Strong Accessibility of Coxeter Groups over Minimal Splittings

M. Mihalik and S. Tschantz

Revised: November 15, 2011

Abstract

Given a class of groups $\mathcal{C}$, a group G is strongly accessible over $\mathcal{C}$ if there is a bound on the number of terms in a sequence $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ of graph of groups decompositions of G with edge groups in $\mathcal{C}$ such that Λ_1 is the trivial decomposition (with 1-vertex) and for $i > 1$, Λ_i is obtained from Λ_{i-1} by non-trivially and compatibly splitting a vertex group of Λ_{i-1} over a group in $\mathcal{C}$, replacing this vertex group by the splitting and then reducing. If H and K are subgroups of a group G then H is *smaller* than K if $H \cap K$ has finite index in H and infinite index in K. The *minimal splitting subgroups of* G, are the subgroups H of G, such that G splits non-trivially (as an amalgamated product or HNN-extension) over H and for any other splitting subgroup K of W, K is not smaller than H. When G is a finitely generated Coxeter group, minimal splitting subgroups are always finitely generated. Minimal splittings are explicitly or implicitly important aspects of Dunwoody's work on accessibility and the JSJ results of Rips-Sela, Dunwoody-Sageev and Mihalik. Our main results are that Coxeter groups are strongly accessible over minimal splittings and if Λ is an irreducible graph of groups decomposition of a Coxeter group with minimal splitting edge groups, then the vertex and edge groups of Λ are Coxeter.

Subject Classifications: 20F65, 20F55, 20E08

Key Words: Coxeter group, accessibility, graph of groups

1 Introduction

In [23], J. Stallings proved that finitely generated groups with more than one end split non-trivially as an amalgamated product $A *_C B$ (where non-trivial means $A \neq C \neq B$) or an HNN-extension $A*_C$ with C a finite group. In about 1970, C. T. C. Wall raised questions about whether or not one could begin with a group A_0 and for $i > 0$, produce an infinite sequence of non-trivial splittings, $A_i *_{C_i} B_i$ or $A_i *_{C_i}$ of A_{i-1}, with C_i finite. When such a sequence could not exist, Wall called the group A_0, *accessible* over such splittings. In [11] M. Dunwoody proved that finitely presented groups are accessible with respect to splittings over finite groups. This implies that for a finitely presented group G there is no infinite sequence $\Lambda_1, \Lambda_2, \ldots$ of graph of groups decomposition of G such that Λ_1 is the trivial decomposition (with 1-vertex) and for $i > 1$, Λ_i is obtained from Λ_{i-1} by non-trivially splitting a vertex group over a finite group, replacing this vertex group by the splitting and then reducing. (For splittings over finite groups there is never a compatibility problem.) Instead, any such sequence of decompositions must terminate in one in which each vertex group is either 1-ended or finite and all edge groups are finite. In [12] Dunwoody gives examples of finitely generated groups that are not accessible over finite splittings. The class of *small* groups is defined in terms of actions on trees and is contained in the class of groups that contain no non-abelian free group as a subgroup. In [1], M. Bestvina and M. Feighn show that for a finitely presented group G there is a bound $N(G)$ on the number of edges in a reduced graph of groups decomposition of G, when edge groups are small. Limits of this sort are generally called "accessibility" results. If $\mathcal{C}$ is a class of groups then call a graph of groups decomposition of a group G with edge groups in $\mathcal{C}$ a $\mathcal{C}$-*decomposition* of G. A group G is called *strongly accessible* over $\mathcal{C}$ if there is a bound on the number of terms in a sequence $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ of $\mathcal{C}$-decompositions of G, such that Λ_1 is the trivial decomposition, and for $i > 1$, Λ_i is obtained from Λ_{i-1} by replacing a vertex group of Λ_{i-1} with a compatible splitting $A *_C B$ or $A*_C$ ($C \in \mathcal{C}$) and then reducing. We call a group G *accessible* over a class of groups $\mathcal{C}$ if there is a bound $N(G)$ on the number of edge groups in a reduced graph of groups decomposition of G with edge groups in $\mathcal{C}$. If $\mathcal{C}$ is a class of groups and there is a bound on the length of a sequence $G \equiv V_0, V_1, \ldots, V_n$ such that V_i is a vertex group of a non-trivial graph of groups decomposition of V_{i-1} then G is called *hierarchical accessible* over $\mathcal{C}$. Hierarchical accessibility is intended to be an algebraic counterpart to W. Haken's process of splitting compact

3-manifolds along incompressible surfaces. In [8], T. Delzant and L. Potyagailo show that finitely presented groups with no 2-torsion are hierarchical accessible over an "elementary" family of subgroups of G. (Here elementary is a technical term that in particular, requires all members of the family to be small.) Certainly hierarchical accessibility implies strong accessibility implies accessibility. Dunwoody's theorem is a hierarchical accessibility result for finitely presented groups over the class of finite groups. We know of no example where accessibility and strong accessibility are different.

Acylindrical accessibility was introduced by Sela in [21]. It plays an important role in his development of JSJ theory for word hyperbolic groups [22] and his solution of the isomorphism problem for torsion free hyperbolic groups [20]. Unlike, most other types of accessibility, Sela showed finitely generated (as opposed to finitely presented) groups are acylindrical accessible. An isometric action of a group G on a simplicial tree T is *D-acylindrical* (for $D \geq 0$) if for any non-trivial $g \in G$, the diameter of the fixed point set of g is $\leq D$. Sela bounds the combinatorics of the resulting graph of group decompositions of G. In [7], T. Delzant obtained a relative version of Sela's theorem for finitely presented groups and in [15] I. Kapovich and R. Weidmann generalize Sela's results to groups acting on $\mathbb{R}$-trees.

In this paper, we produce accessibility results for finitely generated Coxeter groups. In analogy with the 1-ended assumptions of Rips-Sela [19], and the minimality assumptions of Dunwoody-Sageev [13], we consider the class $M(W)$ of minimal splitting subgroups of W. If H and K are subgroups of a group W then H is *smaller* than K if $H \cap K$ has finite index in H and infinite index in K. If W is a group, then define $M(W)$, the set of *minimal splitting subgroups of* W, to be the set of all subgroups H of W, such that W splits non-trivially (as an amalgamated product or HNN-extension) over H and for any other splitting subgroup K of W, K is not smaller than H.

Remark 1. A minimal splitting subgroup of a finitely generated Coxeter group W is finitely generated. This follows from remark 1 of [17]. Suppose $A *_C B$ is a non-trivial splitting of W and C is not finitely generated. There is a reduced visual decomposition of W with (visual and hence finitely generated) edge group E such that a conjugate of E is a subgroup of C. Hence some conjugate of E is smaller than C.

Finite splitting subgroups are always minimal and if a group is 1-ended, then any 2-ended splitting subgroup is minimal. Our main theorem is:

Theorem 1.1 *Finitely generated Coxeter groups are strongly accessible over minimal splittings.*

It should be noted that our accessibility result is different from other accessibility results in the sense that typical accessibility results include algebraic restrictions on the splitting subgroups (e.g. finite, small, ...). Any finitely generated Coxeter group is a minimal splitting subgroup in some other finitely generated Coxeter group. (If W_1, W_2 and W_3 are non-trivial Coxeter groups, then W_3 is a minimal splitting subgroup of $(W_1 * W_2) \times W_3 \equiv (W_1 \times W_3) *_{W_3} (W_2 \times W_3)$.)

Our basic reference for Coxeter groups is Bourbaki [3]. A *Coxeter presentation* is given by

$$\langle S : m(s,t)\ (s,t \in S,\ m(s,t) < \infty)\rangle$$

where $m : S^2 \to \{1, 2, \ldots, \infty\}$ is such that $m(s,t) = 1$ iff $s = t$ and $m(s,t) = m(t,s)$. The pair (W, S) is called a *Coxeter system*. In the group with this presentation, the elements of S are distinct elements of order 2 and a product st of generators has order $m(s,t)$. Distinct generators commute if and only if $m(s,t) = 2$. A subgroup of W generated by a subset S' of S is called *special* or *visual*, and the pair $(\langle S'\rangle, S')$ is a Coxeter system with $m' : (S')^2 \to \{1, 2, \ldots, \infty\}$ the restriction of m. A simple analysis of a Coxeter presentation allows one to construct all decompositions of W with only visual vertex and edge groups from that Coxeter presentation. In [17], the authors show that for any finitely generated Coxeter system (W, S) and any graph of groups decomposition Λ of W, there is an associated "visual" graph of groups decomposition Ψ of W with edge and vertex groups visual, and such that each vertex (respectively edge) group of Ψ is contained in a conjugate of a vertex (respectively edge) group of Λ. This result is called "the visual decomposition theorem for finitely generated Coxeter groups", and we say Ψ is *a visual decomposition for* Λ. Clearly accessibility of finitely generated Coxeter groups is not violated by only visual decompositions. But, we give an example in [17], of a finitely generated Coxeter system (W, S) and a sequence Λ_i $(i \geq 1)$ of (non-visual) reduced graph of groups decompositions of W, such that Λ_i has i-edge groups and, for $i > 1$, Λ_i is obtained by compatibly splitting a vertex group of Λ_{i-1}. Hence, even in the light of the visual decomposition theorem and our accessibility results here, there is no accessibility for Coxeter groups over arbitrary splittings.

Theorem 1.1 implies there are irreducible decompositions of finitely generated Coxeter groups, with minimal splitting edge groups. Our next result implies that any such irreducible decomposition has an "equivalent" visual counterpart. (Recall $M(W)$ is the collection of minimal splitting subgroups of W.)

Theorem 1.2 *Suppose (W, S) is a Coxeter system and Λ is a reduced graph of groups decomposition of W with $M(W)$ edge groups. If Λ is irreducible with respect to $M(W)$ splittings, then each vertex and edge group of Λ is conjugate to a visual subgroup for (W, S).*

Furthermore, if Ψ is a reduced graph of groups decomposition such that each edge group of Ψ is in $M(W)$, each vertex group of Ψ is a subgroup of a conjugate of a vertex group of Λ, and each edge group of Λ contains a conjugate of an edge group of Ψ (in particular if Ψ is a reduced visual graph of groups decomposition for (W, S) derived from Λ as in the main theorem of [17]), then

1. *Ψ is irreducible with respect to $M(W)$ splittings*
2. *There is a (unique) bijection α of the vertices of Λ to the vertices of Ψ such that for each vertex V of Λ, $\Lambda(V)$ is conjugate to $\Psi(\alpha(V))$*

The vertex groups of Λ in theorem 1.2 are Coxeter, and when W is not indecomposable, they have fewer generators than there are in S. Hence they have irreducible decompositions of the same type. As the number of Coxeter generators decreases each time we pass from a non-indecomposable vertex group to a vertex group of an irreducible decomposition with minimal splitting edge groups for that vertex group, eventually this process must terminate with (up to conjugation) irreducible visual subgroups of (W, S). These terminal groups are maximal FA subgroups of W and must be conjugate to the visual subgroups of W determined by maximal complete subsets of the presentation diagram $\Gamma(W, S)$ (see [17]).

The paper is laid out as follows: in §2 we state the visual decomposition theorem and review the basics of graphs of groups decompositions.

In §3, we list several well-known technical facts about Coxeter groups. §3 concludes with an argument that shows an infinite subgroup of a finitely generated Coxeter group W (with Coxeter system (W, S)), containing a visual finite index subgroup $\langle A\rangle$ $(A \subset S)$ decomposes as $\langle A_0\rangle \times F$ where $A_0 \subset A$ and F is a finite subgroup of a finite group $\langle D\rangle$ where $D \subset S$ and D commutes with A_0. This result makes it possible for us to understand arbitrary minimal splitting subgroups of W in our analysis of strong accessibility.

In §4, we begin our analysis of $M(W)$ by classify the visual members of $M(W)$ for any Coxeter system (W, S). Proposition 4.3 shows that for a non-trivial splitting $A *_C B$ of a finitely generated Coxeter group W over a non-minimal group C, there is a splitting of W over a minimal splitting subgroup M, such that M is smaller than C. I.e. all non-trivial splittings of a finitely generated Coxeter group are "refined" by minimal splittings. Theorem 4.5 is the analogue of theorem 2.4 (from [17]), when edge groups of a graph of groups decomposition of a finitely generated Coxeter group are minimal splitting subgroups. The implications with this additional "minimal splitting" hypothesis far exceed the conclusions of theorem 2.4 and supply one of the more important technical results of paper. Roughly speaking, proposition 4.7 says that any graph of groups decomposition of a finitely generated Coxeter group with edge groups equal to minimal splitting subgroups of the Coxeter group is, up to "artificial considerations", visual. Proposition 4.7 gives another key idea toward the proof of the main theorem. It allows us to define a descending sequence of positive integers corresponding to a given sequence of graphs of groups as in the main theorem. Finally, theorem 4.14 is a minimal splitting version of the visual decomposition theorem of [17].

In §5, we define what it means for a visual decomposition of a Coxeter group W, with $M(W)$ edge groups, to look irreducible with respect to $M(W)$ subgroups. We show that a visual decomposition looks irreducible if and only if it is irreducible. This implies that all irreducible visual decompositions of a Coxeter group can be constructed by an elementary algorithm. Our main results, theorems 1.1 and 1.2 are proved in §5.

In the final section, §6, we begin with a list of generalizations of our results that follow from the techniques of the paper. Then, we give an analysis of minimal splitting subgroups of ascending HNN extensions, followed by a complete analysis of minimal splittings of general finitely generated groups that contain no non-abelian free group. This includes an analysis of Thompson's group F. We conclude with a list of questions.

2 Graph of Groups and Visual Decompositions

Section 2 of [17] is an introduction to graphs of groups that is completely sufficient for our needs in this paper. We include the necessary terminology here. A graph of groups Λ consists of a set $V(\Lambda)$ of vertices, a set $E(\Lambda)$ of edges, and maps $\iota, \tau : E(\Lambda) \to V(\Lambda)$ giving the initial and terminal vertices of each edge in a connected graph, together with vertex groups $\Lambda(V)$ for $V \in V(\Lambda)$, edge groups $\Lambda(E)$ for $E \in E(\Lambda)$, with $\Lambda(E) \subset \Lambda(\iota(E))$ and an injective group homomorphism $t_E : \Lambda(E) \to \Lambda(\tau(E))$, called the edge map of E and denoted by $t_E : g \mapsto g^{t_E}$. The fundamental group $\pi(\Lambda)$ of a graph of groups Λ is the group with presentation having generators the disjoint union of $\Lambda(V)$ for $V \in V(\Lambda)$, together with a symbol t_E for each edge $E \in E(\Lambda)$, and having as defining relations the relations for each $\Lambda(V)$, the relations $gt_E = t_E g^{t_E}$ for $E \in E(\Lambda)$ and $g \in \Lambda(\iota(E))$, and relations $t_E = 1$ for E in a given spanning tree of Λ (the result, up to isomorphism, is independent of the spanning tree taken).

If V is a vertex of a graph of groups decomposition Λ of a group G and Φ is a decomposition of $\Lambda(V)$ so that for each edge E of Λ adjacent to V, $\Lambda(E)$ is $\Lambda(V)$-conjugate to a subgroup of a vertex group of Φ, then Φ is *compatible* with Λ. Then V can be replaced by Φ to form a finer graph of groups decomposition of G.

A graph of groups is *reduced* if no edge between distinct vertices has edge group the same as an endpoint vertex group. If a graph of groups is not reduced, then we may collapse a vertex across an edge, giving a smaller graph of groups decomposition of the group.

If there is no non-trivial homomorphism of a group to the infinite cyclic group $\mathbb{Z}$, then a graph of groups decomposition of the group cannot contain a loop. In this case, the graph is a tree. In particular, any graph of groups decomposition of a Coxeter group has underlying graph a tree.

Suppose $\langle S : m(s,t)\ (s,t \in S,\ m(s,t) < \infty)\rangle$ is a Coxeter presentation for the Coxeter group W. The *presentation diagram* $\Gamma(W,S)$ of W with respect to S has vertex set S and an undirected edge labeled $m(s,t)$ connecting vertices s and t if $m(s,t) < \infty$. It is evident from the above presentation that if a subset C of S separates $\Gamma(W,S)$, A is C union some of the components of $\Gamma - C$ and B is C union the rest of the components, then W decomposes as $\langle A\rangle *_{\langle C\rangle} \langle B\rangle$. This generalizes to graphs of groups decompositions of Coxeter groups where each vertex and edge group is generated by a subset of S. We say that Ψ is a *visual graph of groups decomposition of* W (for a given S), if each vertex and edge group of Ψ is a special subgroup of W, the injections of each edge group into its endpoint vertex groups are given simply by inclusion, and the fundamental group of Ψ is isomorphic to W by the homomorphism induced by the inclusion map of vertex groups into W. If C and D are subsets of S, then we say C *separates* D *in* Γ if there are points d_1 and d_1 of $D - C$, such that any path in Γ connecting d_1 and d_2 contains a point of C.

The following lemma of [17] makes it possible to understand when a graph of groups with special subgroups has fundamental group W.

Lemma 2.1 *Suppose (W, S) is a Coxeter system. A graph of groups Ψ with graph a tree, where each vertex group and edge group is a special subgroup and each edge map is given by inclusion, is a visual graph of groups decomposition of W iff each edge in the presentation diagram of W is an edge in the presentation diagram of a vertex group and, for each generator $s \in S$, the set of vertices and edges with groups containing s is a nonempty subtree in Ψ.*

In section 4 we describe when visual graph of groups decompositions with minimal splitting edge groups are irreducible with respect to splittings over minimal splitting subgroups. The next lemma follows easily from lemma 2.1 and helps make that description possible.

Lemma 2.2 *Suppose Ψ is a visual graph of groups decomposition for the finitely generated Coxeter system (W, S), $V \subset S$ is such that $\langle V \rangle$ is a vertex group of Ψ and $E \subset V$ separates V in $\Gamma(W, S)$. Then $\langle V \rangle$ splits over $\langle E \rangle$, non-trivially and compatibly with Ψ to give a finer visual decomposition for (W, S) if and only if there are subsets A and B of S such that A is equal to E union (the vertices of) some of the components of $\Gamma - E$, B is E union the rest of the components of $\Gamma - E$, $A \cap V \neq E \neq B \cap V$, and for each edge D of Ψ which is adjacent to V, and $D_S \subset S$ such that $\langle D_S \rangle = \Psi(D)$, we have $D_S \subset A$ or $D_S \subset B$. The Ψ-compatible splitting of $\langle V \rangle$ is $\langle A \cap V \rangle *_{\langle E \rangle} \langle B \cap V \rangle$.*

The main theorem of [17] is "the visual decomposition theorem for finitely generated Coxeter groups":

Theorem 2.3 *Suppose (W, S) is a Coxeter system and Λ is a graph of groups decomposition of W. Then W has a visual graph of groups decomposition Ψ,* *where each vertex (edge) group of Ψ is a subgroup of a conjugate of a vertex (respectively edge) group of Λ. Moreover, Ψ can be taken so that each special subgroup of W that is a subgroup of a conjugate of a vertex group of Λ is a subgroup of a vertex group of Ψ.*

If (W, S) is a finitely generated Coxeter system, Λ is a graph of groups decomposition of W and Ψ satisfies the conclusion of theorem 2.3 (including the moreover clause) and then Ψ is called *a visual decomposition from* Λ (see [17]). In remark 1 of [17], it is shown that if Λ is reduced and Ψ is a visual decomposition from Λ then for any edge E of Λ there is an edge D of Ψ such that $\Psi(D)$ is conjugate to a subgroup of $\Lambda(E)$.

If a group G decomposes as $A *_C B$ and H is a subgroup of B, then the group $\langle A \cup H \rangle$ decomposes as $A *_C \langle C \cup H \rangle$. Furthermore, G decomposes as $\langle A \cup H \rangle_{\langle C \cup H \rangle} B$, giving a somewhat "artificial" decomposition of G. In [17], this idea is used on a certain Coxeter system (W, S) to produce reduced graph of groups decompositions of W with arbitrarily large numbers of edges.

The following theorem of [17] establishes limits on how far an arbitrary graph of groups decomposition for a finitely generated Coxeter system can stray from a visual decomposition for that system.

Theorem 2.4 *Suppose (W, S) is a finitely generated Coxeter system, Λ is a graph of groups decomposition of W and Ψ is a reduced graph of groups decomposition of W such that each vertex group of Ψ is a subgroup of a conjugate of a vertex group of Λ. Then for each vertex V of Λ, the vertex group $\Lambda(V)$, has a graph of groups decomposition Φ_V such that each vertex group of Φ_V is either*

(1) conjugate to a vertex group of Ψ or

(2) a subgroup of $v\Lambda(E)v^{-1}$ for some $v \in \Lambda(V)$ and E some edge of Λ adjacent to V.

When Ψ is visual, vertex groups of the first type in theorem 2.4 are visual and those of the second type seem somewhat artificial. In section 4 we prove theorem 4.5 which shows that if the edge groups of the decomposition Λ in theorem 2.4 are minimal splitting subgroups of W, then the decompositions Φ_V are compatible with Λ and part (2) of the conclusion can be significantly enhanced.

Lemma 2.5 *If Λ is a reduced graph of groups decomposition of a group G, V and U are vertices of Λ and $g\Lambda(V)g^{-1} \subset \Lambda(U)$ for some $g \in G$, then $V = U$. If additionally Λ is a tree, then $g \in \Lambda(V)$.* $\square$

If W is a finitely generated Coxeter group then since W has a set of order 2 generators, there is no non-trivial homomorphism from W to $\mathbb{Z}$. Hence any graph of groups decomposition of W is a tree. If $C \in M(W)$ and W is finitely generated, then theorem 2.3 implies that C contains a subgroup of finite index which is isomorphic to a Coxeter group and so there is no non-trivial homomorphism of C to $\mathbb{Z}$.

The following is an easy exercise in the theory of graph of groups or more practically it is a direct consequence of the exactness of the Mayer-Vietoris sequence for a pair of groups.

Lemma 2.6 *Suppose the group W decomposes as $A *_C B$ and there is no non-trivial homomorphism of W or C to $\mathbb{Z}$. Then there is no non-trivial homomorphism of A or B to $\mathbb{Z}$.* $\square$

Corollary 2.7 *Suppose W is a finitely generated Coxeter group and Λ is a graph of groups decomposition of W with each edge group in $M(W)$, then any graph of groups decomposition of a vertex group of Λ is a tree.* $\square$

3 Preliminary results

We list some results used in this paper. Most can be found in [3].

Lemma 3.1 *Suppose (W, S) is a Coxeter system and $P = \langle S : (st)^{m(s,t)}$ for $m(s,t) < \infty\rangle$ (where $m : S^2 \to \{1, 2, \ldots, \infty\}$) is a Coxeter presentation for W. If A is a subset of S, then $(\langle A\rangle, A)$ is a Coxeter system with Coxeter presentation $\langle A : (st)^{m'(s,t)}$ for $m'(s,t) < \infty\rangle$ (where $m' = m|_{A^2}$). In particular, if $\{s,t\} \subset S$, then the order of (st) is $m(s,t)$.* $\square$

The following result is due to Tits:

Lemma 3.2 *Suppose (W, S) is a Coxeter system and F is a finite subgroup of W then there is $A \subset S$ such that $\langle A\rangle$ is finite and some conjugate of F is a subgroup of $\langle A\rangle$.* $\square$

If A is a set of generators for a group G, the *Cayley graph* $\mathcal{K}(G, A)$ of G with respect to A has G as vertex set and a directed edge labeled a from $g \in G$ to ga for each $a \in A$. The group G acts on the left of $\mathcal{K}$. Given a vertex g in $\mathcal{K}$, the edge paths in $\mathcal{K}$ at g are in 1-1 correspondence with the words in the letters $A^{\pm 1}$ where the letter a^{-1} is used if an edge labeled a is traversed opposite its orientation. Note that for a Coxeter system (W, S), and $s \in S$, $s = s^{-1}$. It is standard to identify the edges labeled s at x and s at xs in $\mathcal{K}(W, S)$ for each vertex x, of $\mathcal{K}$ and each $s \in S$ and to ignore the orientation on the edges. Given a group G with generators A, an A-*geodesic* for $g \in G$ is a shortest word in the letters $A^{\pm 1}$ whose product is g. A geodesic for G defines a geodesic in $\mathcal{K}$ for each vertex $g \in G$. Cayley graphs provide and excellent geometric setting for many of the results in this section.

The next result is called the deletion condition for Coxeter groups. An elementary proof of this fact, based on Dehn diagrams, can be found in [17].

Lemma 3.3 The Deletion Condition *Suppose (W, S) is a Coxeter system and $a_1 \cdots a_n$ is a word in S which is not geodesic. Then for some $i < j$, $a_i \cdots a_j = a_{i+1} \cdots a_{j-1}$. I.e. the letters a_i and a_j can be deleted.* $\square$

The next collection of lemmas can be derived from the deletion condition.

Lemma 3.4 *Suppose (W, S) is a Coxeter system and A and B are subsets of S. Then for any $w \in W$ there is a unique shortest element, d, of the double coset $\langle A\rangle w\langle B\rangle$. If δ is a geodesic for d, α is an A-geodesic, and β is a B-geodesic, then (α, δ) and (δ, β) are geodesic.* $\square$

Lemma 3.5 *Suppose (W, S) is a Coxeter system, $w \in W$, I and $J \subset S$, and d is the minimal length double coset representative in $\langle I\rangle w\langle J\rangle$. Then $\langle I\rangle \cap d\langle J\rangle d^{-1} = \langle K\rangle$ for $K = I \cap (dJd^{-1})$ and, $d^{-1}\langle K\rangle d = \langle J\rangle \cap (d^{-1}\langle I\rangle d) = \langle K'\rangle$ for $K' = J \cap d^{-1}Id = d^{-1}Kd$. In particular, if $w = idj$ for $i \in \langle I\rangle$ and $j \in \langle J\rangle$ then $\langle I\rangle \cap w\langle J\rangle w^{-1} = i\langle K\rangle i^{-1}$ and $\langle J\rangle \cap w^{-1}\langle I\rangle w = j^{-1}\langle K'\rangle j$.* $\square$

Lemma 3.6 *Suppose (W, S) is a Coxeter system, A is a subset of S and α is an S-geodesic. If for each letter $a \in A$, the word (α, a) is not geodesic, then the group $\langle A\rangle$ is finite.* $\square$

Lemma 3.7 *Suppose (W, S) is a Coxeter system and $x \in S$. If α is a geodesic in $S - \{x\}$, then the word (α, x) is geodesic.* $\square$

If (W, S) is a Coxeter system and $w \in W$ then the deletion condition implies that the letters of S used to compose an S-geodesic for w is independent of which geodesic one composes for w. We define $lett(w)_S$ to be the subset of S used to composes a geodesic for w, or when the system is evident we simply write $lett(w)$.

Lemma 3.8 *Suppose* (W, S) *is a Coxeter system,* $w \in W$*,* $b \in S - lett(w)$*, and* $bwb \in \langle lett(w)\rangle$ *then* b *commutes with* $lett(w)$.□

The next lemma is technical but critical to the main results of the section.

Lemma 3.9 *Suppose* (W, S) *is a finitely generated Coxeter system and* $A \subset S$ *such that* $\langle A\rangle$ *is infinite and there is no non-trivial* $F \subset A$ *such that* $\langle F\rangle$ *is finite and* $A - F$ *commutes with* F*. Then there is an infinite* A*-geodesic* α*, such that each letter of* A *appears infinitely many times in* α*.*

Proof: The case when $\langle A\rangle$ does not (visually) decompose as $\langle A - U\rangle \times \langle U\rangle$ for any non-trivial $U \subset A$, follows from lemma 1.15 of [16]. The general case follows since once the irreducible case is established, one can interleave geodesics from each (infinite) factor of a maximal visual direct product decomposition of $\langle A\rangle$. I.e. if $\langle A\rangle = \langle A - U\rangle \times \langle U\rangle$, $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ are U and $A - U$-geodesics respectively, then the deletion condition implies $(x_1, y_1, x_2, y_2, \ldots)$ is an A-geodesic. □

Remark 2. Observe that if (W, S) is a Coxeter system, and $W = \langle F\rangle \times \langle G\rangle = \langle H\rangle \times \langle I\rangle$ for $F \cup G = S = H \cup I$. Then $W = \langle F \cup H\rangle \times \langle G \cap I\rangle$ and $\langle F \cup H\rangle = \langle F\rangle \times \langle H - F\rangle$. In particular, for $A \subset S$, there is a unique largest subset $C \subset A$ such that $\langle A\rangle = \langle A - C\rangle \times \langle C\rangle$ and $\langle C\rangle$ is finite. Define $T_{(W,S)}(A) \equiv C$ and $E_{(W,S)}(A) \equiv A - C$. When the system is evident we simply write $T_W(A)$ and $E_W(A)$.

For a Coxeter system (W, S) and $A \subset S$, let $lk_2(A, (W, S))$ (*the 2-link of* A in the system (W, S)) be the set of all $s \in S - A$ that commute with A. For consistency we define $lk_2(\emptyset, (W, S)) = S$. When the system is evident we simply write $lk_2(A)$. In the presentation diagram $\Gamma(W, S)$, $lk_2(A)$ is the set of all vertices $s \in S$ such that s is connected to each element of A by an edge labeled 2.

If G is a group with generating set S and u is an S-word, denote by $\bar{u}$ the element of G represented by u.

Lemma 3.10 *Suppose* (W, S) *is a Coxeter system,* $A \subset S$*, and* r *is an* A*-geodesic such that each letter of* A *appears infinitely often in* r*. If* r *can be partitioned as* $(r_1, r_2, \ldots)$ *and* $w \in W$ *is such that* $w\bar{r}_i w^{-1} = s_i$*,* $|s_i| = |\bar{r}_i|$*, and* $(\beta, r_i, r_{i+1}, \ldots)$ *and* $(r_1, \ldots, r_i, \beta^{-1})$ *are geodesic for all* i *where* β *is a geodesic for* w*, then* $w \in \langle A \cup lk_2(A)\rangle$*.*

Proof: If w is a minimum length counter-example, then by lemma 3.8, $|w| > 1$. Say $(w_1, \ldots, w_n)$ is a geodesic for w. For all m, $(w_1, \ldots, w_n, r_1, \ldots, r_m, w_n)$ is not geodesic and the last w_n deletes with one of the initial w_i. For some $i \in \{1, \ldots, n\}$, there are infinitely many m such that the last w_n deletes with w_i. Say this set of such m is $\{m_1, m_2, \ldots\}$ (in ascending order). Then w_n commutes with $\bar{r}_{m_j+1}\bar{r}_{m_j+2}\cdots\bar{r}_{m_{j+1}}$ for all j. By lemma 3.8, $w_n \in A_0 \cup lk_2(A_0)$. Then $w' = w_1 \cdots w_{n-1}$ is shorter than w and satisfies the hypothesis of the lemma with r replaced by $r' = (r'_1, r'_2, \ldots)$ where $r'_i = (r_{m_i+1}, r_{m_i+2}, \ldots, r_{m_{i+1}})$. By the minimality of w , $w' \in \langle A_0 \cup lk_2(A_0)\rangle$, and so $w \in \langle A \cup lk_2(A)\rangle$. $\square$

The next result is analogous to classical results (see V. Deodhar [9]).

Lemma 3.11 *Suppose (W, S) is a finitely generated Coxeter system, A and B are subsets of S, u is a shortest element of the double coset $\langle B\rangle g\langle A\rangle$, and $g\langle A\rangle g^{-1} \subset \langle B\rangle$. Then $uAu^{-1} \subset B$ and $lett(u) \subset lk_2(E_W(A))$. In particular, $uxu^{-1} = x$ for all $x \in E_W(A)$ and $E_W(A) \subset E_W(B)$. If additionally, $g\langle A\rangle g^{-1} = \langle B\rangle$, then $uAu^{-1} = B$ and $E_W(A) = E_W(B)$.*

Proof: Note that $g\langle A\rangle g^{-1} = bua\langle A\rangle a^{-1}u^{-1}b^{-1} \subset \langle B\rangle$ for some $a \in \langle A\rangle$ and $b \in \langle B\rangle$. Then $u\langle A\rangle u^{-1} \subset \langle B\rangle$. By lemma 3.5, $u\langle A\rangle u^{-1} = u\langle A\rangle u^{-1} \cap \langle B\rangle = \langle (uAu^{-1}) \cap B\rangle$ and so $\langle A\rangle = \langle A \cap u^{-1}Bu\rangle$ and $A \subset u^{-1}Bu$ so that $uAu^{-1} \subset B$.

If $E(A) = \emptyset$ there is nothing more to prove. Otherwise, lemma 3.8 implies there is a geodesic α in the letters of $E_W(A)$, such that each letter of $E_W(A)$ appears infinitely often in α. By lemma 3.10 (with partitioning r_i of length 1), $lett(u) \subset E_W(A) \cup lk_2(E_W(A))$. By the definition of u, no geodesic for u can end in a letter of A and so $lett(u) \subset lk_2(E_W(A))$. Then $E_W(A) \subset B$ so $E_W(A) \subset E_W(B)$.

Now assume $g\langle A\rangle g^{-1} = \langle B\rangle$. Then as u^{-1} is the shortest element of the double coset $\langle A\rangle g^{-1}\langle B\rangle$, we have $u^{-1}Bu \subset A$ so $uAu^{-1} = B$, and we have $E_W(B) \subset E_W(A)$ so $E_W(A) = E_W(B)$. $\square$

Proposition 3.12 *Suppose (W, S) is a Coxeter system, B is an infinite subgroup of W and $A \subset S$ such that $\langle A\rangle$ has finite index in B. Then $B = \langle A_0\rangle \times C$ for $A_0 \subset A$ and C a finite subgroup of $\langle lk_2(A_0)\rangle$. (By lemma 3.2, C is a subgroup of a finite group $\langle D\rangle$ such that $D \subset S - A_0$ and D commutes with A_0.)*

Proof: Let $A_0 \equiv E_W(A)$. By lemma 3.9 there is an infinite-length A_0-geodesic r, such that each letter in A_0 appears infinitely often in r. The group $\langle A_0\rangle$ contains a subgroup A' which is a normal finite-index subgroup of B. Let α_i be the initial segment of r of length i, and C_i the B/A' coset containing $\bar\alpha_i$, the element of W represented by α_i. Let i be the first integer such that $C_i = C_j$ for infinitely many j. Replace r by the terminal segment of r that follows α_i. Then r can be partitioned into geodesics $(r_1, r_2, \ldots)$ such that $\bar r_i \in A'$. Hence for any i and any $b \in B$, $b\bar r_i b^{-1} \in A' \subset \langle A_0\rangle$.

It suffices to show that $B \subset \langle A_0\rangle \times \langle lk_2(A_0)\rangle$, since then each $b \in B$ is such that $b = xy$ with $x \in \langle A_0\rangle$ and $y \in \langle lk_2(A_0)\rangle$. As $A_0 \subset B$, $y \in B$ and so $B = \langle A_0\rangle \times (B \cap \langle lk_2(A_0)\rangle)$. (Recall $\langle A_0\rangle$ has finite index in B.)

Suppose b is a shortest element of B such that $b \notin \langle A_0\rangle \times \langle lk_2(A_0)\rangle$. Let β be a geodesic for b.

Claim The path $(\beta, r_1, r_2, \ldots)$ is geodesic.

Proof: Otherwise let i be the first integer such that (β, α_i) (recall α_i is the initial segment of r of length i) is not geodesic. Then $\bar\beta\bar\alpha_i = \bar\gamma\bar\alpha_{i-1}$ where γ is obtained from β by deleting some letter and (γ, α_{i-1}) is geodesic. We have $\bar\gamma\bar\alpha_{i-1} = b\bar\alpha_i$, and $\{b, \bar\alpha_{i-1}, \bar\alpha_i\} \subset B$, so $\bar\gamma \in B$.

We conclude the proof of this claim by showing: If b is a shortest element of B such that $b \notin \langle A_0 \cup lk_2(A_0)\rangle$ and β is a geodesic for b, then a letter cannot be deleted from β to give a geodesic for an element of B.

Otherwise, suppose $\beta = (b_1, \ldots, b_m)$, $\gamma = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_m)$ is geodesic, and $\bar\gamma \in B$. By the minimality hypothesis, $\{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots b_m\} \subset A_0 \cup lk_2(A_0)$. "Sliding" $lk_2(A_0)$-letters of β before b_i "back" and those after b_i "forward", gives a geodesic $(\beta_1, \beta_2, b_i, \beta_3, \beta_4)$ for b, with $lett(\beta_1) \cup lett(\beta_4) \subset lk_2(A_0)$ and $lett(\beta_2) \cup lett(\beta_3) \subset A_0$. Now, $\bar\beta_1\bar\beta_2 b_i\bar\beta_3\bar\beta_4\bar r_1 \cdots \bar r_j\bar\beta_4^{-1}\bar\beta_3^{-1}b_i\bar\beta_2^{-1}\bar\beta_1^{-1} \in A' \subset \langle A_0\rangle$, for each j. This implies $b_i\bar\beta_3\bar r_1 \cdots \bar r_j\bar\beta_3^{-1}b_i \in \langle A_0\rangle$. For large j, $lett(\bar\beta_3\bar r_1 \cdots \bar r_j\bar\beta_3^{-1}) = A_0$. By lemma 3.8, $b_i \in A_0 \cup lk_2(A_0)$, and so $b \in \langle A \cup lk_2(A_0)\rangle$. This is contrary to our assumption and the claim is proved. □

The same proof shows $(\beta, r_k, r_{k+1}, \ldots)$ is geodesic for all k.

Let δ_i be a geodesic for $b\bar r_i b^{-1} \in \langle A_0\rangle$. Next we show $|\delta_i| = |r_i|$. As (β, r_i) is geodesic and $b\bar r_i = \bar\delta_i b$, $|\delta_i| \geq |r_i|$. If $|\delta_i| > |r_i|$ then (δ_i, β) is not geodesic. Say $\delta_i = (x_1, \ldots, x_k)$ for $x_i \in A_0$. Let j be the largest integer such that $(x_j, \ldots, x_k, b_1, \ldots, b_m)$ is not geodesic. Then x_j deletes with say b_i and $(x_{j+1}, \ldots, x_k, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_m)$ is geodesic. As

$$x_{j+1} \ldots x_k b_1 \ldots b_{i-1} b_{i+1} \ldots b_m = x_j \ldots x_k b \in B$$

the word $(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_m)$ is a geodesic for an element of B. This is impossible by the closing argument of our claim.

Since $(\beta, r_1, \ldots, r_i)$ is geodesic for all i, so is $(\delta_1, \ldots, \delta_i, \beta)$. Since

$$(r_1, \ldots, r_i, \beta^{-1})^{-1} = (\beta, r_i^{-1}, \ldots, r_1^{-1})$$

the claim shows $(r_1, \ldots, r_i, \beta^{-1})$ is geodesic for all i. The proposition now follows directly from lemma 3.10. $\square$

4 Minimal Splittings

Recall that a subgroup A of W is a *minimal splitting subgroup* of W if W splits non-trivially over A, and there is no subgroup B of W such that W splits non-trivially over B, and $B \cap A$ has infinite index in A and finite index in B.

For a Coxeter system (W, S) we defined $M(W)$ to be the collection of minimal splitting subgroups groups of W. Observe that if W has more than 1-end, then each member of $M(W)$ is a finite group. Define $K(W, S)$ to be the set of all subgroups of W of the form $\langle A \rangle \times M$ for $A \subset S$, and M a subgroup of a finite special subgroup of $\langle lk_2(A) \rangle$ (including when $\langle A \rangle$ and/or M is trivial). If W is finitely generated, then $K(W, S)$ is finite.

Lemma 4.1 *Suppose (W, S) is a finitely generated Coxeter system and Λ is a non-trivial reduced graph of groups decomposition of W such that each edge group of Λ is in $M(W)$. If Ψ is a reduced visual graph of groups decomposition for W such that each edge group of Ψ is conjugate to a subgroup of an edge group of Λ then each edge group of Ψ is in $M(W)$.* $\square$

Lemma 4.2 *Suppose (W, S) is a finitely generated Coxeter system and G is a group in $M(W)$. Then G is conjugate to a group in $K(W, S)$.*

Proof: By theorem 2.3, there is $E \subset S$ and $w \in W$ such that W splits non-trivially over E and $w\langle E \rangle w^{-1}$ is conjugate to a subgroup of G. By the minimality of G, $\langle E \rangle$ has finite index in $w^{-1}Gw$ and the lemma follows from proposition 3.12. $\square$

Example 1. Consider the Coxeter system (W, S) with $S = \{a, b, c, d, x, y\}$, $m(u, v) = 2$ if $u \in \{a, c, d\}$ and $v \in \{x, y\}$, $m(a, b) = m(b, c) = 2$, $m(c, d) = 3$, $m(x, b) = m(y, b) = 3$ and $m(x, y) = m(a, c) = m(a, d) = m(b, d) = \infty$. The group W is 1-ended since no subset of S separates the presentation diagram

$\Gamma(W,S)$ and also generates a finite group. The group $\langle x,c,y\rangle$ is a member of $M(W)$, since it is 2-ended and $\{x,c,y\}$ separates Γ. The set $\{x,y,b\}$ separates Γ, but $\langle x,b,y\rangle \notin M(W)$ since $\langle x,y\rangle$ has finite index in $\langle x,c,y\rangle$ and infinite index in $\langle x,b,y\rangle$. Note that no subset of $\{x,b,y\}$ generates a group in $M(W)$.

The element cd conjugates $\{x,c,y\}$ to $\{x,d,y\}$. So, $\langle x,d,y\rangle \in M(W)$. Hence a visual subgroup in $M(W)$ need not separate $\Gamma(W,S)$.

Proposition 4.3 *Suppose* (W,S) *is a finitely generated Coxeter system and* $W = A*_C B$ *is a non-trivial splitting of* W*. Then there exists* $D \subset S$ *and* $w \in W$ *such that* $\langle D\rangle \in M(W)$*,* D *separates* $\Gamma(W,S)$ *and* $w\langle E_W(D)\rangle w^{-1} \subset C$ *(so* $w\langle D\rangle w^{-1}\cap C$ *has finite index in* $w\langle D\rangle w^{-1}$*). Furthermore, if* $C \in M(W)$ *then* $w\langle E_W(D)\rangle w^{-1}$ *has finite index in* C*.*

Proof: The second part of this follows trivially from the definition of $M(W)$ and theorem 2.3. Let Ψ_1 be a reduced visual graph of groups decomposition for $A *_C B$. Each edge group of Ψ_1 is a subgroup of a conjugate of C. Say $D_1 \subset S$ and $\langle D_1\rangle$ is an edge group of Ψ_1. Then W splits non-trivially as $\langle E_1\rangle *_{\langle D_1\rangle} \langle F_1\rangle$, where $E_1\cup F_1 = S$ and $E_1\cap F_1 = D_1$. If $\langle D_1\rangle$ is not in $M(W)$, there exists C_1 a subgroup of W, such that W splits non-trivially as $A_1 *_{C_1} B_1$ and such that $C_1 \cap \langle D_1\rangle$ has infinite index in $\langle D_1\rangle$ and finite index in C_1. Let Ψ_2 be a reduced visual decomposition for $A_1 *_{C_1} B_1$, and $D_2 \subset S$ such that $\langle D_2\rangle$ is an edge group of Ψ_2. Then a conjugate of $\langle D_2\rangle$ is a subgroup of C_1, and $W = \langle E_2\rangle *_{\langle D_2\rangle} \langle F_2\rangle$, where $E_2 \cup F_2 = S$ and $E_2 \cap F_2 = D_2$. For $i \in \{1,2\}$, $\langle D_i\rangle = \langle U_i\rangle \times \langle V_i\rangle$ where $U_i = E_W(D_i)$ and $V_i = T_W(D_i)$ (so by remark 2, $U_i \cup V_i = D_i$ and V_i is the (unique) largest such subset of D_i such that $\langle V_i\rangle$ is finite).

It suffices to show that U_2 is a proper subset of U_1. Choose $g \in W$ such that $g\langle D_2\rangle g^{-1} \subset C_1$. Then by lemma 3.5, $g\langle D_2\rangle g^{-1} \cap \langle D_1\rangle = d\langle K\rangle d^{-1}$ for $d \in \langle D_1\rangle$ and $K = D_1 \cap mD_2m^{-1}$ where m is the minimal length double coset representative of $\langle D_1\rangle g\langle D_2\rangle$. Write $\langle K\rangle = \langle U_3\rangle \times \langle V_3\rangle$ with $U_3 = E_W(K)$ and $V_3 = T_W(K)$. As $K \subset D_1$, $E_W(K) \subset E_W(D_1)$, so $U_3 \subset U_1$. As $m^{-1}Km \subset D_2$, lemma 3.11 implies $E_W(K) \subset E_W(D_2)$ so $U_3 \subset U_2$. Hence $U_3 \subset U_1 \cap U_2$. Since $C_1 \cap \langle D_1\rangle$ has infinite index in $\langle D_1\rangle$, $d\langle K\rangle d^{-1}$ has infinite index in $\langle D_1\rangle$. As $d_1 \in \langle D_1\rangle$, $\langle K\rangle$ has infinite index in $\langle D_1\rangle$. Hence U_3 is a proper subset of U_1.

Recall that $g\langle D_2\rangle g^{-1} \subset C_1$ and $C_1 \cap \langle D_1\rangle$ has finite index in C_1 so $d\langle K\rangle d^{-1} = g\langle D_2\rangle g^{-1} \cap \langle D_1\rangle$ has finite index in $g\langle D_2\rangle g^{-1}$ and $g^{-1}d\langle U_3\rangle d^{-1}g$

has finite index in $\langle D_2\rangle$. Thus, for u the minimal length double coset representative of $\langle D_2\rangle g^{-1}d\langle U_3\rangle$, $u\langle U_3\rangle u^{-1}$ has finite index in $\langle D_2\rangle$.

Since $E_W(U_3) = U_3$, lemma 3.11 implies $U_3 = uU_3u^{-1} \subset D_2$. Hence $\langle U_3\rangle$ has finite index in $\langle U_2\rangle$. By proposition 3.12, $\langle U_2\rangle = \langle U_3\rangle \times C$ for C a finite subgroup of $\langle lk_2(U_3)\rangle$. If $s \in U_2 - U_3$ then as $U_2 \subset U_3 \cup lk_2(U_3)$, $s \in lk_2(U_3)$. Hence $\langle U_2\rangle = \langle U_3\rangle \times \langle U_2 - U_3\rangle$. As $\langle U_3\rangle$ has finite index in $\langle U_2\rangle$, $\langle U_2 - U_3\rangle$ is finite. By the definition of U_2, $U_2 = U_3$ and so U_2 is a proper subset of U_1. $\square$

We can now easily recognize separating special subgroups in $M(W)$.

Corollary 4.4 *Suppose (W, S) is a Coxeter system and $C \subset S$ separates $\Gamma(W, S)$. Then $\langle C\rangle \in M(W)$ iff there is no $D \subset S$ such that D separates $\Gamma(W, S)$ and $E_W(D)$ is a proper subset of $E_W(C)$.*

Proof: If $\langle C\rangle \in M(W)$, $D \subset S$ such that D separates Γ and $E_W(D)$ is a proper subset of $E_W(C)$, then by proposition 3.12, $\langle E_W(D)\rangle$ has infinite index in $\langle E_W(C)\rangle$. But then $\langle D\rangle \cap \langle C\rangle$ has finite index in $\langle D\rangle$ and infinite index in $\langle C\rangle$, contrary to the assumption $\langle C\rangle \in M(W)$.

If $\langle C\rangle \notin M(W)$, then by proposition 4.3, there is $D \subset S$ and $w \in W$ such that $\langle D\rangle \in M(W)$, D separates Γ, and $w\langle E_W(D)\rangle w^{-1} \subset \langle C\rangle$. By lemma 3.11, $E_W(D) \subset E_W(C)$. Since $\langle C\rangle \notin M(W)$, $E_W(D)$ is a proper subset of $E_W(C)$. $\square$

Theorem 4.5 *Suppose (W, S) is a finitely generated Coxeter system, Λ is a reduced graph of groups decomposition for W with each edge group a minimal splitting subgroup of W, and Ψ is a reduced graph of groups decomposition of W such that each vertex group of Ψ is conjugate to a subgroup of a vertex group of Λ and for each edge E of Λ, there is an edge D of Ψ such that $\Psi(D)$ is conjugate to a subgroup of $\Lambda(E)$. (E.g. if Ψ is a visual decomposition from Λ.) If A is a vertex of Λ, and Φ_A is the reduced decomposition of $\Lambda(A)$ given by the action of $\Lambda(A)$ on the Bass-Serre tree for Ψ, then*

1) For each edge E of Λ adjacent to A, $\Lambda(E) \subset a\Phi_A(K)a^{-1}$, for some $a \in \Lambda(A)$ and some vertex K of Φ_A. In particular, the decomposition Φ_A is compatible with Λ.

2) Each vertex group of Φ_A is conjugate to a vertex group of Ψ (and so is Coxeter), or is $\Lambda(A)$-conjugate to $\Lambda(E)$ for some edge E adjacent to A.

3) If each edge group of Ψ is in $M(W)$, then each edge group of Φ_A is a minimal splitting subgroup of W.

Proof: Suppose E is an edge of Λ adjacent to A. By hypothesis, there is an edge D of Ψ and $w \in W$ such that $w\Psi(D)w^{-1} \subset \Lambda(E)$. Since $\Lambda(E)$ is minimal, $\Psi(D)$ has finite index in $w^{-1}\Lambda(E)w$ and so corollary 4.8 of [10] implies $\Lambda(E)$ stabilizes a vertex of T_Ψ, the Bass-Serre tree for Ψ. Thus $\Lambda(E)$ is a subgroup of $a\Phi_A(K)a^{-1}$, for some vertex K of Φ_A, and some $a \in \Lambda(A)$. Part 1) is proved.

By theorem 2.4, each vertex group of Φ_A is either conjugate to a vertex group of Ψ or $\Lambda(A)$-conjugate to a subgroup of an edge group $\Lambda(E)$, for some edge E of Λ adjacent to A. Suppose Q is a vertex of Φ_A and $a_1\Phi_A(Q)a_1^{-1} \subset \Lambda(E)$ for some $a_1 \in \Lambda(A)$. By part 1), $\Lambda(E) \subset a_2\Phi_A(K)a_2^{-1}$, for some $a_2 \in \Lambda(A)$ and K a vertex of Φ_A. Thus, $a_1\Phi_A(Q)a_1^{-1} \subset \Lambda(E) \subset a_2\Phi_A(K)a_2^{-1}$. Lemma 2.5 implies $Q = K$ and $a_2^{-1}a_1 \in \Phi_A(Q)$, so $\Phi_A(Q) = a_2^{-1}\Lambda(E)a_2$ and part 2) is proved.

By part 1) W splits non-trivially over each edge group of Φ_A and part 3) follows. $\square$

Proposition 4.6 *Suppose (W,S) is a finitely generated Coxeter system, Λ is a reduced graph of groups decomposition of W and E is an edge of Λ such that $\Lambda(E)$ is conjugate to a group in $K(W,S)$. Then there is $Q \subset S$ such that a conjugate of $\langle Q\rangle$ is a subgroup of a vertex group of Λ and a conjugate of $\Lambda(E)$ has finite index in $\langle Q\rangle$.*

Proof: The group $\Lambda(E)$ is conjugate to $\langle B\rangle \times F$ for $B \subset S$ and $F \subset \langle D\rangle$ where $D \subset lk_2(B)$ and $\langle D\rangle$ is finite. Let T_Λ be the Bass-Serre tree for Λ and set $B = \{b_1, \ldots, b_n\}$. It suffices to show that $\langle B \cup D\rangle$ stabilizes a vertex of T_Λ. Otherwise, let $i \in \{0, 1, \ldots, n-1\}$ be large as possible so that $\langle D \cup \{b_1, \ldots, b_i\}\rangle$ stabilizes a vertex of T_Λ. As $\langle D \cup \{b_{i+1}\}\rangle$ is finite, it stabilizes some vertex V_1 of T_Λ. The group $\langle B\rangle$ stabilizes a vertex V_2 of T_Λ and $\langle D \cup \{b_1, \ldots, b_i\}\rangle$ stabilizes a vertex V_3 of T_Λ. Since T_Λ is a tree, there is a vertex V_4 common to the three T_Λ-geodesics connecting pairs of vertices in $\{V_1, V_2, V_3\}$. Then $\langle D \cup \{b_1, \ldots, b_{i+1}\}\rangle$ stabilizes V_4, contrary to the minimality of i. Instead, $\langle D \cup B\rangle$ stabilizes a vertex of T_Λ. $\square$

The next result combines theorem 4.5 and proposition 4.6 to show that any graph of groups decomposition of a Coxeter group with edge groups equal to minimal splitting subgroups of the Coxeter group is, up to "artificial considerations", visual.

Proposition 4.7 *Suppose (W, S) is a finitely generated Coxeter system, Λ is a reduced graph of groups decomposition for W with each edge group a minimal splitting subgroup of W, and Ψ is a reduced visual decomposition from Λ. If Φ' is the graph of groups obtained from Λ by replacing each vertex A of Λ by Φ_A, the graph of groups decomposition of $\Lambda(A)$ given by the action of $\Lambda(A)$ on the Bass-Serre tree for Ψ, and Φ is obtained by reducing Φ', then there is a bijection τ, from the vertices of Φ to those of Ψ so that for each vertex V of Φ, $\Psi(\tau(V))$ is conjugate to $\Phi(V)$.*

Proof: Part 1) of theorem 4.5 implies the decomposition Φ is well-defined. If Q is a vertex of Ψ then a conjugate of $\Psi(Q)$ is a subgroup of $\Lambda(B)$ for some vertex B of Λ, and corollary 7 of [17] (an elementary corollary of theorem 2.4) implies this conjugate of $\Psi(Q)$ is a vertex group of Φ_B. Hence each vertex group of Ψ is conjugate to a vertex group of Φ'. Suppose A is a vertex of Λ and U is a vertex of Φ_A such that $\Phi_A(U)$ is $\Lambda(A)$-conjugate to $\Lambda(E)$ for some edge E adjacent to A. If $\Lambda(E)$ is not conjugate to a special subgroup of (W, S), then as $\Lambda(E)$ is conjugate to a group in $K(W, S)$, proposition 4.6 implies there is a vertex V of Λ and a vertex group of Φ_V properly containing a conjugate of $\Lambda(E)$. Hence $\Phi_A(U)$ is eliminated by reduction when Φ is formed. If $\Lambda(E)$ is conjugate to a special subgroup of (W, S), then as $\Lambda(E)$ is also conjugate to a subgroup of a vertex group of Ψ, either $\Lambda(E)$ is conjugate to a vertex group of Ψ or $\Lambda(E)$ is eliminated by reduction when Φ is formed. Hence by part 2) of theorem 4.5, every vertex group of Φ is conjugate to a vertex group of Ψ. No two vertex groups of Ψ are conjugate, so if V is a vertex of Φ, let $\tau(V)$ be the unique vertex of Ψ such that $\Phi(V)$ is conjugate to $\Psi(\tau(V))$. As no two vertex groups of Φ are conjugate, τ is injective. If Q is a vertex of Ψ, then as noted above $\Psi(Q)$ is conjugate to a vertex group of Φ' and so $\Psi(Q) \subset w\Phi(V)w^{-1}$ for some $w \in W$ and V a vertex of Φ. Choose $x \in W$ such that $\Phi(V) = x\Psi(\tau(V))x^{-1}$. Then $\Psi(Q) \subset wx\Psi(\tau(V))x^{-1}w^{-1}$. Lemma 2.5 implies $Q = \tau(V)$ and so τ is onto. $\Box$

In the previous argument it is natural to wonder if a vertex group of Ψ might be conjugate to a vertex group of Φ_A and to a vertex group of Φ_B for A and B distinct vertices of Λ. Certainly such a group would be conjugate to an edge group of Λ. The next example show this can indeed occur.

Example 2. Consider the Coxeter presentation $\langle a, b, c, d : \ a^2 = b^2 = c^2 = d^2 = 1\rangle$. Define Λ to be the graph of groups decomposition $\langle a, cdc\rangle *_{\langle cdc\rangle} \langle b, cdc\rangle * \langle d\rangle$. Then Λ has graph with a vertex A and $\Lambda(A) = \langle a, cdc\rangle$, edge

C with $\Lambda(C) = \langle cdc \rangle$ vertex B with $\Lambda(B) = \langle b, cdc \rangle$ edge E with $\Lambda(E)$ trivial and vertex D with $\Lambda(D) = \langle d \rangle$. The visual decomposition for Λ is $\Psi = \langle a \rangle * \langle b \rangle * \langle c \rangle * \langle d \rangle$, a graph of groups decomposition with each vertex group isomorphic to $\mathbb{Z}_2$ and each edge group trivial. Now Φ_A has decomposition $\langle a \rangle * \langle cdc \rangle$, Φ_B has decomposition $\langle b \rangle * \langle cdc \rangle$ and Φ_D has decomposition $\langle d \rangle$. Observe that the Ψ vertex group $\langle d \rangle$ is conjugate to a vertex group of both Φ_A and Φ_B. The group Φ of the previous theorem would have decomposition $\langle a \rangle * \langle b \rangle * \langle c \rangle * \langle cdc \rangle$.

Lemma 4.8 *Suppose (W, S) is a finitely generated Coxeter system and C is a subgroup of W conjugate to a group in $K(W, S)$. If D is a subgroup of W and $wDw^{-1} \subset C \subset D$ for some $w \in W$, then $wDw^{-1} = C = D$.*

Proof: Conjugating we may assume $C = \langle U \rangle \times F$, for $U \subset S$, $E_W(U) = U$ and F a finite group. Let $K \subset lk_2(U)$ such that $\langle K \rangle$ is finite and $F \subset \langle K \rangle$. Now, $w\langle U \rangle w^{-1} \subset wCw^{-1} \subset wDw^{-1} \subset C \subset \langle U \cup K \rangle$. Write $w = xdy$ for $x \in \langle U \cup K \rangle$, $y \in \langle U \rangle$, and d the minimal length double coset representative of $\langle U \cup K \rangle w \langle U \rangle$. Then $dCd^{-1} \subset dDd^{-1} \subset x^{-1}Cx$. By lemma 3.11, $dUd^{-1} = U$ and by the definition of x, $x^{-1}\langle U \rangle x = \langle U \rangle$. The index of $\langle U \rangle$ in dCd^{-1} is $|F|$ and the index of $\langle U \rangle$ in $x^{-1}Cx$ is $|F|$. Hence $dCd^{-1} = dDd^{-1} = x^{-1}Cx$ and $wCw^{-1} = wDw^{-1} = C$. $\square$

Remark 3. The argument in the first paragraph below shows that if Λ is a reduced graph of groups decomposition of a Coxeter group W, V is a vertex of Λ and Φ is a reduced graph of groups decomposition of $\Lambda(V)$, compatible with Λ then when replacing V by Φ to form Λ_1, no vertex group of Φ is W-conjugate to a subgroup of another vertex group of Φ. In particular, each edge of Φ survives reduction in Λ_1.

Proposition 4.9 *Suppose (W, S) is a finitely generated Coxeter system and Λ is a reduced graph of groups decomposition of W with $M(W)$ edge groups. Suppose a vertex group of Λ splits nontrivially and compatibly as $A *_C B$ over an $M(W)$ group C. Then there is a group in $K(W, S)$ contained in a conjugate of B which is not also contained in a conjugate of A (and then also with A and B reversed).*

Proof: Let V be the vertex group such that $\Lambda(V)$ splits as $A *_C B$ and let Λ_1 be the graph of groups resulting from replacing $\Lambda(V)$ by this splitting. If there is $w \in W$ such that $wBw^{-1} \subset A$, then (by considering the Bass-Serre tree for Λ_1) a W-conjugate of B is a subgroup of C. Lemma 4.8 then implies $B = C$, which is nonsense. Hence no W-conjugate of B (respectively A) is a subgroup of A (respectively B). This implies that if Λ_2 is obtained by reducing Λ_1, then there is an edge $\bar{C}$ of Λ_2 with vertices $\bar{A}$ and $\bar{B}$, such that $\Lambda_2(\bar{C}) = C$, and $\Lambda_2(\bar{A})$ is $\hat{A}$ where $\hat{A}$ is either A or a vertex group (other than $\Lambda_1(V)$) of Λ_1 containing A as a subgroup. Similarly for $\Lambda_2(\bar{B})$.

If B collapses across an edge of Λ_1 then B is conjugate to a group in $K(W, S)$ and B satisfies the conclusion of the proposition. If B does not collapse across an edge of Λ_1 (so that $\hat{B} = B$), then let Φ_B be the reduced graph of groups decomposition of B induced from the action of B on Ψ, the visual decomposition of W from Λ_2. By theorem 4.5, each vertex group of Φ_B is conjugate to a group in $K(W, S)$ and the decomposition Φ_B is compatible with Λ_2. Let Λ_3 be the graph of groups decomposition of W obtained from Λ_2 by replacing the vertex for B by Φ_B. In Λ_3, the edge $\bar{C}$ connects the vertex $\bar{A}$ to say the Φ_B-vertex $\tilde{B}$. If $\Lambda_3(\tilde{B})$ is not conjugate to a subgroup of A, then $\Lambda_3(\tilde{B})$ satisfies the conclusion of our proposition. Otherwise, (as before) lemma 4.8 implies $\Lambda_3(\bar{C}) = \Lambda_3(\tilde{B})$ and we collapse $\tilde{B}$ across $\bar{C}$ to form Λ_4. Note that if $\bar{C}$ does collapse, then Φ_B has more than one vertex. There is an edge of Λ_4 (with edge group some subgroup of C which is also an edge group of Φ_B) separating the vertex $\bar{A}$ from some vertex K of Φ_B. The group $\Lambda_4(K)$ satisfies the conclusion of the proposition, since otherwise a W-conjugate of $\Lambda_4(K)$ is a subgroup of A. But then lemma 4.8 implies $\Lambda_4(K)$ is equal to an edge group of Φ_B which is impossible. □

Proposition 4.9 is the last result of this section needed to prove our main theorem. The remainder of the section is devoted to proving theorem 4.14, a minimal splitting version of the visual decomposition theorem of [17]. In order to separate this part of the paper from the rest, some lemmas are listed here that could have been presented in earlier sections. The next lemma follows directly from theorem 3.12.

Lemma 4.10 *Suppose (W, S) is a finitely generated Coxeter system and $A \subset S$. If B is a proper subset of $E(A)$ then $\langle B\rangle$ has infinite index in $\langle E(A)\rangle$.* □

Lemma 4.11 *Suppose (W, S) is a finitely generated Coxeter system, A and B are subsets of S such that $\langle A\rangle$ and $\langle B\rangle$ are elements of $M(W)$. If $E(A) \subset B$ then $E(A) = E(B)$.*

Proof: If $E(A) \subset B$, then the definitions of $E(A)$ and $E(B)$, imply $E(A) \subset$

$E(B)$. As $\langle B\rangle \in M(W)$, lemma 4.10 implies $E(A)$ is not a proper subset of $E(B)$. $\square$

Lemma 4.12 *Suppose (W,S) is a finitely generated Coxeter system, $C \subset S$ is such that $\langle C\rangle \in M(W)$ and C separates $\Gamma(W,S)$. If $K \subset S$ is a component of $\Gamma - C$, then for each $c \in E(C)$, there is an edge connecting c to K.*

Proof: Otherwise, $C - \{c\}$ separates Γ. This is impossible by lemma 4.10 and the fact that $\langle C\rangle \in M(W)$. $\square$

In the remainder of this section we simplify notation for visual graph of groups decompositions by labeling each vertex of such a graph by A, where $A \subset S$ and $\langle A\rangle$ is the vertex group. It is possible for two distinct edges of such a decomposition to have the same edge groups so we do not extend this labeling to edges.

Lemma 4.13 *Suppose (W,S) is a finitely generated Coxeter system and Ψ is a reduced (W,S)-visual graph of groups decomposition with $M(W)$-edge groups. If $A \subset S$ is a vertex of Ψ, and $M \subset S$ is such that $\langle M\rangle \in M(W)$, M separates $\Gamma(W,S)$ and $E(M) \subset A$, then*

1) either $E(M) = E(C)$ for some $C \subset S$ and $\langle C\rangle$ the edge group of an edge of Ψ adjacent to A, or $M \subset A$ and M separates A in Γ, and

2) for each $C \subset S$ such that $\langle C\rangle$ is the edge group of an edge of Ψ adjacent to A, $C - M$ is a subset of a component of $\Gamma - M$.

In particular, if $E(M) \neq E(C)$ for each $C \subset S$ such that $\langle C\rangle$ is the edge group of an edge adjacent to A in Ψ, then $\langle A\rangle$ visually splits over $\langle M\rangle$, compatibly with Ψ, such that each vertex group of the splitting is generated by M union the intersection of A with a component of $\Gamma - M$.

Proof: First we show that if $M \not\subset A$, then $E(M) = E(C)$ for some C such that $\langle C\rangle$ is the edge group of an edge adjacent to A in Ψ. If $E(M) = \emptyset$ then $\langle M\rangle$ is finite and $E(C) = \emptyset$ for every $C \subset S$ such that $\langle C\rangle \in M(W)$. Hence we may assume $E(M) \neq \emptyset$. As $E(M) \subset A$, there is $m \in M - E(M)$ such that $m \notin A$. Say $m \in B$ for $B \subset S$ a vertex of Ψ. If E is the first edge of the Ψ-geodesic from A to B and $\Psi(E) = C$, then $m \notin C$. But in Γ, there is an edge between m and each vertex of $E(M)$. Hence $E(M) \subset C$ and lemma 4.11 implies $E(M) = E(C)$.

To complete part 1), it suffices to show that if $E(M) \neq E(C)$ for all $C \subset S$ such that $\langle C\rangle$ is the edge group of an edge of Ψ adjacent to the vertex A of Ψ, then M separates A in Γ. We have shown that $M \subset A$. Write $W = \langle D_C\rangle *_{\langle C\rangle} \langle B_C\rangle$ where $C \subset S$ is such that $\langle C\rangle = \Psi(E)$ for E an edge of Ψ adjacent to A, and B_C (respectively D_C) the union of the S-generators of vertex groups for all vertices of Ψ on the side of E opposite A (respectively, on the same side of C as A). In particular, $M \subset D_C$ and $M \cap (B_C - C) = \emptyset$. Then B_C is the union of C and some of the components of $\Gamma - C$ (and D_C is the union of C and the rest of the components of $\Gamma - C$). By lemma 4.11, $E(C) \not\subset M$. Choose $c \in E(C) - M$. If B' is a component of $\Gamma - C$ and $B' \subset B_C$, then by lemma 4.12, there is an edge of Γ connecting c and B'. Hence $(B_C - C) \cup (E(C) - M) \subset K_C$ for some component K_C of $\Gamma - M$. In particular, $c \in A \cap K_C$. Also note that if $c' \in C - M$ then either $c' \in E(C) - M$ or there is an edge of Γ connecting c' to c. In either case $c' \in K_C$ and $(B_C - C) \cup (C - M) \subset K_C$.

For $i \in \{1, \ldots, n\}$, let E_i be the edges of Ψ adjacent to A and let $\langle C_i\rangle = \Psi(E_i)$ for $C_i \subset S$. Since $\langle A\rangle$ is a vertex group of Ψ, $\Gamma - A = \cup_{i=1}^n (B_{C_i} - C_i) \subset \cup_{i=1}^n K_{C_i}$. We have argued that there is $c_i \in A \cap K_{C_i}$ for the component K_{C_i} of $\Gamma - M$. If $K_{C_i} \neq K_{C_j}$, then M separates the points c_i and c_j of A, in Γ. If all K_{C_i} are equal (e.g. when $n = 1$), then $\Gamma - K_{C_i} \subset A$. Since M separates Γ, $\Gamma \neq K_{C_i} \cup M$, so M separates c_i from a point of $A - (K_{C_i} \cup M)$. In any case part 1) is proved.

Part 2): As noted above, if $E(M) \neq E(C)$, then for any $C \subset S$ such that $\langle C\rangle$ is the edge group of an edge of Ψ adjacent to A we have $(B_C - C) \cup (C - M) \subset K_C$ for K_C a component of $\Gamma - M$ and B_C some subset of S. If $E(M) = E(C)$ then $\langle C - M\rangle$ is finite, so $C - M$ is a complete subset of Γ and hence a subset of a component of $\Gamma - M$. $\square$

The next result is a minimal splitting version of the visual decomposition theorem. While part 2) of the conclusion is slightly weaker than the corresponding conclusion of the visual decomposition theorem, part 3) ensures that all edge groups of a given graph of groups decomposition of a finitely generated Coxeter group are "refined" by minimal visual edge groups of a visual decomposition. The example following the proof of this theorem shows that part 2) cannot be strengthened.

Theorem 4.14 *Suppose (W, S) is a finitely generated Coxeter system and Λ is a reduced graph of groups decomposition for W. There is a reduced visual decomposition Ψ of W such that*

1) each vertex group of Ψ is a subgroup of a conjugate of a vertex group *of* Λ*,*

2) if D *is an edge of* Ψ *then either* $\Psi(D)$ *is conjugate to a subgroup of an edge group of* Λ*, or* $\Psi(D)$ *is a minimal splitting subgroup for* W *and a visual subgroup of finite index in* $\Psi(D)$ *is conjugate to a subgroup of an edge group of* Λ*.*

3) for each edge E *of* Λ *there is an edge* D *of* Ψ *such that* $\Psi(D)$ *is a minimal splitting subgroup for* W*, and a visual subgroup of finite index of* $\Psi(D)$ *is conjugate to a subgroup of* $\Lambda(E)$*.*

Proof: Let C_1 be an edge group of Λ. By proposition 4.3 there exists $M_1 \subset S$ and $w \in W$ such that $\langle M_1 \rangle \in M(W)$, M_1 separates $\Gamma(W,S)$ and $w\langle M_1\rangle w^{-1} \cap C_1$ has finite index in $w\langle M_1\rangle w^{-1}$. Then W visually splits as $\Psi_1 \equiv \langle A_1 \rangle *_{\langle M_1 \rangle} \langle B_1 \rangle$ (so $A_1 \cup B_1 = S$, $M_1 = A_1 \cap B_1$, and A_1 is the union of M_1 and some of the components of $\Gamma - M_1$ and B_1 is M_1 union the other components of $\Gamma - M_1$). Suppose C_2 is an edge group of Λ other than C_1. Then $W = K_2 *_{C_2} L_2$ where K_2 and L_2 are the subgroups of W generated by the vertex groups of Λ on opposite sides of C_2. Let T_2 be the Bass-Serre tree for this splitting.

Suppose $\langle A_1 \rangle$ and $\langle B_1 \rangle$ stabilize the vertices X_1 and Y_1 respectively of T_2. Then $X_1 \neq Y_1$, since W is not a subgroup of a conjugate of K_2 or L_2. Now, $\langle M_1 \rangle$ stabilizes the T_2-geodesic connecting X_1 and Y_1 and so $\langle M_1 \rangle$ is a subgroup of a conjugate of C_2. In this case we define $\Psi_2 \equiv \Psi_1$.

If $\langle A_1 \rangle$ does not stabilize a vertex of T_2 then there is a non-trivial visual decomposition Φ_1 of $\langle A_1 \rangle$ from its action on T_2 as given by the visual decomposition theorem. Since a conjugate of $\langle M_1 \rangle \cap w^{-1}C_1 w$ has finite index in $\langle M_1 \rangle$ and at the same time stabilizes a conjugate of a vertex group of Λ (and hence a vertex of T_2), corollary 4.8 of [10] implies $\langle M_1 \rangle$ stabilizes a vertex of T_2, and so Φ_1 is visually compatible with the visual splitting $\Psi_1 = \langle A_1 \rangle *_{\langle M_1 \rangle} \langle B_1 \rangle$. If $\langle E_2 \rangle$ is an edge group of Φ_1, then a conjugate of $\langle E_2 \rangle$ is a subgroup of C_2. By corollary 4.4, there is $M_2 \subset S$ such that M_2 separates $\Gamma(W,S)$, $\langle M_2 \rangle \in M(W)$ and $E(M_2) \subset E_2$ and so $\langle E(M_2) \rangle$ is a subgroup of a conjugate of C_2. If $E(M_2) \neq E(M_1)$, then lemma 4.13 implies $M_2 \subset A_1$ and $\langle A_1 \rangle$ visually splits over $\langle M_2 \rangle$ compatibly with the splitting Ψ_1. Reducing produces a visual decomposition Ψ_2. Similarly if $\langle A_1 \rangle$ stabilizes a vertex of T_2 and $\langle B_1 \rangle$ does not.

Inductively, assume $C_1, \ldots, C_n$ are distinct edge groups of Λ, Ψ_{n-1} is a reduced visual graph of groups decomposition, each edge group of Ψ_{n-1} is in $M(W)$ and contains a visual subgroup of finite index conjugate to a subgroup of C_i for some $1 \leq i \leq n-1$, and for each $i \in \{1, 2, \ldots, n-1\}$ there is an edge group $\langle M_i \rangle$ ($M_i \subset S$) of Ψ_{n-1} such that a visual subgroup of finite index of $\langle M_i \rangle$ is conjugate to a subgroup of C_i. Write $W = K_n *_{C_n} L_n$ as above, and let T_n be the Bass-Serre tree for this splitting. Either two adjacent vertex groups of Ψ_{n-1} stabilize distinct vertices of T_n (in which case we define $\Psi_n \equiv \Psi_{n-1}$) or some vertex $V_i \subset S$ of Ψ_{n-1} does not stabilize a vertex of T_n. In the latter case $\langle V_i \rangle$ visually splits (as above) to give Ψ_n. Hence, we obtain a reduced visual decomposition Ψ' such that for each edge group $\langle M \rangle$ ($M \subset S$) of Ψ', $\langle M \rangle$ is a group in $M(W)$, a subgroup of finite index in $\langle M \rangle$ is conjugate to a subgroup of an edge group of Λ, and for each edge D of Λ there is an edge group $\langle M \rangle$ of Ψ' such that $\langle E(M) \rangle$ (a subgroup of finite index in $\langle M \rangle$) is conjugate to a subgroup of $\Lambda(D)$.

Suppose $V \subset S$ is a vertex of Ψ'. Consider Φ_V, the visual decomposition of $\langle V \rangle$ from its action on T_Λ, the Bass-Serre tree for Λ. If $\langle D \rangle$ ($D \subset S$) is an edge group for an edge of Ψ' adjacent to V, then a subgroup of finite index in $\langle D \rangle$ stabilizes a vertex of T_Λ. By corollary 4.8 of [10], $\langle D \rangle$ stabilizes a vertex of T_Λ and Φ_V is compatible with Ψ'. Replacing each vertex V of Ψ' by Φ_V and reducing gives the desired decomposition of W. $\square$

The following example exhibits why one cannot expect a stronger version of theorem 4.14 with visual decomposition Ψ having only minimal edge groups, or so that all minimal edge groups of Ψ are conjugate to subgroups of edge groups of Λ.

Example 3. Consider the Coxeter presentation $\langle a_1, a_2, a_3, a_4, a_5 : a_i^2 = 1, (a_1a_2)^2 = (a_2a_3)^2 = (a_3a_4)^2 = (a_4a_5)^2 = (a_5a_1^2) = (a_2a_5)^2 = 1 \rangle$ and the splitting $\Lambda = \langle a_2, a_3, a_4 \rangle *_{\langle a_2, a_4 \rangle} \langle a_1, a_2, a_4, a_5 \rangle$. The subgroup $\langle a_2, a_5 \rangle$ is the only minimal visual splitting subgroup for this system, and it is smaller than $\langle a_2, a_4 \rangle$. Then no subgroup of $\langle a_2, a_4 \rangle$ is a minimal splitting subgroup for our group. The only visual decomposition for this splitting satisfying the conclusion of theorem 4.14 is: $\langle a_1, a_2, a_5 \rangle *_{\langle a_2, a_5 \rangle} \langle a_2, a_4, a_5 \rangle *_{\langle a_2, a_4 \rangle} \langle a_2, a_3, a_4 \rangle$.

5 Accessibility

We prove prove our main theorem in this section, a strong accessibility result for splittings of Coxeter groups over groups in $M(W)$. For a class of groups $\mathcal{V}$, we call a graph of groups decomposition of a group *irreducible with respect to*

$\mathcal{V}$-*splittings* if for any vertex group V of the decomposition, every non-trivial splitting of V over a group in $\mathcal{V}$ is not compatible with the original graph of groups decomposition.

The following simple example describes a non-trivial compatible splitting of a vertex group of a graph of groups decomposition Λ, of a Coxeter group followed by a reduction to produce a graph of groups with fewer edges than those of Λ. This illustrates potential differences between accessibility and strong accessibility.

Example 4.

$$W \equiv \langle s_1, s_2 : s_i^2 \rangle \times \langle s_3, s_4, s_5, s_6 : s_i^2 \rangle$$

First consider the splitting of W as:

$$\langle s_1, s_2, s_3, s_4 \rangle *_{\langle s_1, s_2, s_4 \rangle} \langle s_1, s_2, s_4, s_5 \rangle *_{\langle s_1, s_2, s_5 \rangle} \langle s_1, s_2, s_5, s_6 \rangle$$

The group $\langle s_1, s_2, s_4, s_5 \rangle$ splits as $\langle s_1, s_2, s_4 \rangle *_{\langle s_1, s_2 \rangle} \langle s_1, s_2, s_5 \rangle$. Replacing this group in the above splitting with this amalgamated product and collapsing gives the following decomposition of W:

$$\langle s_1, s_2, s_3, s_4 \rangle *_{\langle s_1, s_2 \rangle} \langle s_1, s_2, s_5, s_6 \rangle$$

Proposition 5.1 *Suppose (W, S) is a finitely generated Coxeter system, Ψ is a reduced visual graph of groups decomposition of (W, S), with $M(W)$ edge groups and V is a vertex of Ψ such that $\Psi(V)$ decomposes compatibly as a nontrivial amalgamated product $A *_C B$ where C is in $M(W)$. Then $\Psi(V)$ is a nontrivial amalgamated product of special subgroups over an $M(W)$ special subgroup U, with U a subgroup of a conjugate of C, and such that any special subgroup contained in a conjugate of A or B is a subgroup of one of the factors of this visual splitting. In particular, the vertex group $\Psi(V)$ visually splits, compatibly with Ψ, to give a finer visual decomposition of (W, S).*

Proof: Applying theorem 2.3 to the amalgamated product $A *_C B$, we get that there is a reduced visual graph of groups decomposition Ψ' of $\Psi(V)$ such that each vertex group of Ψ' is a subgroup of a conjugate of A or B and each edge group a subgroup of a conjugate of C. Then Ψ' has more than one vertex since $A *_C B$ being nontrivial means $\Psi(V)$ is not a subgroup of a conjugate of A or B. Fix an edge of Ψ', say with edge group U, and collapse the other edges in Ψ' to get a nontrivial visual splitting of $\Psi(V)$ over

U a subgroup of a conjugate of C. By theorem 2.3, a special subgroup of $\Psi(V)$ contained in a conjugate of A or B is contained in a vertex group of Ψ' and so is contained in one of the factors of the resulting visual splitting of $\Psi(V)$ derived from partially collapsing Ψ'. Hence this visual decomposition of $\Psi(V)$ is compatible with Ψ, giving a finer visual decomposition of (W, S). Since C is in $M(W)$ and a conjugate of U is a subgroup of C, U is in $M(W)$. $\square$

A visual decomposition Ψ of a Coxeter system (W, S) *looks irreducible with respect to $M(W)$ splittings* if each edge group of Ψ is in $M(W)$ and for any subset V of S such that $\langle V\rangle$ is a vertex group of Ψ, $\langle V\rangle$ cannot be split visually, non-trivially and Ψ-compatibly over $\langle E\rangle \in M(W)$ for $E \subset S$, to give a finer visual decomposition of W. By lemma 2.2, it is elementary to see that every finitely generated Coxeter group has a visual decomposition that looks irreducible with respect to $M(W)$ splittings. The following result is a direct consequence of Proposition 5.1.

Corollary 5.2 *A visual decomposition of a Coxeter group looks irreducible with respect to $M(W)$ splittings, iff it is irreducible with respect to $M(W)$ splittings.* $\square$

Hence any visual graph of groups decomposition of a Coxeter group with $M(W)$ edge groups can be refined to a visual decomposition that is irreducible with respect to $M(W)$ splittings.

Corollary 5.3 *Suppose (W, S) is a finitely generated Coxeter system and W is the fundamental group of a graph of groups Λ where each edge group is in $M(W)$. Then W has an irreducible with respect to $M(W)$ splittings visual decomposition Ψ where each vertex group of Ψ is a subgroup of a conjugate of a vertex group of Λ.*

Proof: Applying theorem 2.3 to Λ, we get a reduced visual graph of groups Ψ from Λ. If Ψ looks irreducible with respect to $M(W)$ splittings, then we are done. Otherwise, some vertex group of Ψ visually splits nontrivially and compatibly over an $M(W)$ special subgroup and we replace the vertex with this visual splitting in Ψ. We can repeat, replacing some special vertex group by special vertex groups with fewer generators, until we must reach a visual graph of groups which looks irreducible with respect to $M(W)$ splittings. $\square$

Theorem 1.2 describes how "close" a decomposition with $M(W)$ edge groups, which is irreducible with respect to $M(W)$ splittings, is to a visual one.

Theorem 1.2 *Suppose (W,S) is a finitely generated Coxeter system and Λ is a reduced graph of groups decomposition of W with $M(W)$ edge groups. If Λ is irreducible with respect to $M(W)$ splittings, then each vertex and edge group of Λ is conjugate to a visual subgroup for (W,S).*

Furthermore, if Ψ is a reduced graph of groups decomposition such that each edge group of Ψ is in $M(S)$, each vertex group of Ψ is a subgroup of a conjugate of a vertex group of Λ, and each edge group of Λ contains a conjugate of an edge group of Ψ (in particular if Ψ is a reduced visual graph of groups decomposition for (W,S) derived from Λ as in the main theorem of [17]), then

1. *Ψ is irreducible with respect to $M(W)$ splittings*

2. *There is a (unique) bijection α of the vertices of Λ to the vertices of Ψ such that for each vertex V of Λ, $\Lambda(V)$ is conjugate to $\Psi(\alpha(V))$.*

Proof: Consider a vertex V of Λ with vertex group $A=\Lambda(V)$. By theorem 4.5, $\Lambda(V)$ has a graph of groups decomposition Φ_V such that Φ_V is compatible with Λ, each edge group of Φ_V is in $M(W)$ and each vertex group of Φ_V is conjugate to a vertex group of Ψ or conjugate to $\Lambda(E)$ for some edge E of Λ adjacent to V. Since Λ is reduced and irreducible with respect to $M(W)$ splittings, Φ_V has a single vertex and $\Lambda(V)$ is conjugate to $\Psi(V')$ for some vertex V' of Ψ.

Since no vertex group of Ψ is contained in a conjugate of another, V' is uniquely determined, and we set $\alpha(V)=V'$. No vertex group of Λ is conjugate to another so α is injective. Since each vertex group $\Psi(V')$ is contained in a conjugate of some $\Lambda(V)$ which is in turn conjugate to $\Psi(\alpha(V))$ we must have $V'=\alpha(V)$ and each V' is in the image of α.

If Ψ is not irreducible with respect to $M(W)$ splittings, then it does not look irreducible with respect to $M(W)$ splittings and some vertex group W_1 of Ψ visually splits nontrivially and compatibly over an $M(W)$ special subgroup U_1. Reducing gives a visual graph of groups decomposition Ψ_1 of W satisfying the hypotheses on Ψ in the statement of the theorem. Now W_1 is conjugate to a vertex group A of Λ and the above argument shows A is conjugate to a vertex group of Ψ_1. But then, W_1 is conjugate to a vertex group of Ψ_1, which is nonsense. Instead, Ψ is irreducible with respect to $M(W)$ splittings.

Since Λ is a tree, we can take each edge group of Λ as contained in its endpoint vertex groups taken as subgroups of W. Hence each edge group is simply the intersection of its adjacent vertex groups (up to conjugation). Since vertex groups of Λ are conjugates of vertex groups in Ψ, their intersection is conjugate to a special subgroup (by lemma 3.5) when Ψ is visual. □

Example 5. Let W have the Coxeter presentation:

$$\langle s_1, s_2, s_3, s_4, s_5 : s_k^2, (s_1s_2)^2, (s_2s_3)^2, (s_3s_4)^2, (s_4s_5)^2\rangle \times \langle s_6, s_7 : s_k^2\rangle$$

Then W is 1-ended and has the following visual $M(W)$-irreducible decomposition (each edge group is 2-ended):

$$\langle s_1, s_2, s_6, s_7\rangle *_{\langle s_2,s_6,s_7\rangle} \langle s_2, s_3, s_6, s_7\rangle *_{\langle s_3,s_6,s_7\rangle} \langle s_3, s_4, s_6, s_7\rangle *_{\langle s_4,s_6,s_7\rangle} \langle s_4, s_5, s_6, s_7\rangle$$

There is an automorphism of W sending s_5 to $s_3s_5s_3$ and all other s_i to themselves. This gives another $M(W)$-irreducible decomposition of W where the last vertex group $\langle s_4, s_5, s_6, s_7\rangle$ of the above graph of groups decomposition is replaced by $\langle s_4, s_3s_5s_3, s_6, s_7\rangle$. As s_3 does not commute with s_1 we see that in regard to part 2 of theorem 1.2, a single element of W cannot be expected to conjugate each vertex group of an arbitrary $M(W)$-irreducible decomposition to a corresponding vertex group of a corresponding visual $M(W)$-irreducible decomposition.

Theorem 1.1 *Finitely generated Coxeter groups are strongly accessible over minimal splittings.*

Proof: Suppose (W, S) is a finitely generated Coxeter system. There are only finitely many elements of $K(W, S)$ (which includes the trivial group). For G a subgroup of W let $n(G)$ be the number of elements of $K(W, S)$ which are contained in any conjugate of G (so $1 \leq n(G) \leq n(W)$). For Λ a finite graph of groups decomposition of W, let $c(\Lambda) = \Sigma_{i=1}^{n(W)} 3^i c_i(\Lambda)$ where $c_i(\Lambda)$ is the count of vertex groups G of Λ with $n(G) = i$.

If Λ reduces to Λ' then clearly $c_i(\Lambda') \leq c_i(\Lambda)$ for all i, and for some i, $c_i(\Lambda')$ is strictly less than $c_i(\Lambda)$. Hence, $c(\Lambda') < c(\Lambda)$.

If Λ is reduced with $M(W)$ edge groups, and a vertex group G of Λ splits non-trivially and compatibly as $A*_C B$ to produce the decomposition Λ' of W, then every subgroup of a conjugate of A or B is a subgroup of a conjugate of G, but, by proposition 4.9, some element of $K(W,S)$ is contained in a conjugate of B, and so of G, but not in a conjugate of A. Hence $n(A) < n(G)$, and similarly $n(B) < n(G)$. This implies that $c(\Lambda') < c(\Lambda)$ since $c_{n(G)}$ decreases by 1 in going from Λ to Λ' and the only other c_i that change are $c_{n(A)}$ and $c_{n(B)}$, which are both increased by 1 if $n(A) \neq n(B)$ and $c_{n(A)}$ increases by 2 if $n(A) = n(B)$, but $c_{n(A)}$ and $c_{n(B)}$ have smaller coefficients than $c_{n(G)}$ in the summation c. More specifically, $c(\Lambda) - c(\Lambda') = 3^{n(G)} - (3^{n(A)} + 3^{n(B)}) > 0$.

If Λ is the trivial decomposition of W, then $c(\Lambda) = 3^{|K(W,S)|}$ and we define this number to be $C(W,S)$. Suppose $\Lambda_1, \ldots, \Lambda_k$ is a sequence of reduced graph of groups decompositions of W with $M(W)$ edge groups, such that Λ_1 is the trivial decomposition and Λ_i is obtained from Λ_{i-1} by splitting a vertex group G of Λ_{i-1} non-trivially and compatibly as $A *_C B$, for $C \in M(W)$ and then reducing. We have shown that $c(\Lambda_i) < c(\Lambda_{i-1})$ for all i, and so $k \leq C(W,S)$. In particular, W is strongly accessible over $M(W)$ splittings $\square$

6 Generalizations, Ascending HNN extensions (and a group of Thompson) and Closing questions

Recall that if G is a group and H and K are subgroups of G then H is smaller than K if $H \cap K$ has finite index in H and infinite index in K. Suppose W is a finitely generated Coxeter group and $\mathcal{C}$ is a class of subgroups of W such that for each $G \in \mathcal{C}$, any subgroup of G is in $\mathcal{C}$, e.g. the virtually abelian subgroups of W. Define $M(W, \mathcal{C})$, *the minimal* $\mathcal{C}$ *splitting subgroups of* W, to be the set of all subgroups H of W such that $H \in \mathcal{C}$, W splits non-trivially over H and for any $K \in \mathcal{C}$ such that W splits non-trivially over K, K is not smaller than H. Then the same line of argument as used in this paper shows that W is strongly accessible over $M(W, \mathcal{C})$ splittings.

If (W,S) is a finitely generated Coxeter system and Ψ is an $M(W)$-irreducible graph of groups decomposition of W with $M(W)$-edge groups, then by theorem 1.2, each vertex group V of Ψ is a Coxeter group with Coxeter system (V,A) where A is conjugate to a proper subset of S. The collection $M(V)$ is not, in general, a subset of $M(W)$, and so V has an $M(V)$-irreducible graph of groups decomposition with $M(V)$-edge groups.

As $|A| < |S|$, there cannot be a sequence $\Psi = \Psi_0, \Psi_1, \ldots, \Psi_n$, with $n > |S|$, of distinct graph of groups decompositions where Ψ is $M(W)$-indecomposable with $M(W)$-edge groups, for $i > 0$, V_i a vertex group of Ψ_{i-1} and Ψ_i is $M(V_i)$-indecomposable with edge groups in $M(V_i)$. Such a sequence must terminate with a special subgroup of W that has no non-trivial decomposition. By the FA results of [17], that group must have a complete presentation diagram.

Suppose B is a group, and $\phi : A_1 \to A_2$ is an isomorphism of subgroups of B. The group G with presentation $\langle t, B : t^{-1}at = \phi(a) \text{ for } a \in A_1\rangle$ is called an *HNN extension* with *base group* B, *associated subgroups* A_i and *stable letter* t. If $A_1 = B$ then the HNN extension is *ascending* and if additionally, A_2 is a proper subgroup of B (i.e. $A_2 \neq B$), then the HNN extension is *strictly ascending.*

The bulk of this section is motivated by an example of Richard Thompson. Thompson's group F is finitely presented and is an ascending HNN extension of a group isomorphic to F. Hence F is not "hierarchical accessible" over such splittings (see question 1 below). If a group G splits as an ascending HNN extension, then (by definition) there is no splitting of the base group which is compatible with the first splitting, so standard accessibility is not an issue. The only question is that of minimality of such splittings.

Theorem 6.1 *Suppose A is a finitely generated group and $\phi : A \to A$ is a monomorphism. Let $G \equiv \langle t, A : t^{-1}at = \phi(a) \text{ for } a \in A\rangle$ be the resulting ascending HNN extension. Then:*

1) If $\phi(A)$ has infinite index in A, this splitting of G is not minimal and there is no finitely generated subgroup B of G such that B is smaller than A, G splits as an ascending HNN extension over B and this splitting over B is minimal.

2) If $\phi(A)$ has finite index in A, then there is no finitely generated subgroup B of G such that B is smaller than A and G splits as an ascending HNN extension over B.

Proof: First note that G is also an ascending HNN extension over $\phi(A)$, (with presentation $\langle t, \phi(A) : t^{-1}at = \phi(a) \text{ for } a \in \phi(A)\rangle$. Hence if $\phi(A)$ has infinite index in A, the splitting over A is not minimal. Part 5) of lemma 6.2 implies the second assertion of part 1) of the theorem. Part 4) of lemma 6.2 implies part 2) of the theorem. □

Lemma 6.2 *Suppose $\phi : A \to A$ and $\tau : B \to B$ are monomorphisms of finitely generated subgroups of G, and the corresponding ascending HNN extensions are isomorphic to G.*

$$G \equiv \langle A, t : t^{-1}at = \phi(a) \text{ for all } a \in A\rangle \equiv \langle s, B : s^{-1}bs = \tau(b) \text{ for all } b \in B\rangle$$

If $A \cap B$ has finite index in B (so B is potentially smaller than A). Then:

1) The normal closures $N(A)$ and $N(B)$ in G are equal.

2) If $\phi(A) \neq A$ then $s = at$ for some $a \in N(A)$

3) If $\phi(A) = A$, then $\tau(B) = B$ (so $N(A) = A = B = N(B)$) and $s = at^{\pm 1}$ for some $a \in A$.

4) If $\phi(A)$ has finite index in A, then $A \cap B$ has finite index in A (so B is not smaller than A) and $\tau(B)$ has finite index in B.

5) If $\phi(A)$ has infinite index in A, then $\tau(B)$ has infinite index in B.

Proof: Let $A_0 = A$ and let $A_i = t^i A_0 t_i^{-i}$. Then $t^{-1}A_i t = A_{i-1} < A_i$. Note that $N(A) = \cup_{i=0}^{\infty} A_i$. Let $\pi : G \to G/N(A) \equiv \mathbb{Z}$ be the quotient map. Since $A \cap B$ has finite index in B, $\pi(B)$ is finite (and hence trivial). This implies $B < N(A)$. As B is finitely generated, $B < A_m$ for some m. This also implies that $\langle \pi(s)\rangle = \langle \pi(t)\rangle = \mathbb{Z}$ and so $N(B) = N(A)$, completing 1).

Normal forms in ascending HNN extensions imply $s = t^p a_1 t^{-q}$ for some $p, q \geq 0$ and $a_1 \in A$. This implies $|p - q| = 1$. Hence $s = at^{\pm 1}$ for $a \in A_p$.

Suppose $s = at^{-1}$. Let r be the maximum of m and p. Note that

$$N(A) = N(B) = \cup_{i=0}^{\infty} s^i B s^{-i} = \cup_{i=0}^{\infty} (at^{-1})^i B (ta^{-1})^i < A_r$$

(since, $t^{-1}Bt < t^{-1}A_m t = A_{m-1} < A_r$ and (as $a \in A_r$) $aA_r a^{-1} = A_r$). But if $\phi(A) \neq A$, $A_{r+1} \not< A_r$. Instead, $s = at$, completing 2).

If $\phi(A) = A$, then $N(A) = A$. As $N(B) = A$ is finitely generated, $N(B) = \cup_{i=0}^{n} s^i B s^{-i} = s^n B s^{-n}$ for some $n > 0$. So $N(B) = B$ completing 3).

If $\phi(A)$ has finite index in A then A has finite index in A_i for all $i \geq 0$. Since $N(A) = N(B)$, and A and B are finitely generated, there are positive integers $p < p'$ and $q < q'$ such that

$$A < s^p B s^{-p} < t^q A t^{-q} < s^{p'} B s^{-p'} < t^{q'} A t^{-q'}$$

Hence B has finite index in $s^{p'-p} B s^{-(p'-p)}$. This implies B has finite index in $s^i B s^{-i}$ for all $i \geq 0$ and also $\tau(B)$ has finite index in B. Similarly, there are positive integers j and k such that

$$B < t^k A t^{-k} < s^j B s^{-j}$$

Hence B and A (and so $A \cap B$) have finite index in $t^k A t^{-k}$. This implies $A \cap B$ has finite index in A and 4) is complete.

Assume $\phi(A)$ has infinite index in A. As A and B are finitely generated subgroups of $N(A) = N(B)$, there positive integers k and j such that

$$B < t^k A t^{-k} < s^j B s^{-j}$$

The group B does not have finite index in $t^k A t^{-k}$ since otherwise $A \cap B$ (and then A) would have finite index in $t^k A t^{-k}$. This implies B has infinite index in $s^j B s^{-j}$. This in turn implies B has infinite index in $s^i B s^{-i}$ for all $i \geq 0$. This also implies $\tau(B)$ has infinite index in B. □

Example 6. (Thompson's Group) In unpublished work, R. J. Thompson introduced a group, traditionally denoted F, in the context of finding infinite finitely presented simple groups. This group is now well studied in a variety of other contexts. The group F has presentation

$$\langle x_1, x_2, \ldots : x_i^{-1} x_j x_i = x_{j+1} \text{ for } i < j \rangle$$

Well know facts about this group include: F is FP_∞ ([6]), in particular, F is finitely presented (with generators x_1 and x_2), the commutator subgroup of F is simple ([5]), and F contains no free group of rank 2 ([4]). Clearly, F is an ascending HNN extension of itself (with base group $\langle x_2, x_3, \ldots \rangle$ and stable letter x_1 - called the "standard" splitting of F).

We are interested in understanding "minimal" splittings of F and more generally minimal splittings of finitely generated groups containing no non-abelian free group. We list some elementary facts.

Fact 1. *If G contains no non-abelian free group and G splits as an amalgamated free product $A *_C B$ then C is of index 2 in both A and B and hence is normal in G. If G splits as an HNN-extension, then this splitting is ascending.*

Fact 2. *The group F does not split non-trivially as $A *_C B$*

Proof: Otherwise C is normal in F and F/C is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. Since the commutator subgroup K of G is simple, $K \cap C$ is either trivial or K. The intersection is not K since F/C is not abelian. The intersection is non-trivial, since otherwise K would inject under the quotient map $F \to F/C \equiv \mathbb{Z}_2 * \mathbb{Z}_2$. □

By theorem 6.1 and the previous facts we have:

Fact 3. *The only non-trivial splittings of F are as ascending HNN extensions $\langle t, A : t^{-1}at = \phi(a)$ for all $a \in A\rangle$. For A finitely generated, this splitting is minimal iff the image of the monomorphism $\phi : A \to A$ has finite index in A.*

R. Bieri, W. D. Neumann and R. Strebel have shown that if G is a finitely presented group containing no free group of rank 2 and G maps onto $\mathbb{Z} \oplus \mathbb{Z}$, then G contains a finitely generated normal subgroup H such that $G/H \cong \mathbb{Z}$ (see theorem D of [2] or theorem 18.3.8 of [14]). Hence, there is a short exact sequence $1 \to H \to F \to \mathbb{Z} \to 1$ with H finitely generated.

Fact 4. *The ascending HNN extensions given by the short exact sequence $1 \to H \to F \to \mathbb{Z} \to 1$ (with H finitely generated) are minimal splittings.*

Theorem 6.3 *Suppose G is a finitely generated group containing no non-abelian free subgroup. Suppose G can be written as an ascending HNN extension $\langle t, A : t^{-1}at = \phi(a)$ for all $a \in A\rangle$ and as non-trivial amalgamated products $C *_D E$ and $H *_K L$ where all component groups are finitely generated, then:*

1) $D \cap A$ does not have finite index in A or D (so neither A nor D is smaller than the other),

2) if $D \cap K$ has finite index in K then $K = D$ (so neither D nor K is smaller than the other),

3) $C *_D E$ is a minimal splitting and $\langle t, A : t^{-1}at = \phi(a)$ for all $a \in A\rangle$ is minimal iff $\phi(A)$ has finite index in A.

Proof: Let $q : G \to G/N(A) \equiv \mathbb{Z}$ and $p : G \to G/D \equiv \mathbb{Z}_2 * \mathbb{Z}_2$ be the quotient maps. If $D \cap A$ has finite index in D then $q(D)$ is finite, so $q(D)$ is trivial and $D < N(A)$. But this implies there is a homomorphism from $\mathbb{Z}_2 * \mathbb{Z}_2$ onto $\mathbb{Z}$, which is nonsense.

If $D \cap A$ has finite index in A, then $p(A)$ is a finite subgroup of $\mathbb{Z}_2 * \mathbb{Z}_2 \equiv \langle x : x^2 = 1\rangle * \langle y : y^2 = 1\rangle$. Then $p(A)$ is a subgroup of a conjugate of $\langle x\rangle$ or $\langle y\rangle$. Without loss, assume $p(A) < \langle x\rangle$. If $p(A) = 1$, then $A < D$ and so $N(A) < D$. But this implies there is a homomorphism of $\mathbb{Z}$ onto $\mathbb{Z}_2 * \mathbb{Z}_2$ which is nonsense. Hence $p(A) = \langle x\rangle$. But then $p(t)$ commute with x. This is implies $p(t)$ is trivial. This is impossible as $p(t)$ and $p(A)$ generate $\mathbb{Z}_2 * \mathbb{Z}_2$. Part 1) is finished.

Suppose $D \cap K$ has finite index in K. Then as above we can assume that $p(K) < \langle x \rangle$. If $p(K) = 1$, then $K < D$. If additionally $K \neq D$ then there is a homomorphism from $\mathbb{Z}_2 * \mathbb{Z}_2$ onto $\mathbb{Z}_2 * \mathbb{Z}_2$ with non-trivial kernel. This is impossible. Hence, either, $K = D$ or $p(K) = \langle x \rangle$. We conclude that $K = D$ since $p(K)$ is normal in $\mathbb{Z}_2 * \mathbb{Z}_2$, and 2) is finished.

Fact 1, and part 2) implies $C *_D E$ is a minimal splitting. Fact 1, theorem 6.1 and part 1) imply $\langle t, A : t^{-1}at = \phi(a) \text{ for all } a \in A \rangle$ is minimal iff $\phi(A)$ has finite index in A. □

We conclude this paper with some questions of interest.

1. Are finitely generated Coxeter groups hierarchical accessible over minimal splittings? If so then do last terms of splitting sequences have no splittings of any sort (are they FA)? Would such a last term always be visual?

2. Is there a JSJ theorem for Coxeter groups over minimal splittings? In [18], we produce a JSJ result for Coxeter groups over virtually abelian splitting subgroups that relies on splittings over minimal virtually abelian subgroups.

   For the standard strictly ascending HNN splitting of Thompson's group F (given by $\langle x_1, x_2, \ldots : x_i^{-1} x_j x_i = x_{j+1} \text{ for } i < j \rangle$ - with base group $B \equiv \langle x_2, x_3, \ldots \rangle$ and stable letter x_1) there is no minimal splitting subgroup C of F with C smaller than B. Hence, for finitely presented groups, there is no analogue for proposition 4.3. Still F, and in fact all finitely generated groups containing no non-abelian free group, are strongly accessible over finitely generated minimal splitting subgroups.

3. Are finitely presented groups (strongly) accessible over finitely generated minimal splittings?

   Finitely generated groups are not accessible over finite splitting subgroups (see [12]), and hence finitely generated groups are not accessible over minimal splittings.

4. Does Thompson's group split as a *strict* ascending HNN extension with finitely generated base A and monomorphism $\phi : A \to A$ such that $\phi(A)$ has finite index in A?

References

[1] M. Bestvina, M. Feighn, Stable actions of groups on real trees, Invent. Math. **121** (2) (1995), 287–321.

[2] R. Bieri, W. D. Neumann, R. Strebel, A geometric invariant of discrete groups, Invent. Math. **90** (1987) 451-477.

[3] N. Bourbaki, *Groupes et Algebres de Lie* Chapters IV-VI, Hermann, Paris, 1968.

[4] M. G. Brin, C. C. Squire, Groups of piecewise smooth linear homeomorphisms of the real line, Invent. Math. **79** (3) (1985), 485-498.

[5] K. S. Brown, Finiteness properties of groups, J. Pure Appl. Algebra, **44** (1-3) (1987), 45-75.

[6] K. S. Brown, R. Geoghegan, An infinite-dimensional torsion-free FP_∞ group, Invent. Math. **77** (2) (1984), 367-381.

[7] T. Delzant, Sur l'accessibilité acylindrique des groupes de presentation finie, Ann. Inst. Fourier **49** (4) (1999), 1215-1224.

[8] T. Delzant, L. Potyagailo, Accessibilité hièrarchique des groupes de présentation finie, Topology **40** (3) (2001), 617-629.

[9] V. V. Deodhar, On the root system of a Coxeter group, Commun. Algebra **10** (1982), 611-630.

[10] W. Dicks, M. J. Dunwoody, *Groups Acting on Graphs*, Cambridge University Press, 1989.

[11] M. J. Dunwoody, The accessibility of finitely presented groups, Invent. Math. **81** (1985), 449–457.

[12] M. J. Dunwoody, An inaccessible group, from *Geometric group theory* Vol. 1 (Sussex, 1991), in: London Math. Soc. Lecture Note Series **181**, Cambridge Univ. Press, 1993.

[13] M. J. Dunwoody, M. Sageev, JSJ-Splittings for finitely presented groups over slender groups, Invent. Math. **135** (1999) 25-44.

[14] R. Geoghegan *Topological Methods in Group Theory*, Graduate Texts in Mathematics **243**, Springer, 2008.

[15] I. Kapovich, R. Weidmann, Acylindrical accessibility for groups acting on $\mathbb{R}$-trees, Math. Z. **249** (4) (2005), 773-782.

[16] M. L. Mihalik, K. Ruane, S. Tschantz, Local connectivity of right-angled Coxeter group boundaries, J. Group Theory **10** (2007) 531-560.

[17] M. L. Mihalik, S. Tschantz, Visual decompositions for Coxeter groups, Groups Geom. Dyn. **3** (2009) 173-198.

[18] M. L. Mihalik, JSJ-decompositions of Coxeter groups over virtually abelian splittings, preprint 2005. arXiv.math.GR/0804.3963 (4/24/08).

[19] E. Rips, Z. Sela, Cyclic splittings of finitely presented groups and the cannonical JSJ decomposition, Ann. of Math (2) **146** (1) (1997), 53-109.

[20] Z. Sela, The isomorphism problem for hyperbolic groups I, Ann. of Math. (2) **141** (2) (1995), 217-283.

[21] Z. Sela, Acylindrical Accessibility, Invent. Math. **129** (1997) 527-565.

[22] Z. Sela, Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II, Geom. Funct. Anal. **7** (1997) no. 3, 561-593.

[23] J. Stallings, *Group Theory and 3-dimensional Manifolds*, Yale Univ. Press, 1971.

M. Mihalik and S. Tschantz, Department of Mathematics, Vanderbilt University, Nashville TN. 37240, USA

E-mail: michael.l.mihalik@vanderbilt.edu, tschantz@math.vanderbilt.edu