A Comparison of Extended Kalman Filter, Particle Filter, and Least Squares Localization Methods for a High Heat Flux Point Source

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Abstract

State estimation procedures using the extended Kalman filter, particle filter, and least squares are investigated for a transient heat transfer problem in which a high heat flux point source is applied on one side of a thin plate and ultrasonic pulse time of flight is measured between spatially separated transducers on the opposite side of the plate. This work is an integral part of an effort to develop a system capable of locating the boundary layer transition region on a hypersonic vehicle aeroshell. Results from thermal conduction experiments involving one-way ultrasonic pulse time of flight measurements are presented. Comparisons of heating source localization measurement models are conducted where ultrasonic pulse time of flight readings provide the measurement update to the extended Kalman filter, particle filter, and least squares. Two different measurement models are compared: 1) directly using

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the one-way ultrasonic pulse time of flight as the measurement vector and 2) indirectly obtaining distance from the one-way ultrasonic pulse time of flight and then using these obtained distances as the measurement vector. For the direct model, the Jacobian required by the extended Kalman filter and least squares is obtained numerically using finite differences and a finite element forward conduction solution. For the indirect model, the derivatives with respect to the state variables are obtained in closed form. Heating source localization results and convergence behavior are compared for the three inverse methods and the two measurement models. The extended Kalman filter, least squares, and particle filter methods using the one-way ultrasonic pulse time of flight measurement model (direct model) produced similar results when considering accuracy of converged solution, ability to converge to the correct solution, and smoothness of convergence behavior. The results provide quantified justification for moving forward with development of an extended Kalman filter-based localization solution.

Keywords: , ultrasonic thermometry, ultrasound, localization

1. Nomenclature

\[ A \] area (m²);

- amplitude (m);
- state Jacobian

\[ a \] state model;

- material thickness

\[ B \] measurement Jacobian
\textit{b} expected measurement

\textit{C} heat capacity (J/K)

\textit{c}_p specific heat (J/kg K)

\textit{c} distance from ellipse center to ellipse edge along the major axis

\textit{d} distance from ellipse center to ellipse edge along the minor axis

\textit{E} energy (J); Young’s modulus (GPa)

\textit{F} cumulative density function

\textit{G} ultrasonic time of flight (s)

\textit{\overline{G}} expected ultrasonic time of flight (s)

\textit{gain} particle filter gain

\textit{h} convection heat transfer coefficient (W/m$^2$-K)

\textit{I}_n \, n \times n \text{ identity matrix}

\textit{K} Kalman gain

\textit{k} thermal conductivity (kg/m$^3$)

\textit{L} length (m)

\textit{l} length (m)

\textit{m} number of particles

\textit{Nu} Nusselt Number

\textit{P} probability

\textit{Pr} Prandtl Number

\textit{p} probability
$Q$ heat source; state covariance
$q''$ heat flux (W/m$^2$)
$R$ thermal resistance (K/W);
measurement covariance; radius (m)
$Ra$ Rayleigh Number
$r$ radius (m)
$\bar{r}$ expected radius (m)
$S$ sensitivity
$T$ temperature (K)
$t$ time (s)
$U$ control input
$v$ sound speed (m/s)
$w$ width (m); particle weight
$X$ state
$\bar{X}$ predicted state
$x, y, z$ rectangular coordinates (m)
$\bar{x}, \bar{y}$ predicted rectangular coordinates (m)
$Z$ actual measurements
Greek Letters

\(\alpha\)  
thermal diffusivity \((m^2/s)\)

\(\beta\)  
volumetric thermal expansion \((1/K)\)

\(\gamma\)  
temperature dependence of Young’s modulus \((1/K)\)

\(\Delta\)  
normalized temperature difference

\(\nabla\)  
Laplacian operator

\(\delta\)  
coefficient

\(\theta\)  
temperature change relative to reference \((K)\)

\(\overline{\theta}\)  
predicted temperature change relative to reference \((K)\)

\(\kappa\)  
thermal conductivity \((W/m-K)\)

\(\xi\)  
ultrasonic time of flight temperature factor \((1/K)\)

\(\rho\)  
density \((kg/m^3)\)

\(\Sigma\)  
covariance

\(\overline{\Sigma}\)  
predicted covariance

\(\sigma^2\)  
variance for a Gaussian probability density function

\(\tau\)  
period \((s)\)

\(\phi\)  
information vector

\(\overline{\phi}\)  
predicted information vector

\(\Omega\)  
information matrix

\(\overline{\Omega}\)  
predicted information matrix
2. Introduction

Knowledge of where air flowing across a body transitions from laminar flow to turbulent flow can provide numerous benefits to air vehicle design, thermal protection system design, and air vehicle in-flight control [1]. Of particular interest in this work is the transition region for hypersonic vehicles (Mach 5+). Directly observing and measuring the transition region in an operational vehicle is difficult because the harsh environment presents numerous challenges including high-speed airflow and high surface temperatures [2, 3, 4, 5]. The mechanisms leading to transition are still poorly understood [6, 7]. Further complicating hypersonic research, ground tests
of the boundary layer transition generally produce unsatisfactory results [8]. For example, sound fields radiated from the turbulent tunnel-wall boundary layers cause undesirable noise in ground facilities forcing researchers to pursue costly flight experiments.

We propose a novel measurement system that leverages the hypersonic body-surface heating profile to locate the boundary layer transition region. The heat flux and body-surface temperatures in the turbulent region after the transition are significantly higher than in the laminar region ahead of the transition [9, 10, 11, 12]. Several measurement strategies have been studied for similar applications including using thermocouples [13], thermopiles [14], thin-film temperature gauges [15], optical sensors coupled with thermographic phosphors [16, 17], infrared sensors [18], and ultrasonic transducers. In this work, we focus on ultrasonic sensing as the measurement strategy. Ultrasonic pyrometry has proven effective for gases, fluids, and solids as long as direct access to the material where the temperature being measured is available [19, 20, 21]. Furthermore, ultrasonic pyrometry has been used for decades in many process control systems [22, 23] and in non-destructive evaluation and defect detection with a great deal of success [24, 25, 26]. Therefore, many technological advances in ultrasonic thermometry exist that we can leverage for measurement of heating loads. The proposed measurement system leverages temperature differences rather than absolute temperatures to measure heating source location.

For a boundary layer localization system using ultrasonic sensors, the sensors would be located on the inside surface of the aeroshell away from the harsh external conditions. Consequently, the phenomenon that is being
measured is not disturbed and the sensor is not exposed to deleterious environments. In addition, the sample rate is limited only by the speed of sound through the medium, and the body-surface temperature is proportional to an easily measured quantity, time of flight [27]. Measuring temperature via ultrasonic time of flight at multiple locations on the aeroshell, however, does not by itself locate and characterize the transition region. A method is needed to correlate ultrasonic time of flight measurements at multiple locations to the location of the transition region. Here, we investigate using state estimation techniques where the system state consists of the transition location and heating magnitude. We hypothesize that incorporating state estimation into the solution will require fewer sensors than otherwise possible. The solution envisioned in this work involves a forward conduction solution and an inverse procedure based on the extended Kalman filter. Kalman filters construct a framework of predicting the state based on an input to the system and correcting the predicted state based on sensor observations [28, 29]. Kalman filters were invented by Swerling (1958) and Kalman (1960) as a technique for filtering and prediction in linear Gaussian systems [30]. Kalman filters have been used extensively in guidance and navigation systems [30], have been used in various state estimation scenarios [31], but have not seen much activity in heat transfer applications [32].

Development of the proposed measurement method based on ultrasound and the extended Kalman filter is accomplished using simple, controlled experiments involving concentrated high heat flux sources on a large flat metal plate. Both high heat flux point sources and high heat flux step sources are considered; however, the work presented here is restricted to the high heat
flux point source. Once the method is developed and is proven for the simpler flat plate environment, development can proceed into the hypersonic flight testing realm. Previous work details a thermocouple experiment with a high heat flux point source [33, 34], a forward conduction solution [33, 34], six different measurement models for the inverse procedure [35, 34], and examined the sensitivity to heating source location, noise, boundary conditions, and thermal conductivity, and assessed the ability to localize a heating source using a one-way ultrasonic pulse sensor array with the extended Kalman filter [36]. This work extends the previous research by comparing extended Kalman filter [30], particle filter [30, 37], and least squares [38] localization techniques. The goal is to provide tangible justification for moving forward with development of an extended Kalman filter-based localization solution.

3. Flat Plate Experiment Using Ultrasonic Transducers

This work concentrates first on a large flat plate heated over a small area with a known heat source. Consider a 6 cm x 30.5 cm x 0.635 cm stainless steel 316L plate (Figure 1) with constant properties (Table 1). The plate is large enough so the plate edges do not affect the temperature profile in the plate during the experiment. The heating source, a Research, Inc. SpotIR 4150 heater with focusing cone, is positioned approximately 2 mm from the plate surface such that its beam strikes a fixed position on the plate and is applied at $t = 300$ s and removed at $t = 600$ s. A parameter estimation study concluded the SpotIR heater has a heating profile of $q'' = 0.930 \text{ MW/m}^2$ over 0.635 cm diameter circular area with a secondary heating modeled as a Gaussian with a profile of $q_g'' = 100 \text{ W/m}^2$ and a variance of $\sigma_g^2 = 0.0009 \text{ m}^2$. 
Two ultrasonic sensors consisting of 2 MHz direct deposit transducers using Ferroperm Piezoceramics Pz46 are attached to the non-heated side of the plate. The direct deposit transducers are 1 cm diameter and 1 mm thick. With plate center on the heated side being the origin and the x-axis being the length (Figure 1), transducers are attached at \((x = -4 \text{ cm}, y = 0 \text{ cm})\) and \((x = 4 \text{ cm}, y = 0 \text{ cm})\) locations on the non-heated side \((z = 0.635 \text{ cm})\). One transducer transmits ultrasonic pulses while the other transducer receives the pulses and time of flight is recorded. Separate experiments are conducted.
Figure 2: Illustration of boundary conditions on the flat plate.

with the source positioned on the heated side of the plate at \((x, y)\) locations of \((0 \text{ cm}, 0 \text{ cm})\), \((0 \text{ cm}, 2 \text{ cm})\), \((0 \text{ cm}, 4 \text{ cm})\), \((0 \text{ cm}, 6 \text{ cm})\), \((0 \text{ cm}, 8 \text{ cm})\), and \((0 \text{ cm}, 10 \text{ cm})\). Black Zynolyte® Hi-Temp Paint is applied to a 1.5 cm wide strip at the plate center to maximize energy absorption from the heater. The plate is oriented vertically with the positive \(y\)-axis pointing up. Data acquisition equipment employing cross-correlation techniques is used to determine and record ultrasonic pulse time of flight readings once per second during the experiment.
Table 1: Material properties for the stainless steel 316L test sample used in the conduction experiments.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>density ($\rho$)</td>
<td>8,000 kg/m$^3$</td>
</tr>
<tr>
<td>thermal conductivity ($k$)</td>
<td>14.6 W/m K</td>
</tr>
<tr>
<td>specific heat ($c_p$)</td>
<td>500 J/kg K</td>
</tr>
<tr>
<td>sound speed ($v_0$)</td>
<td>5,100 m/s @ 293 K</td>
</tr>
<tr>
<td>ultrasonic TOF temperature factor ($\xi$)</td>
<td>$110 \times 10^{-6}$ 1/K</td>
</tr>
<tr>
<td>sample length</td>
<td>61 cm</td>
</tr>
<tr>
<td>sample width</td>
<td>30.5 cm</td>
</tr>
<tr>
<td>sample height</td>
<td>0.635 cm</td>
</tr>
</tbody>
</table>

4. Forward Conduction Solution

The forward conduction solution used in this localization study was developed in previous work [33, 35, 34, 36] and leverages COMSOL Multiphysics® by the COMSOL Group and MATLAB® by The Mathworks, Inc. The solution uses a finite element mesh with smaller elements near the heat source and larger elements near the plate edges to conserve computing resources. A grid convergence study was performed to ensure grid independence [39]. Both the number of elements in the plate’s $x - y$ plane and the number of layers in the plate’s thickness were considered. The grid convergence study led to the selection of three mesh layers through the plate’s thickness dimension, 9,780 total elements, and 45,983 degrees of freedom. Independent verification of the COMSOL® solution was performed using a closed-form,
analytical solution of heating through a circular domain without convection [40]. Agreement between the COMSOL solution and the closed-form solution is acceptable with mean absolute error less than 0.5 K. The maximum temperature rise is approximately 80 K.

The measured time of flight is related to the average temperature between the transducers by [41, 42, 33, 35, 34]

\[ G_{ij} = \frac{R_{ij}}{v_0} \left(1 + \xi \theta_{avg|_i^j}\right) \]  

(1)

where \( R_{ij} \) is the distance between transducers (m), \( v_0 \) is the sound speed in the material at a reference temperature, \( \xi \) is the ultrasonic time of flight factor, which is material dependent (Table 1), and \( \theta_{avg} \) is the change in temperature from the reference temperature between the two sensors. Since \( R_{ij} \) is known with insufficient accuracy to compare the time of flight from the model to the measured values, the time of flight can be normalized to the initial state.

\[ \frac{G_{ij}}{G_0} = \frac{R_{ij}}{v_0} \left[1 + \xi \left(T_{avg|_i^j} - T_0\right)\right] = 1 + \xi \left(T_{avg|_i^j} - T_0\right) \]  

(2)

where \( G_0 \) is the average time of flight recorded at 1 s intervals from \( t = 1 \) to 299 s before the heater is turned on. Figures 3 and 4 illustrate the agreement between the COMSOL® model and the ultrasonic time of flight measured during the experiment. The residuals [43, 44] provide valuable insight into the accuracy of the model and indicate that the solution is somewhat biased. Agreement between the model and the experiment is acceptable; however, the magnitude with the heat source located at \((x = 0 \text{ cm}, y = 0 \text{ cm})\) and when the heat source is located at \((x = 0 \text{ cm}, y = 2 \text{ cm})\) are both underestimated.
Figure 3: Comparison of the COMSOL® model with the one-way ultrasonic pulse experiment with heat source located between the sensors (top curve) and offset by 2 cm, 4 cm, 6 cm, 8 cm, and 10 cm. The model uses temperatures along the non-heated surface of the plate.

by the model. Figure 5 illustrates the time of flight measurements during the beginning part of the experiment and highlights the time needed for the heat to reach the sensors.

5. Development of Measurement Models

This section examines heating source localization using four ultrasonic transducers in an 8 cm square pattern (Figure 6). Multiplexing equipment was not available when this study was conducted; therefore, data from separate experiments detailed above are used together in this section to simulate the four sensor array.

Locating and characterizing the boundary layer transition depends upon
Figure 4: Residuals between the COMSOL® model and the one-way ultrasonic pulse experiment. The model uses temperatures along the non-heated surface of the plate.

Figure 5: One-way ultrasonic pulse time of flight measurements for the beginning part of the heating phase.
many factors such as heating source movements in time, heating source magnitude changes in time, and other transient behaviors. Fairly restrictive assumptions can be imposed that simplify the problem. Analysis and algorithm development can proceed using these restrictive assumptions and then assumptions can be relaxed in stages to achieve the end result of source localization and characterization. The assumptions for this work are:

1. Source in fixed position (location unknown)
2. Source applied at time $t = 300$ s and removed at $t = 600$ s

3. Main heat flux $q'' = 0.930 \text{ MW/m}^2$ over 0.00635 m diameter circular area while source applied (value obtained in previous study [33, 35, 34])

4. Secondary heating is characterized by a Gaussian with magnitude $q''_g = 100 \text{ W/m}^2$ and spread $\sigma^2_g = 0.0009 \text{ m}^2$ while source applied

5. Convection coefficient $h = 3.20 \text{ W/m}^2\text{K}$ on both sides of the plate (value obtained in previous study [33, 35, 34])

6. Convection coefficient $h = 3 \text{ W/m}^2\text{K}$ on the plate edges

7. Thermal conductivity $k = 14.6 \text{ W/mK}$

8. Specific heat $C_p = 500 \text{ J/kgK}$ and density $\rho = 8,000 \text{ kg/m}^3$

9. Positions of sensors are ($\pm4$ cm, $\pm4$ cm) on the non-heated side

The three inverse methods compared in this work are: extended Kalman filter, particle filter, and least squares. The two measurement models studied are: ultrasonic pulse one-way time of flight measurement model, and ellipse from ultrasonic pulse one-way time of flight measurement model. With the particle filter, only the ultrasonic pulse one-way time of flight measurement model is considered because, as will become clear later in this work, the particle filter does not depend upon a Jacobian, and the ellipse model was developed specifically as an alternative way of expressing the Jacobian for those methods that require a Jacobian. Comparison of the five methods is performed in locating the heating source on the plate in the $x - y$ plane ($x_q, y_q$). For all methods, the state therefore is $X_t = [x_q, y_q]^T$. In all methods considered in this work, the ultrasonic time of flight is normalized by the time of flight before the heating source is applied to the plate ($G_{ij}/G_0$).
Table 2: Extended Kalman filter algorithm.

<table>
<thead>
<tr>
<th>Step</th>
<th>Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\overline{X}<em>t = a(U_t, X</em>{t-1})$</td>
</tr>
<tr>
<td>2</td>
<td>$\Sigma_t = A_t \Sigma_{t-1} A_t^T + Q_t$</td>
</tr>
<tr>
<td>3</td>
<td>$K_t = \Sigma_t B_t^T (B_t \Sigma_t B_t^T + R_t)^{-1}$</td>
</tr>
<tr>
<td>4</td>
<td>$X_t = \overline{X}_t + K_t (Z_t - b(\overline{X}_t))$</td>
</tr>
<tr>
<td>5</td>
<td>$\Sigma_t = (I - K_t B_t) \Sigma_t$</td>
</tr>
<tr>
<td>6</td>
<td>Return to Step 1 for next time step</td>
</tr>
</tbody>
</table>

5.1. Extended Kalman Filter with Ultrasonic Pulse One-Way Time of Flight Measurement Model

The extended Kalman filter algorithm to locate the source can be found in Table 2. The state is $X_t = [x_q, y_q]^T$, and there is no input ($U_t$) to the state; thus the extended Kalman filter state model is $a = I_2$ and the state Jacobian is $A = I_2$, where $I_2$ is a $2 \times 2$ identity matrix. Sensitivity of the state variance was compared for values from $\sigma^2 = 0.01 \text{ m}^2$ to $\sigma^2 = 0.000001 \text{ m}^2$ with the smaller values providing a damping effect on the convergence. It was determined that a state variance of $\sigma^2 = 0.0001 \text{ m}^2$ provides a good compromise between damping and stability and this value is used in this work. Thus, the state covariance matrix is $Q_t = 0.0001 \text{ m}^2 \times I_2$.

This measurement model consists of obtaining expected temperatures from COMSOL®, computing the average temperature between the transducers, and then computing an expected time of flight to form $a(U_t, X_{t-1})$ (equation 3). For the current analysis, the average temperature is computed
along the path on the non-heated plate surface between the two sensors. The Jacobian partial derivatives are obtained using finite difference when moving the source in the $x$ and $y$ directions independently (equation 4).

$$
\begin{align*}
    b(\mathbf{x}_t) &= \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}; \\
    B_t &= \begin{bmatrix}
        -\frac{\partial G_1}{\partial x_1} & -\frac{\partial G_1}{\partial y_1} \\
        -\frac{\partial G_2}{\partial x_2} & -\frac{\partial G_2}{\partial y_2} \\
        -\frac{\partial G_3}{\partial x_3} & -\frac{\partial G_3}{\partial y_3} \\
        -\frac{\partial G_4}{\partial x_4} & -\frac{\partial G_4}{\partial y_4}
    \end{bmatrix},
\end{align*}
$$

where $t$ is time in seconds with a time step of 1 s, $\overline{G}_i$ with $i = 1, 2, 3, 4$ is the ultrasonic pulse time of flight with the heating source located at $(x_s, y_s)$, and $(x_i, y_i)$ with $i = 1, 2, 3, 4$ are the locations of four transducers. The Jacobian $B_t$ is constructed using the derivatives with respect to sensor position for convenience because this information can be obtained with one COMSOL® simulation. The derivatives are obtained from COMSOL® using finite differences by independently varying the $x$ and $y$ positions of all sensors by $0.0001$ m. Based on the flat plate experiment above, the sensor noise is assumed be $\pm 6 \times 10^{-5}$ (a non-dimensional number based on $G_{ij}/G_0$) and is normally distributed ($\sigma^2 = ((6 \times 10^{-5})/3)^2 = 4 \times 10^{-10}$). Solution instabilities were present when using this variance, which were reduced by increasing the variance to $4 \times 10^{-7}$. This larger variance effectively dampens the solution and prevents large changes from one iteration to the next. The measurement
covariance matrix, therefore, is $R = 4 \times 10^{-7} \times I_4$.

5.2. Extended Kalman Filter with Ellipse from Ultrasonic Pulse One-Way Time of Flight Measurement Model

In an attempt to simplify the sensitivity calculation the lines of constant time of flight around the sensor pairs form approximate ellipses. With this approximation, the sensitivities can be calculated algebraically. Figure 7 illustrates the geometry of an ellipse, where the two sensors are assumed to be the foci for the ellipse. Since the distance between sensors is known, ellipse parameters $c$ and $d$ can be related to each other and the ellipse can be represented with just one parameter $c$.

$$r_{is} + r_{js} = 2c = \sqrt{r_{ij}^2 + 4d^2}$$  \hspace{1cm} (5)
where \( i \) and \( j \) are sensors and \( s \) is heat source.

\[
c = \frac{1}{2} \sqrt{r_{ij}^2 + 4d^2} = \frac{r_{is} + r_{js}}{2} \quad (6)
\]

\[
r_{is} = \sqrt{(x_i - x_s)^2 + (y_i - y_s)^2} \quad (7)
\]

\[
r_{js} = \sqrt{(x_j - x_s)^2 + (y_j - y_s)^2} \quad (8)
\]

\[
\frac{\partial c_i}{\partial x_s} = \frac{1}{2} \left[ \frac{x_i - x_s}{r_{is}} + \frac{x_i - x_j}{r_{js}} \right] \quad (9)
\]

\[
\frac{\partial c_i}{\partial y_s} = \frac{1}{2} \left[ \frac{y_i - y_s}{r_{is}} + \frac{y_i - y_j}{r_{js}} \right] \quad (10)
\]

The parameter \( c \) is measured indirectly by first measuring the one-way ultrasonic pulse time of flight. The forward conduction solution is used to get time of flight for a range of \( c \) values and interpolated using the spline method to obtain \( c \) for the measured time of flight. The measurement transition function \( b(\mathbf{X}_t) \) and the Jacobian \( B_t \) are then

\[
b(\mathbf{X}_t) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} ;
\]

\[
B_t = \begin{bmatrix} \frac{\partial c_1}{\partial x_s} & \frac{\partial c_1}{\partial y_s} \\ \frac{\partial c_2}{\partial x_s} & \frac{\partial c_2}{\partial y_s} \\ \frac{\partial c_3}{\partial x_s} & \frac{\partial c_3}{\partial y_s} \\ \frac{\partial c_4}{\partial x_s} & \frac{\partial c_4}{\partial y_s} \end{bmatrix}, \quad (12)
\]

where \( t \) is time in seconds with a time step of 1 second, \( c_i \) with \( i = 1, 2, 3, 4 \) is the ellipse parameter if the source is located at \((x_s, y_s)\). Based on the flat
plate experiment above, sensor noise is assumed be $\pm 1.05 \times 10^{-8}$ s and is
normally distributed ($\sigma^2 = 1.22 \times 10^{-17}$ sec$^2$). The sensor noise in terms of
temperature can be expressed as

$$\theta_{\text{noise}} = \frac{G_{\text{noise}} v_0}{L \xi} = 6.09 \text{ K}$$

(13)

Using the average slope of $0.015$ m/K determined in previous work [33, 35, 34], ellipse noise for the $c$ parameter from ultrasonic pulse time of flight measurement model is $\pm 0.0914$ m and is normally distributed ($\sigma^2 = 9.28 \times 10^{-4}$ m$^2$). Since the measurement covariance matrix $R$ represents the measurement noise of the $c$ parameter, the measurement covariance is $3.19 \times 10^{-10}$ m$^2$ ($[1.05 \times 10^{-8}$ s$/3 \times 5, 100$ m/sec sound speed$]^2$).

5.3. Particle Filter with Ultrasonic Pulse One-Way Time of Flight Measurement Model

The particle filter is an alternative nonparametric implementation of the
Bayes filter and is a Monte Carlo technique used for the solution of state
estimation problems. The main idea is to represent the required posterior
density function by a set of random samples with associated weights and to
compute the estimates based on these samples and weights [37]. Because it
is nonparametric, the particle filter can represent a much broader space of
distributions than Gaussians and has the ability to model nonlinear trans-
formations of random variables [30]. The particle filter algorithm to locate
the source can be found in Table 3.

Implementation of the particle filter for heating source localization starts
with defining the area of possible source locations on the plate. The number
of particles $m$ to use in the algorithm must also be defined. A large number
Table 3: Particle filter algorithm.

<table>
<thead>
<tr>
<th>Step</th>
<th>Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Generate $m$ random possible source locations across the plate (particles)</td>
</tr>
<tr>
<td>2</td>
<td>Obtain expected measurements for each particle $i$ for the current time $t$ ($Z_{i,t}$)</td>
</tr>
<tr>
<td>3</td>
<td>Obtain actual measurements for the current time ($Z_{true}$)</td>
</tr>
<tr>
<td>4</td>
<td>Weight each particle $i$ according to $N(Z_{true}, I)$</td>
</tr>
<tr>
<td>5</td>
<td>Normalize weights for each particle $i$ into bins from 0 to 1</td>
</tr>
<tr>
<td>6</td>
<td>Resample best particles using normal distribution</td>
</tr>
<tr>
<td>7</td>
<td>Add position noise to each particle</td>
</tr>
<tr>
<td>8</td>
<td>Return to Step 2 for next time step</td>
</tr>
</tbody>
</table>

of particles yields a higher probability that one or more particles will be located near the actual source but the downside is higher computational cost. For this study, the area is defined as the $8\text{ cm} \times 8\text{ cm}$ sensor grid and the number of particles is $m = 40$. The algorithm starts with Step 1 and the generation of $m$ random particles within the defined area of possible source locations (Figure 8). Step 2 involves obtaining expected temperatures from COMSOL® with the heating source at each particle location, computing the average temperature between the transducers, and then computing an expected time of flight to form $Z_{i,t}$, a $4 \times 1$ matrix for each particle $i$ at the current time $t$. For the current analysis, the average temperature is computed along the path on the non-heated plate surface between the two sensors. Step 3 entails obtaining actual ultrasonic time of flight measurements at the
current time \( t \) for all four sensor pairs to form \( Z_{true_t} \), a \( 4 \times 1 \) matrix. The particle filter relies on an importance factor, or weight \( w_{i,t} \), to incorporate the measurement \( Z_{true_t} \) into the particle set. The weight \( w_{i,t} \) for each particle is computed in Step 4 using

\[
    w_{i,t} = N(Z_{true_t}, I) = \det(2\pi I)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}((Z_{i,t} - Z_{true_t}) \times \text{gain})^T I^{-1}((Z_{i,t} - Z_{true_t}) \times \text{gain})\right).
\]

Gain is discussed in detail later in this section. A number of resampling techniques have been devised [30, 32, 45]. This work employs the sampling importance resampling technique. In Step 5, the particle weights are normalized into bins from 0 to 1, which gives the particles with the highest weight the largest bins and then in Step 6, the particles are resampled using a normal distribution. By resampling across the bins from 0 to 1 using a normal distribution, a higher probability exists that the best particles will be chosen but some particles that are not the best will be chosen too. It is important to note that the number of particles \( m \) remains constant through the resampling process, thus some particles will have identical locations on the plate after resampling. Degeneracy is common with particle filters, a situation where the solution converges to the one best particle within the current particle set without considering locations nearby [46]. This fact necessitates Step 7 where position noise or roughness is added to each particle [47]. In this work, uniform position noise of \( \pm 0.5 \text{ cm} \) is used. Figures 9 and 10 illustrate the particle distribution before and after adding noise to each particle location. After completion of Step 7, the algorithm returns to Step 2 and the process is repeated for the next step in time. For this work, a time step of 1 s is used.
Figure 8: Particle locations \((m = 40)\) at 300 s.

Figure 9: Particle locations \((m = 40)\) at 315 s.
Figure 10: Particle locations ($m = 40$) at 315 s after adding noise to each particle location.

Figure 11: Particle locations ($m = 40$) at 330 s.
Because the magnitude of the non-dimensional values in the matrix \( Z_{i,t} - Z_{true_t} \) in equation 14 ranges from 0 to 0.008, using no gain \((gain = 1)\) produces a value of 1 in the exponent portion of the equation for all particles. Thus, identical weights are computed for all particles (Figure 12). For the particle filter to function properly, it is imperative that the particles close to the actual heating source location receive the highest weight \((Z_{i,t} - Z_{true_t} \) close to zero in Figure 12) and those particles far from the actual heating source location receive a weight close to zero. The non-dimensional sensor noise measured in the experiment above is approximately \( 6 \times 10^{-5} \). Therefore, the applied gain should yield the largest weights for \( Z_{i,t} - Z_{true_t} \) magnitudes between 0 and the noise of \( 6 \times 10^{-5} \) and should yield weights close to 0 for \( Z_{i,t} - Z_{true_t} \) magnitudes larger than the sensor noise. Illustrated in Figure 12, a gain of \( 1 \times 10^4 \) is too small to produce weights close to zero for \( Z_{i,t} - Z_{true_t} \) magnitudes larger than the sensor noise, but a gain of \( 6 \times 10^5 \) precipitates the desired effect. Figure 13 illustrates the effect of the number of particles \( m \) on the convergence behavior and performance of the particle filter. The filter is quite robust and Figure 13 demonstrates the filter’s ability to converge to the correct location with only 10 particles. The particle filter used in the comparisons with the other localization methods in the next section is based on 40 particles.

5.4. Least Squares with Ultrasonic Pulse One-Way Time of Flight Measurement Model

The ordinary least squares method is sometimes called the Gauss method of minimization [38]. For the current localization, the estimated time of flight \( G \) for each sensor pair for a particular time \( t \), a \( 4 \times 1 \) matrix, depends on a
Figure 12: Comparison of selected particle filter gain values with sensor noise.

Figure 13: Particle filter convergence for selected numbers of particles with heating source located at \((x = 2 \text{ cm}, y = 0 \text{ cm})\) and an initial guess of \((x = 0 \text{ cm}, y = 0 \text{ cm})\).
vector of two unknowns in the state $X$ and the value of $G$ at $X = X + \Delta X$ is obtained through the truncated Taylor’s series as

$$G|_{X + \Delta X} \approx G|_X + \frac{\partial G}{\partial X}|_X \Delta X. \quad (15)$$

The gradient coefficient in equation 15 is the $4 \times 2$ sensitivity matrix

$$B_t = \frac{\partial G}{\partial X} = \begin{bmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial y_1} \\ \frac{\partial G_2}{\partial x_2} & \frac{\partial G_2}{\partial y_2} \\ \frac{\partial G_3}{\partial x_3} & \frac{\partial G_3}{\partial y_3} \\ \frac{\partial G_4}{\partial x_4} & \frac{\partial G_4}{\partial y_4} \end{bmatrix} \quad (16)$$

where $t$ is time in seconds with a time step of 1 s, $\overline{G}_i$ with $i = 1, 2, 3, 4$ is the ultrasonic pulse time of flight with the heating source located at $(x_s, y_s)$, and $(x_i, y_i)$ with $i = 1, 2, 3, 4$ are the locations of four transducers. The experimentally obtained time of flight measurements are designated as $Z_t$, a $4 \times 1$ matrix. The desire is to improve the estimate for the state $X_t$ based on the observations $Z_t$. The ordinary least squares objective function is

$$S = (Z_t - G|_{X_t} - B_t \Delta X_t)^T (Z_t - G|_{X_t} - B_t \Delta X). \quad (17)$$

The minimizer of equation 17 is found by forcing to zero the derivative with respect to $\Delta X$ resulting in the estimator

$$\Delta X_t = (B_t^T B_t)^{-1} B_t^T (Z_t - G|_{X_t}). \quad (18)$$

This method consists of obtaining expected temperatures from COMSOL®, computing the average temperature between the transducers, and then com-
puting an expected time of flight from equation 1. For the current analysis, the average temperature is computed along the path on the non-heated plate surface between the two sensors. The Jacobian $B_t$ is constructed using the derivatives with respect to sensor position for convenience since this information can be obtained with one COMSOL® simulation. The derivatives are obtained from COMSOL® using finite differences by independently varying the $x$ and $y$ positions of all sensors by 0.0001 m.

5.5. Least Squares with Ellipse from Ultrasonic Pulse One-Way Time of Flight Measurement Model

This method uses the same ellipse model detailed above with the extended Kalman filter and employs least squares method detailed above to locate the source. Figure 7 illustrates the geometry of an ellipse, where the two sensors are assumed to be the foci for the ellipse. Since the distance between sensors is known, ellipse parameters $c$ and $d$ can be related to each other and the ellipse can be represented with just one parameter $c$.

\[
r_{is} + r_{js} = 2c = \sqrt{r_{ij}^2 + 4d^2}
\]  
(19)

where $i$ and $j$ are sensors and $s$ is heat source.

\[
c = \frac{1}{2} \sqrt{r_{ij}^2 + 4d^2} = \frac{r_{is} + r_{js}}{2}
\]  
(20)

\[
r_{is} = \sqrt{(x_i - x_s)^2 + (y_i - y_s)^2}
\]  
(21)

\[
r_{js} = \sqrt{(x_j - x_s)^2 + (y_j - y_s)^2}
\]  
(22)

\[
\frac{\partial c_i}{\partial x_s} = \frac{1}{2} \left[ \frac{x_s - x_i}{r_{is}} + \frac{x_s - x_j}{r_{js}} \right]
\]  
(23)

\[
\frac{\partial c_i}{\partial y_s} = \frac{1}{2} \left[ \frac{y_s - y_i}{r_{is}} + \frac{y_s - y_j}{r_{js}} \right]
\]  
(24)
The parameter $c$ is measured indirectly by first measuring the one-way ultrasonic pulse time of flight. The forward conduction solution is used to get time of flight for a range of $c$ values and interpolated using the spline method to obtain $c$ for the measured time of flight. The sensitivity matrix $B_t$ is then

$$B_t = \begin{bmatrix}
      \frac{\partial c_3}{\partial x_s} & \frac{\partial c_3}{\partial y_s} \\
      \frac{\partial c_2}{\partial x_s} & \frac{\partial c_2}{\partial y_s} \\
      \frac{\partial c_1}{\partial x_s} & \frac{\partial c_1}{\partial y_s} \\
      \frac{\partial c_4}{\partial x_s} & \frac{\partial c_4}{\partial y_s}
\end{bmatrix},$$

where $t$ is time in seconds with a time step of 1 second, $c_i$ with $i = 1, 2, 3, 4$ is the ellipse parameter if the source is located at $(x_s, y_s)$. The desire is to improve the estimate for the state $X_t$ based on the observations $c$.

6. Results

Convergence behavior for the three inverse methods and both measurement models is compared in Figures 14 through 16. With the heating source located inside the sensor grid (Figure 14), the extended Kalman filter and the least squares with the direct model and the particle filter converge to the correct location while the extended Kalman filter and the least squares with the ellipse model do not. With the heating source located at the edge of the sensor grid (Figure 15), the extended Kalman filter and the least squares with the direct model and the particle filter converge to the correct location while the extended Kalman filter and the least squares with the ellipse model do not. With the heating source located outside of the sensor grid (Figure 16), none of the methods converge to the correct location. These examples
Figure 14: Least squares, extended Kalman filter, and particle filter convergence for both one-way ultrasonic pulse measurement models with the heating source located at \((x = 2 \text{ cm}, y = 0 \text{ cm})\) and an initial guess of \((x = 0 \text{ cm}, y = 0 \text{ cm})\).

started with an initial guess of \((x = 0 \text{ cm}, y = 0 \text{ cm})\) for the heating source location.

Sensitivity to heating source location, explored in previous work [36], can help explain these results. With the heating source outside the sensor grid, only one sensor pair would have sufficient sensitivity to heating source location and only then if the heating source is located close to the sensor pair. With only one sensor pair receiving usable information, the inverse routine has insufficient information to converge on the correct heating source location. With the heating source located inside the sensor grid, all sensor pairs receive usable information and the inverse routine is able to converge to the correct location.

Repeating the experiment using multiplexing equipment and a complete
Figure 15: Least squares, extended Kalman filter, and particle filter convergence for both one-way ultrasonic pulse measurement models with the heating source located at \((x = 4\, \text{cm}, \, y = 0\, \text{cm})\) and an initial guess of \((x = 0\, \text{cm}, \, y = 0\, \text{cm})\).

sensor grid of four transducers instead of simulating the sensor grid with separate experiments using two transducers might produce different convergence behavior, especially for heating source locations outside the sensor grid. While the experiment is reproducible, simulating a sensor grid with separate experiments introduces uncertainties that could effect the results. Additionally, sensor and heating source placement introduce uncertainties when simulating the sensor grid.

7. Conclusions

The extended Kalman filter, least squares, and particle filter methods using the one-way ultrasonic pulse time of flight measurement model (direct model) produced similar results when considering accuracy of converged
solution, ability to converge to the correct solution, and smoothness of convergence behavior. Kalman filter successes in robotics and other fields lies in its simplicity, computational efficiency, and the existence of a control input. The results presented here provide quantified justification for moving forward with development of an extended Kalman filter-based heating source localization solution. Whereas this work had no inputs to the state model, the ability to add inputs to the extended Kalman filter is anticipated to be more robust for heat source localization and in turn for boundary layer transition localization and characterization.
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