtually free group with graph of groups decomposition such that each vertex group is either \mathbb{Z}_2 or finite dihedral and each edge group is either trivial or \mathbb{Z}_2).

Proof: As Ψ is JSJ, the only visual subgroups of $M(\Psi, \langle V \rangle)$ are crossing. Say $\langle A \rangle$ and $\langle B \rangle$ are minimal rank elements of $M(\Psi, \langle V \rangle)$ that cross one another. If $C \subset V$ such that $\langle C \rangle$ is virtually abelian, and C separates V in Γ then $\text{Rank}(C) \geq \text{Rank}(A)$. If additionally $\text{Rank}(C) = \text{Rank}(A)$ then $\langle C \rangle \in M(\Psi, \langle V \rangle)$ and so $\langle C \rangle$ is crossing in $M(\Psi, \langle V \rangle)$.

By lemma 22, $A = \{a_1, a_2\} \cup M$ and $B = \{b_1, b_2\} \cup M$ where $\langle M \rangle$ is virtually abelian and commutes with $\{a_1, a_2, b_1, b_2\}$, A separates b_1 and b_2 in Γ and B separates a_1 and a_2 in Γ . By Proposition 23, $\Gamma - A$ has exactly two components which intersect V non-trivially, C_{b_1} containing b_1 and C_{b_2} containing b_2 . Then $V - M \subset \{a_1\} \cup C_{b_1} \cup \{a_2\} \cup C_{b_2}$. By lemma 20 there is an edge connecting (cyclically) adjacent members of this union, and no edge connecting non-adjacent members. In particular, $C_{b_1} \cup \{a_1, a_2\}$ is a connected subset of Γ . We show:

(i) $(C_{b_1} \cup \{a_1, a_2\}) - \{b_1\}$ has exactly two components which intersect V non-trivially, one containing a_1 and the other containing a_2 .

A path connecting a_1 and a_2 in $C_{b_1} \cup \{a_1, a_2\}$ avoiding b_1 also avoids b_2 and M , but this is impossible as B separates a_1 and a_2 in Γ . Hence b_1 separates a_1 and a_2 in $(C_{b_1} \cup \{a_1, a_2\})$. Let C_{b_1, a_1} (respectively C_{b_1, a_2}) be the component of $(C_{b_1} \cup \{a_1, a_2\}) - \{b_1\}$ containing a_1 (respectively a_2). If $z \in V$ is an element of a third component of $(C_{b_1} \cup \{a_1, a_2\}) - \{b_1\}$, then $z \in \Gamma - B$ and by proposition 23, there is a (shortest) path α in $\Gamma - B$ from z to either a_1 or a_2 . Without loss, say α connects z to a_1 . Then a_2 is not a point of α . Note that α is not contained in $C_{b_1} \cup \{a_1, a_2\}$ since b_1 separates z and a_1 in $C_{b_1} \cup \{a_1, a_2\}$ and α avoids b_1 . If β is the longest initial segment of α in $C_{b_1} \cup \{a_1, a_2\}$ with end point t , and s follows t on α , then β avoids A and $s \notin M \cup \{a_1, a_2\} \cup C_{b_1}$. Then s is connected to $z \in V$ by a path in $\Gamma - A$, implying $s \in C_{b_1} \cup C_{b_2}$. So $s \in C_{b_2}$. But then the edge $[ts]$ connects points of C_{b_1} and C_{b_2} , which is impossible (recall A separates C_{b_1} and C_{b_2} in Γ). So (i) is proved.

Observe that V is contained in the union $C_{b_1, a_1} \cup C_{b_1, a_2} \cup C_{b_2, a_1} \cup C_{b_2, a_2} \cup$

$M \cup \{b_1, b_2\}$. The only overlap among these sets is $C_{b_1, a_i} \cap C_{b_2, a_i} = \{a_i\}$. Next we show:

The set $\{b_1, a_1\} \cup M$ separates $C_{b_1, a_1} - \{a_1\}$ from $C_{b_2} \cup C_{b_1, a_2}$ in Γ . By the symmetry of the situation (see figure 4), the same argument implies:

(ii) For $u \in \{b_1, b_2\}$ and $v \in \{a_1, a_2\}$, $\{u, v\} \cup M$ separates $C_{u, v} - \{v\}$ from $C_t \cup C_{u, s}$ in Γ , where $\{t\} = \{b_1, b_2\} - \{u\}$ and $\{s\} = \{a_1, a_2\} - \{v\}$.

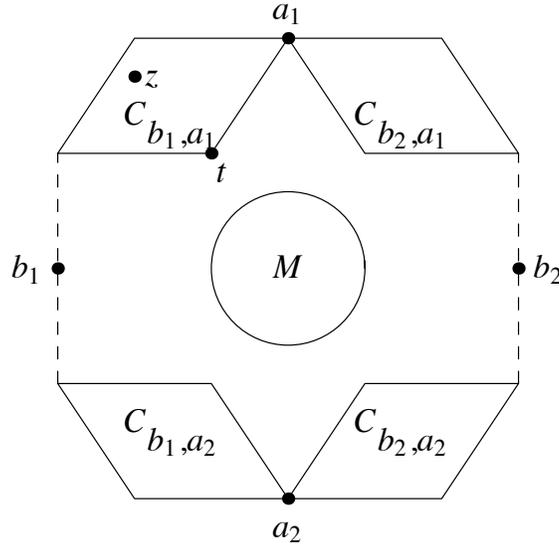


Figure 4: Circular decomposition of V in $\Gamma(W, S)$

By lemma 20 there is an edge from a_2 to C_{b_2} so $C_{b_2} \cup C_{b_1, a_2}$ is a connected subset of $\Gamma - (\{b_1, a_1\} \cup M)$. Suppose $z \in C_{b_1, a_1} - \{a_1\}$ and α is a path from z to a_2 in $\Gamma - (\{a_1, b_1\} \cup M)$. Let t be the last vertex of $\beta \equiv$ the longest initial segment of α in $C_{b_1, a_1} - \{a_1\}$ (then $t \neq a_2$) and s the vertex of α following t . Note that $s \notin \{a_1, b_1\} \cup M$ by the definition of α , $s \notin C_{b_1, a_1} - \{a_1\}$ by the definition of β , and s is not an element of a component of $(C_{b_1} \cup \{a_1, a_2\}) - \{b_1\}$ other than C_{b_1, a_1} , since b_1 separates C_{b_1, a_1} from those components in $C_{b_1} \cup \{a_1, a_2\}$. Hence $s \notin C_{b_1} \cup \{a_1, a_2\} \cup M$. The path β followed by the edge from t to s begins in C_{b_1} and avoids A . This implies $s \in C_{b_1}$, which is nonsense. So, (ii) is proved.

For u, v as in (ii), if $V \cap C_{u, v} \neq \{v\}$, then $\{u, v\} \cup M \equiv C$ separates V in Γ . There is no edge connecting u and v since otherwise, $\text{Rank}(\langle C \rangle) < \text{Rank}(\langle A \rangle)$. Hence $\text{Rank}(\langle C \rangle) = \text{Rank}(\langle A \rangle)$, and $\langle C \rangle$ is crossing in $M(\Psi, \langle V \rangle)$. Say $\langle C \rangle$

crosses $\langle D \rangle$. If D does not separate u and v in Γ then D must separate some other unrelated pair, m_1 and m_2 in M . But then D separates A in Γ and D separates B in Γ . By lemma 22, $D - \{d_1, d_2\} = A - \{m_1, m_2\}$ and $D - \{d'_1, d'_2\} = B - \{m_1, m_2\}$. Hence $\{a_1, a_2, b_1, b_2\} \subset D$. As D is virtually abelian and the pairs a_1, a_2 and b_1, b_2 are unrelated pairs, there is an edge (labeled 2) in Γ connecting a_i to b_j for all $i, j \in \{1, 2\}$. This is nonsense since we have assumed $u \in \{b_1, b_2\}$ and $v \in \{a_1, a_2\}$ are unrelated. Instead D separates u and v in Γ .

If $V \cap C_{b_1, a_1} \neq \{a_1\}$, then proposition 23 implies $\Gamma - (\{a_1, b_1\} \cup M)$ has exactly 2 components which intersect V non-trivially. Now, $\{b_1, a_1\} \cup M$ separates $C_{b_1, a_1} - \{a_1\}$ from (the connected set) $C_{b_2} \cup C_{b_1, a_2}$ in Γ . Hence we have:

(iii) If $V \cap C_{b_1, a_1} \neq \{a_1\}$ then $V \cap (C_{b_1, a_1} - \{a_1\})$ is contained in one of the two components of $\Gamma - (\{a_1, b_1\} \cup M)$ which intersect V non-trivially (call it Q_1) and $C_{b_2} \cup C_{b_1, a_2}$ is contained in the other (call it Q_2).

Next we improve (iii) by proving:

(iv) If $V \cap C_{b_1, a_1} \neq \{a_1\}$ then Q_1 , the component of $\Gamma - (\{a_1, b_1\} \cup M)$ containing $V \cap (C_{b_1, a_1} - \{a_1\})$, is a component of $C_{b_1, a_1} - \{a_1\}$. (In particular, the set $V \cap (C_{b_1, a_1} - \{a_1\})$ is contained in a single component of $C_{b_1, a_1} - \{a_1\}$.)

If K_1 is a component of $\Gamma - A$ other than C_{b_1} or C_{b_2} and α is a path connecting K_1 to $C_{b_1, a_1} - \{a_1\} (\subset C_{b_1})$ in $\Gamma - (\{a_1, b_1\} \cup M)$, then α contains $a_2 \in C_{b_1, a_2}$ which is impossible by (iii). Hence $K_1 \cap Q_1 = \emptyset$. This implies $Q_1 \subset A \cup C_{b_1} \cup C_{b_2}$. But $C_{b_1, a_2} \cup C_{b_2} \subset Q_2$ and $(\{a_1\} \cup M) \cap Q_1 = \emptyset$, hence $Q_1 \subset C_{b_1} - (\{b_1\} \cup C_{b_1, a_2})$. Suppose K_2 is a component of $(C_{b_1} \cup \{a_1, a_2\}) - \{b_1\}$ other than C_{b_1, a_1} or C_{b_1, a_2} and α is a path from K_2 to $V \cap (C_{b_1, a_1} - \{a_1\}) (\subset Q_1)$ avoiding $\{a_1, b_1\} \cup M$. If α leaves C_{b_1} , then α contains a_2 which is impossible by (iii). Then α is a path in C_{b_1} passing through b_1 , which is also impossible. So, $Q_1 \subset C_{b_1, a_1} - \{a_1\} \subset \Gamma - (\{a_1, b_1\} \cup M)$, and (iv) is verified.

If $V \cap C_{b_i, a_j} \neq \{a_j\}$, define \hat{C}_{b_i, a_j} to be the component of $C_{b_i, a_j} - \{a_j\}$ (equivalently by (iv), the component of $\Gamma - (\{a_j, b_i\} \cup M)$) containing $V \cap (C_{b_i, a_j} - \{a_j\})$. As $\hat{C}_{b_i, a_j} \cap (A \cup B) = \emptyset$, \hat{C}_{b_i, a_j} is a component of $\Gamma - (A \cup B)$. If $V \cap C_{b_i, a_j} = \{a_j\}$, then define $\hat{C}_{b_i, a_j} \equiv \emptyset$.

As C_{b_i, a_j} is connected, a_j is connected by an edge to each component of $C_{b_i, a_j} - \{a_j\}$. Each component of $C_{b_i, a_j} - \{a_j\}$ is also a component of $C_{b_i} - \{b_i\}$ and so is connected by an edge to b_j . In particular, if $\hat{C}_{b_i, a_j} \neq \emptyset$ then it is connected to both a_j and b_i by edges.

We conclude that $V - (A \cup B)$ is contained in the following disjoint union of sets, each of which is either trivial, or has nontrivial intersection with V and is a component of $\Gamma - (A \cup B)$.

$$\hat{C}_{b_1, a_1} \cup \hat{C}_{b_1, a_2} \cup \hat{C}_{b_2, a_2} \cup \hat{C}_{b_2, a_1}$$

If $\hat{C}_{b_i, a_j} = \emptyset$, then there may be an edge connecting b_i and a_j , but if no such set is empty, then two members of the following collection are connected by an edge iff they are cyclically adjacent. I.e. they form a “loop” of sets in Γ :

$$\{a_1\}, \hat{C}_{b_1, a_1}, \{b_1\}, \hat{C}_{b_1, a_2}, \{a_2\}, \hat{C}_{b_2, a_2}, \{b_2\}, \hat{C}_{b_2, a_1}$$

By the symmetry of the above results, we could define C_{a_1, b_1} . If $V \cap C_{a_1, b_1} \neq \{b_1\}$ then $\hat{C}_{a_1, b_1} = \hat{C}_{b_1, a_1}$ and is the only component of $\Gamma - (A \cup B)$ containing points of V and connected to each of a_1 and b_1 by edges. Similarly for the other such sets. Remove $\hat{C}_{u, v}$ from the above loop if $\hat{C}_{u, v} = \emptyset$ and then sets are connected by an edge only if they are cyclically adjacent.

If $\{u, v\} \in \{\{a_1, b_1\}, \{b_1, a_2\}, \{a_2, b_2\}, \{b_2, a_1\}\}$ and $V \cap C_{u, v} \neq \{v\}$, then $\langle \{u, v\} \cup M \rangle$ crosses say $\langle \{d_1, d_2\} \cup M \rangle$ in $M(\Psi, \langle V \rangle)$. By (iii) and proposition 23, for some i , $d_i \in C_{u, v} - \{v\}$. By the above analysis, d_i separates u and v in $C_{u, v} \cup \{u\}$. Form the sets $C_{d_i, u}$ and $C_{d_i, v}$ as before. Note that d_i commutes with M . Continue until all remaining $C_{u, v}$ are such that $V \cap C_{u, v} = \{v\}$. The final collection of sets are all of the singletons of $V - M$ and each commutes with M . Then $T (\equiv V - M)$ and M satisfies the conclusion of our theorem.

If the presentation diagram for T is a loop of length ≥ 4 , then theorem 7.16.2 of Beardon’s book [1] guarantees the existence of an n -sided hyperbolic polygon whose vertex angles are $\frac{\pi}{m_i}$ (in cyclic order) for m_i the edge labels of the presentation diagram (in cyclic order). Theorem 7.1.4 of [14] concludes that the reflection group in this Coxeter polygon is a Coxeter group with cyclic presentation diagram and edge labels m_i (in cyclic order). Selberg’s lemma implies this Coxeter group has a torsion free subgroup of finite index and so $\langle T \rangle$ has a closed surface subgroup of finite index.

If T is not a loop, a visual decomposition of $\langle T \rangle$ produces the desired graph of groups decomposition of $\langle T \rangle$. \square

Note that for a Coxeter system (W, S) $A \subset S$ may separate $\Gamma(W, S)$ and generate a visual virtually abelian group, but A may not be a subset of a vertex of a JSJ-decomposition with virtually abelian edge groups for W .

Example 4. Let (W, S) be the Coxeter system with presentation diagram shown in figure 5 (with all edge labels equal to 2). Then W has virtually abelian JSJ-decomposition:

$$\langle a, b, c \rangle *_{\langle a, c \rangle} \langle a, c, d \rangle_{\langle a, c \rangle} \langle a, c, e \rangle_{\langle a, c \rangle} \langle a, c, f \rangle$$

The set $\{a, b, c, d\}$ separates Γ , and generates a *rank-2* virtually abelian subgroup of W . Each vertex group of the JSJ-decomposition is *rank-1* and so cannot contain a conjugate of $\langle a, b, c, d \rangle$.

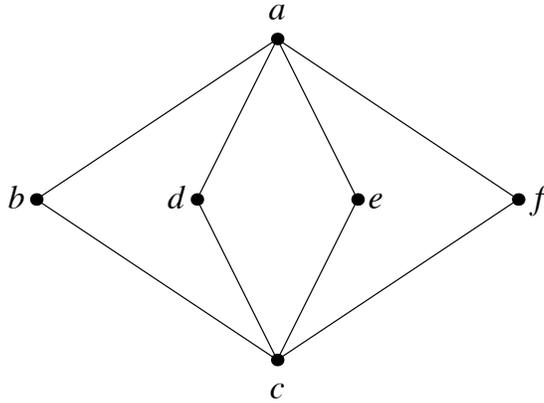


Figure 5: $\Gamma(W, S)$

The group of example 3 (figure 3) has orbifold vertex group $\langle x, y, u, v \rangle$, a virtually free group with presentation diagram two disjoint edges. This group is isomorphic to $\langle x, u \rangle * \langle y, v \rangle$, the free product of two (finite) dihedral groups.

Example 5. Let (W, S) be the Coxeter system with presentation diagram shown in figure 6 (with all edge labels connecting $\{7, 8\}$ to $\{1, 2, 3, 4\}$ equal to 2 and other labels arbitrary). Then W has virtually abelian JSJ-decomposition:

$$(\langle 7, 8 \rangle \times \langle 1, 2, 3, 4 \rangle) *_{\langle 7, 8 \rangle} \langle 5, 6, 7, 8 \rangle$$

In the orbifold vertex group $\langle 7, 8 \rangle \times \langle 1, 2, 3, 4 \rangle$ the virtually abelian splitting subgroups $\langle 1, 2 \rangle \times \langle 7, 8 \rangle$ and $\langle 3, 4 \rangle \times \langle 7, 8 \rangle$ are crossing. The loop determined by vertices in $\{1, 2, 3, 4\}$ generates a virtual closed surface group.

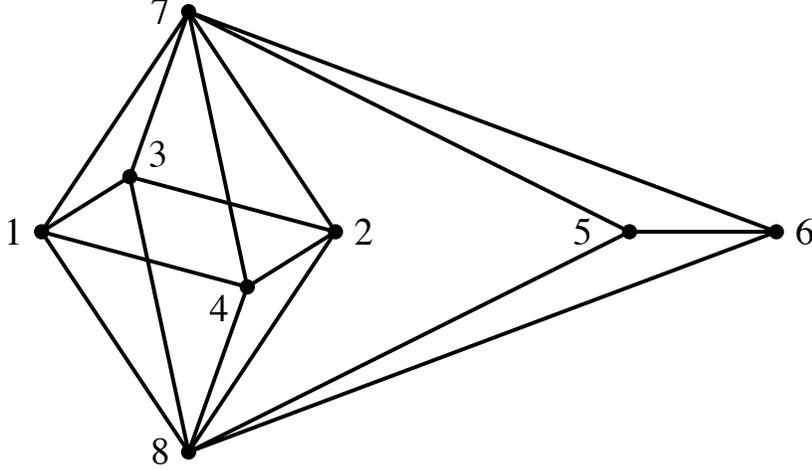


Figure 6: $(\langle 7, 8 \rangle \times \langle 1, 2, 3, 4 \rangle) *_{\langle 7, 8 \rangle} \langle 5, 6, 7, 8 \rangle$

If in the preceding example the loop determined by the vertices $\{1, 2, 3, 4\}$ is replaced by a loop of length ≥ 4 with arbitrary edge labels and one considers the direct product of such a Coxeter group with an arbitrary virtually abelian Coxeter group then the resulting Coxeter groups have JSJ decompositions with orbifold groups that (non-trivially) realizes all possible orbifold groups of the type as described in part 3) of theorem 1.

Example 6. Suppose (W', T) has presentation diagram that is a union of isolated vertices and simple paths. Our goal is to realize $\langle T \rangle$ as an orbifold group in the JSJ decomposition of a 1-ended Coxeter group. Decompose T as $\cup_{i=0}^n T_i$ where the T_i are disjoint and each T_i is the set of vertices of a component of the presentation diagram $\Gamma(W', T)$. If T_i is a singleton z_i , then let $a_i = b_i = z_i$. Otherwise, let a_i and b_i be the end points of the simple path of T_i . Let $S = T \cup (\cup_{i=0}^n C_i)$ where $(\text{mod } n+1)$ $C_i = \{b_i, a_{i+1}, x_i, y_i\}$ for $i \in \{0, \dots, n\}$ and (W_i, C_i) is the Coxeter system with group W_i isomorphic to $\mathbb{D}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (where $\{x_i\}$ generates one of the \mathbb{Z}_2 , $\{y_i\}$ generates the other, and $\{b_i, a_{i+1}\}$ generates the infinite dihedral group \mathbb{D}_2).

The desired Coxeter system (W, S) has the relations of (W', T) and (W_i, C_i) . The JSJ decomposition of (W, S) has irreducible vertex groups $\langle C_i \rangle$ and orbifold vertex group $\langle T \rangle$. The tree for this decomposition has a vertex V (with vertex group $W' \cong \langle T \rangle$), $n+1$ edges E_i , each incident to V (the edge group of E_i is the infinite dihedral group $\langle b_i, a_{i+1} \rangle$), and V_i , the vertex of E_i opposite

V , has edge group $W_i \equiv \langle C_i \rangle$. See figure 7.

The vertices of T can now be ordered as t_0, \dots, t_m respecting the ordering of the T_i and the internal ordering of the simple paths with a_i preceding b_i . If t_i and t_j are not adjacent (mod $m+1$) then $\{t_i, t_j\}$ separates T in $\Gamma(W, S)$ and defines a crossing splitter. The group W is 1-ended since no subset A of S both separates $\Gamma(W, S)$ and generates a finite subgroup of W .

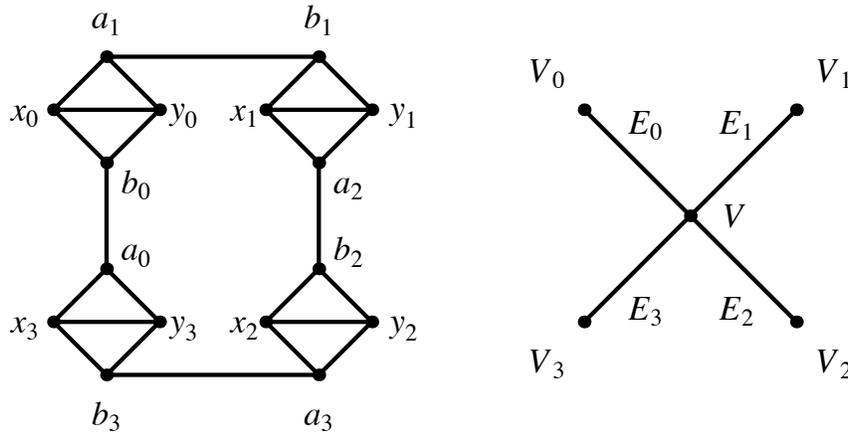


Figure 7: Virtually free orbifold group

References

- [1] A. F. Beardon, *The Geometry of Discrete Groups*, Graduate Texts in Math., vol. **91**, Springer Verlag, Berlin, Heidelberg, and New York, 1983.
- [2] N. Bourbaki, *Groupes et Algebres de Lie* Chapters IV-VI, Hermann, Paris, 1968.
- [3] H. M. S. Coxeter *Regular Polytopes*, Dover, New York, 1973.
- [4] V. V. Deodhar, *On the root system of a Coxeter group*, Commun. Algebra **10** (1982), 611-630.
- [5] W. Dicks and M. J. Dunwoody, *Groups Acting on Graphs*, Cambridge University Press, 1989.

- [6] M. J. Dunwoody and M. E. Sageev, *JSJ-Splittings for Finitely Presented Groups over Slender Groups*, Invent. Math. **135** (1999), 25-44.
- [7] M. Forester, *Deformation and rigidity of simplicial group actions on trees*, Geom. & Topol. **6** (2002), 219-267.
- [8] K. Fujiwara and P. Papasoglu, *JSJ-decompositions of finitely presented groups and complexes of groups*, Geom. Funct. Anal. **16** (2006) no.1, 70-125.
- [9] V. Guirardel and G. Levitt, *A general construction of JSJ decompositions*, Geometric Group Theory, Trends in Mathematics, 65-73, 2007 Birkhäuser Verlag, Basel, Switzerland.
- [10] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Memoirs of the AMS **220**, 1979.
- [11] K. Johannson, *Homotopy equivalences of 3-manifolds with boundary*, LNM Springer Verlag, 1978.
- [12] D. Krammer, *The conjugacy problem for Coxeter groups*, Thesis, Universiteit Utrecht, Netherlands, 1994.
- [13] M. L. Mihalik and S. Tschantz, *Visual Decompositions of Coxeter groups*, Groups Geom. Dyn. **3** (2009), 173-198. .
- [14] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Graduate Texts in Math., vol. **149**, Springer-Verlag, Berlin, Heidelberg, and New York, 1994.
- [15] E. Rips and Z. Sela, *Cyclic Splittings of Finitely Presented Groups and the Canonical JSJ-Decomposition*, Ann. of Math (2) **146** (1997) no. 1, 53-109.
- [16] P. Scott and G. A. Swarup, *Regular neighbourhoods and canonical decompositions for groups*, Astérisque **289** (2003).
- [17] Z. Sela, *Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups II*, Geom. Funct. Anal. **7**, (1997) no. 3, 561-594.
- [18] J. P. Serre, *Trees*, Springer-Verlag, New York, 1980.

- [19] F. Waldhausen, *On the determination of some bounded 3-manifolds by their fundamental groups alone*, Proc. Sympos. on Topology and its Applications, 331-332, Beograd.