Distributions on integer partitions

Michael Griffin (with K. Ono, L. Rolen, and W.-L. Tsai)

The Partition function $p(n)$

Definition

A **partition** of an integer n is any nonincreasing sequence

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of positive integers which sum to n.

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$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1 \implies p(4) = 5.$$

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General Problem

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(1) Are there any nice natural examples?

(2) ....examples with **normalized limits** *independent of n?*

Dyson's Rank

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The **rank** of a partition is its largest part minus its number of parts.

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Example

The ranks of the partitions of 4:

Partition	Largest Part	# Parts	Rank
4	4	1	$3 \equiv 3 \pmod 5$
$3+1$	3	2	$1 \equiv 1 \pmod 5$
$2+2$	2	2	$0 \equiv 0 \pmod 5$
$2+1+1$	2	3	$-1 \equiv 4 \pmod 5$
$1+1+1+1$	1	4	$-3 \equiv 2 \pmod 5$

Dyson's ranks are equidistributed

Theorem (Atkin and Swinnerton-Dyer, 1954)

If $0 \leq a < b$ *are integers and*

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$$N(a, 5; 5n+4) = p(5n+4)/5,$$
$$N(a, 7; 7n+5) = p(7n+5)/7.$$

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Theorem (Bringmann, 2008)

For all $0 \le a < b$ *we have*

$$\lim_{n \to +\infty} \frac{N(a, b; n)}{p(n)} = \frac{1}{b}.$$

Number of Parts

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The **"number of parts" polynomials** $P_{\#}(n;T)$ *are defined by*

$$\sum_{n=0}^{\infty} P_{\#}(n;T)q^n := \prod_{n=1}^{\infty} \frac{1}{(1-Tq^n)}.$$

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Example (Asymmetry)

$$\begin{aligned}
P_{\#}(4;T) &= T + 2T^2 + T^3 + T^4 \\
P_{\#}(5;T) &= T + 2T^2 + 2T^3 + T^4 + T^5 \\
P_{\#}(6;T) &= T + 3T^2 + 3T^3 + 2T^4 + T^5 + T^6 \\
&\vdots \\
P_{\#}(15;T) &= T + 7T^2 + 19T^3 + 27T^4 + 30T^5 + \cdots + 2T^{13} + T^{14} + T^{15}.
\end{aligned}$$

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If k is a positive integer, then let

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If $C := \pi\sqrt{2/3}$ *and* $k_n(x) := C^{-1}\sqrt{n}\log n + \sqrt{n}x,$

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Theorem (Erdös and Lehner (1941))

If $C := \pi\sqrt{2/3}$ and $k_n(x) := C^{-1}\sqrt{n}\log n + \sqrt{n}x$, then as a function in x we have

$$\lim_{n\to+\infty} \frac{p_{\leq k_n(x)}(n)}{p(n)} = \exp\left(-\frac{2}{C} \cdot e^{-\frac{1}{2}Cx}\right).$$

Remarks

(1) **Normal order** *for the number of parts is*

$$\frac{\sqrt{n}\log n}{C} = \frac{\sqrt{3n}\log n}{\sqrt{2\pi}}.$$

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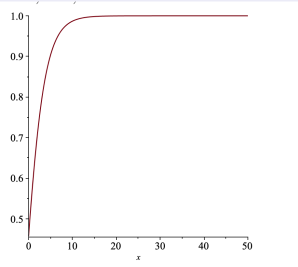
*(2) The graph of the "***Gumbel** *cumulative distribution function"*



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Partitions of $n = 750$

x	$\lfloor k_{750}(x) \rfloor$	$\delta_{k_{750}}(x)$	$\mathrm{Gumbel}(x)$
0.5	84	$0.656\ldots$	$0.663\ldots$
1.0	98	$0.814\ldots$	$0.805\ldots$
1.5	111	$0.899\ldots$	$0.892\ldots$
2.0	125	$0.949\ldots$	$0.941\ldots$
2.5	139	$0.975\ldots$	$0.969\ldots$
3.0	152	$0.987\ldots$	$0.983\ldots$

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Question (Precise Form)

If $A \geq 2$, *then let*

$$p_{\leq k}(A; n) \ := \ \#\{\lambda \vdash n \text{ with } \leq k \text{ parts in } A\mathbb{N}\}.$$

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If $A \geq 2$, then let

$$p_{\leq k}(A;n) \;:=\; \#\{\lambda \vdash n \text{ with } \leq k \text{ parts in } A\mathbb{N}\}.$$

What is the **cumulative distribution function** *for*

$$\frac{p_{\leq k}(A;n)}{p(n)} \;?$$

Solution to Problem 1

Theorem (G, Ono, Rolen, Tsai (2021))

If $C := \pi\sqrt{2/3}$ *and* $k_n = k_n(x) := \frac{1}{AC}\sqrt{n}\log n + \sqrt{n}x$,

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Remarks

(1) These are Gumbel distributions.

(2) The mean and variance of the limiting distribution are:

$$\text{Mean} := \frac{2}{AC}\left(\log\left(\frac{2}{AC}\right) + \gamma_{Euler}\right),$$
$$\text{Variance} := 1/A^2.$$

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Distribution of even parts for $n = 600$

x	$\lfloor k_{600}(x) \rfloor$	$\delta_{k_{600}}(x)$	$\mathrm{Gumbel}(x)$
-0.1	28	$0.597\ldots$	$0.604\ldots$
0.0	30	$0.663\ldots$	$0.677\ldots$
0.1	32	$0.721\ldots$	$0.739\ldots$
0.2	35	$0.791\ldots$	$0.792\ldots$
0.3	37	$0.830\ldots$	$0.835\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1.5	67	$0.994\ldots$	$0.992\ldots$
2.0	79	$0.998\ldots$	$0.998\ldots$

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If $A \geq 2$ *and* k **is fixed**, *then as* $n \to +\infty$ *we have*

$$p_{\leq k}(A; n) \sim \frac{24^{\frac{k}{2}-\frac{1}{4}} n^{\frac{k}{2}-\frac{3}{4}}}{\sqrt{2}\left(1-\frac{1}{A}\right)^{\frac{k}{2}-\frac{1}{4}} k! A^{k+\frac{1}{2}} (2\pi)^k} e^{2\pi\sqrt{\frac{1}{6}\left(1-\frac{1}{A}\right)n}},$$

$$p_k(A; n) \sim \frac{24^{\frac{k}{2}-\frac{1}{4}} (n-Ak)^{\frac{k}{2}-\frac{3}{4}}}{\sqrt{2}\left(1-\frac{1}{A}\right)^{\frac{k}{2}-\frac{1}{4}} k! A^{k+\frac{1}{2}} (2\pi)^k} e^{2\pi\sqrt{\frac{1}{6}\left(1-\frac{1}{A}\right)(n-Ak)}}.$$

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Remarks

(1) This theorem is proved by Wright's "circle method."

(2) Error terms are too large to imply the Gumbel distributions.

Example $A = 3$ and $k = 1$

The previous theorem gives

$$p_1(3;n) \sim \frac{1}{6\pi(n-3)^{\frac{1}{4}}} \cdot e^{\frac{2\pi\sqrt{n-3}}{3}}.$$

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n	$p_1(3;n)$	$p_1^*(3;n)$	$p_1(3;n)/p_1^*(3;n)$
200	93125823847	≈ 82738081118	≈ 1.126
400	$\approx 1.718 \times 10^{16}$	$\approx 1.579 \times 10^{16}$	≈ 1.088
600	$\approx 1.928 \times 10^{20}$	$\approx 1.799 \times 10^{20}$	≈ 1.071
800	$\approx 5.058 \times 10^{23}$	$\approx 4.764 \times 10^{23}$	≈ 1.062
1000	$\approx 5.232 \times 10^{26}$	$\approx 4.959 \times 10^{26}$	≈ 1.055

Problem 2: t-hooks

Example (Hook lengths)

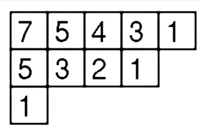

Figure: Hook lengths for $\lambda = (5, 4, 1)$

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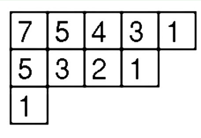

Figure: Hook lengths for $\lambda = (5, 4, 1)$

Problem

Does the sequence $\{Y_t(n)\}$ of distributions of the number of t-hooks in the partitions of integers n have a limiting behavior?

Example $t = 2$ and $n = 5000$

$$\sum_{\lambda \vdash 5000} T^{\#\{2 \in \mathcal{H}(\lambda)\}} = 704\,T + 9211712\,T^2 + \cdots + 1805943379138\,T^{98} + 2\,T^{99}.$$

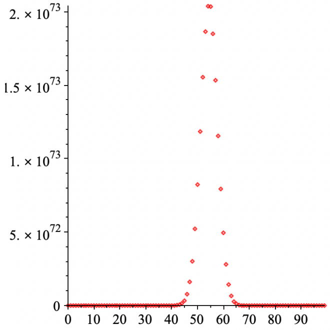

Figure: $Y_2(5000)$

Solution to Problem 2

Theorem (G, Ono, Tsai (2022))

(1) The sequence $\{Y_t(n)\}$ *is asymptotically* **normal** *with mean* $\mu_t(n) \sim \frac{\sqrt{6n}}{\pi} - \frac{t}{2}$ *and variance* $\sigma_t^2(n) \sim \frac{(\pi^2-6)\sqrt{6n}}{2\pi^3}$.

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(2) If $k_{t,n}(x) := \mu_t(n) + \sigma_t(n)x$, *then we have*

$$\lim_{n\to+\infty} D_t(k_{t,n}(x); n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy =: E(x).$$

Example $t = 2$ and $n = 5000$ continued

Illustration of the cumulative distribution approximation

$$D_2(k_{2,5000}(x); 5000) \approx E(x).$$

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x	$D_2(k_{2,5000}(x), 5000)$	$E(x)$	$D_2(k_{2,5000}(x), 5000)/E(x)$
-1.5	$0.0658\ldots$	$0.0668\ldots$	$0.9849\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.0	$0.5055\ldots$	$0.5000\ldots$	$1.0011\ldots$
1.0	$0.8246\ldots$	$0.8413\ldots$	$0.9802\ldots$
2.0	$0.9685\ldots$	$0.9772\ldots$	$0.9911\ldots$

Problem 3: Hook lengths in $t\mathbb{N}$

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$$\sum_{\lambda \vdash 1000} T^{\#\mathcal{H}_{11}(\lambda)} = 811275879 + 7892635410T + \cdots + 29672185525213602280791828408T^{90}.$$

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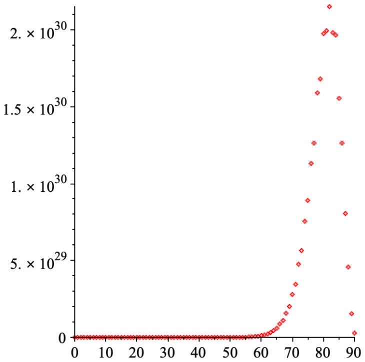

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If $t \geq 4$, then $\widehat{Y}_t(n)$ is a **shifted Gamma distribution.**

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Definition

A random variable $X_{k,\theta}$ is **Gamma distributed with parameter $k > 0$ and scale $\theta > 0$** if its probability distribution function is

$$F_{k,\theta}(x) := \frac{1}{\Gamma(k)\theta^k} \cdot x^{k-1} e^{-\frac{x}{\theta}}.$$

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Theorem (G, Ono, Tsai (2022))

(1) If $t \geq 4$, then

$$\widehat{Y}_t(n) \sim \frac{n}{t} - \frac{\sqrt{3(t-1)n}}{\pi t} \cdot X_{\frac{t-1}{2}, \sqrt{\frac{2}{t-1}}},$$

and has mean $\widehat{\mu}_t(n) \sim \frac{n}{t} - \frac{(t-1)\sqrt{6n}}{2\pi t}$ and variance $\widehat{\sigma}_t^2(n) \sim \frac{3(t-1)n}{\pi^2 t^2}$.

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(2) If $\widehat{k}_{t,n}(x) := \widehat{\mu}_t(n) + \widehat{\sigma}_t(n)x$, *then in the lower incomplete* γ*-function*

$$\lim_{n \to +\infty} \widehat{D}_t(\widehat{k}_{t,n}(x); n) = \frac{\gamma\left(\frac{t-1}{2}; \sqrt{\frac{t-1}{2}}x + \frac{t-1}{2}\right)}{\Gamma\left(\frac{t-1}{2}\right)}.$$

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Remark

No continuous limit for $t \in \{2, 3\}$ as there are always vanishing terms as in

$$\sum_{\lambda \vdash 19} T^{\#\mathcal{H}_t(\lambda)} = 300\,T^9 + 185\,T^8 + 0\,T^7 + 0\,T^6 + 0\,T^5 + 0\,T^4 + 0\,T^3 + 5\,T^2.$$

Example $t = 11$ and $n = 1000$

We illustrates the approximation

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x	$\widehat{D}_{11}(k(x); 1000)$	$\widehat{E}_{11}(x)$	$\widehat{D}_{11}(k(x); 1000)/\widehat{E}_{11}(x)$
-1.00	$0.1319\ldots$	$0.1467\ldots$	$0.8993\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.75	$0.7410\ldots$	$0.7954\ldots$	$0.9315\ldots$
1.00	$0.8226\ldots$	$0.8474\ldots$	$0.9707\ldots$
1.25	$0.8872\ldots$	$0.8880\ldots$	$0.9991\ldots$

Solving Problem 1

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Proposition

If $A \geq 2$, then for every positive integer n we have

$$p_{\leq k}(A; n) = \sum_{j=0}^{\lfloor \frac{n}{A} \rfloor} p_{\leq k}(j) \cdot p_{\mathrm{reg}}(A; n - Aj),$$

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where $|A\lambda'| = Aj$ and λ' is counted by $p_{\leq k}(j)$. □

Erdös-Lehner Formula for $p_{\leq k}(j)$

Proposition (Erdös-Lehner (1941))

If k and j are positive integers, then

$$p_{\leq k}(j) = p(j) - \sum_{m=1}^{\infty} (-1)^m S_k(m; j),$$

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where

$$S_k(m;j) := \sum_{\substack{1 \leq r_1 < r_2 < \cdots < r_m \\ T_m \leq r_1 + r_2 + \cdots + r_m \leq j - mk}} p\left(j - \sum_{i=1}^{m} (k + r_i)\right)$$

and $T_m := m(m+1)/2.$

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$$p(j) - S_k(1;j) \leq p_{\leq k}(j) \leq p(j) - S_k(1;j) + S_k(2;j).$$

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- Inclusion-Exclusion. □

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- Dividing by $p(n)$ we get

$$\frac{p_{\leq k}(A; n)}{p(n)} = \frac{\sum_{j=0}^{\lfloor \frac{n}{A} \rfloor} \left(\sum_{m=0}^{\infty} (-1)^m S_k(m; j)\right) p_{\text{reg}}(A; n - Aj)}{p(n)}.$$

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- Erdös and Lehner proved

$$S_k^*(m;j) \sim \frac{1}{m!} \left(\frac{2}{C} \sqrt{j} \exp\left(-\frac{Ck}{2\sqrt{j}} \right) \right)^m.$$

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- For every m this means $S_k^*(m;j) \sim \frac{1}{m!} \cdot S_k^*(1;j)^m$, giving

$$\sum_{m=0}^{\infty} (-1)^m S_{k_n}^*(m;j) \sim \exp(-S_{k_n}^*(1;j)).$$

- Therefore, **as a sum in** j we have

$$\frac{p_{\leq k}(A;n)}{p(n)} \sim \sum_{j=0}^{\lfloor \frac{n}{A} \rfloor} \exp(-S^*_{k_n}(1;j)) \cdot \frac{p(j)p_{\mathrm{reg}}(A;n-Aj)}{p(n)}.$$

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- Hagis proved that

$$p_{\text{reg}}(A;n) \sim C_A(24n-1+A)^{-\frac{3}{4}} \exp\left(C\sqrt{\frac{A-1}{A}\left(n+\frac{A-1}{24}\right)}\right).$$

- Therefore, each jth summand has the "factor"

$$\frac{p(j)p_{\mathrm{reg}}(A;n-Aj)}{p(n)}$$
$$= \frac{C_A}{(24n-24Aj-1+A)^{\frac{3}{4}}}\frac{n}{j}\exp\left(C\left(\sqrt{j}-\sqrt{n}+\sqrt{\frac{A-1}{A}\left(n-Aj+\frac{A-1}{24}\right)}\right)\right)\cdot\left(1+O_j(n^{-\frac{1}{2}})\right)$$

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- The convenient change of variable $j = \lfloor n/A^2 \rfloor + y$ gives

$$= \frac{C_A}{(24n-24n/A-24Ay-1+A)^{\frac{3}{4}}}\frac{A^2 n}{n+A^2 y}$$
$$\times \exp\left(C\left(\sqrt{n/A^2+y}-\sqrt{n}+\sqrt{\frac{A-1}{A}\left(n-n/A-Ay+\frac{A-1}{24}\right)}\right)\right)\cdot\left(1+O_y(n^{-\frac{1}{2}})\right).$$

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- Letting $n \to +\infty$, this converges to the limit of integrals

$$\lim_{n\to+\infty} \frac{A^2}{96^{1/4}\sqrt{A-1}} \int_{-\log(n)}^{\log(n)} \exp\left(-\frac{CA^4}{8(A-1)}t^2 - \frac{2}{AC}\exp\left(-\frac{1}{2}xAC\right)\right) dt$$

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$$\lim_{n \to +\infty} \frac{p_{\leq k_n}(A; n)}{p(n)} = \exp\left(-\frac{2}{AC} \exp\left(-\frac{1}{2} xAC\right)\right). \qquad \square$$

Counting hooks

Theorem (Han, 2008)

$$G_t(T;q) = \sum_{n=0}^{\infty} P_t(n;T)q^n =:= \sum_{\lambda} q^{|\lambda|} T^{\#\{t \in \mathcal{H}(\lambda)\}} = \prod_{n=1}^{\infty} \frac{(1+(T-1)q^{tn})^t}{1-q^n},$$

$$\widehat{G}_t(T;q) = \sum_{n=0}^{\infty} \widehat{P}_t(n;T)q^n := \sum_{\lambda} q^{|\lambda|} T^{\#\mathcal{H}_t(\lambda)} = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-(Tq^t)^n)^t(1-q^n)}.$$

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$$P_t(n;T) = \frac{c(T)}{2\sqrt{2}\pi n T^{\frac{t}{2}}} \cdot e^{c(T)\left(2\sqrt{n} - \frac{1}{\sqrt{n}}\right)} \cdot \left(1 + O_\eta(n^{-\frac{1}{7}})\right),$$

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If t *is a positive integer and* $T := \{T_n\}$ *is a positive real sequence for which* $T_n = e^{\frac{\alpha(T)+\varepsilon_T(n)}{\sqrt{n}}}$, *where* $\alpha(T)$ *is real and* $\varepsilon_T(n) = o_T(1)$,

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$$\widehat{P}_t(n;T_n) \sim \frac{1}{2^{\frac{7}{4}} 3^{\frac{1}{4}} n} \cdot \sqrt{\frac{1}{\sqrt{6}} + \frac{\alpha(T)}{\pi t}} \left(\frac{\pi t}{\pi t + \sqrt{6}\alpha(T))}\right)^{\frac{t}{2}} \cdot e^{\pi\sqrt{n}\left(\sqrt{\frac{2}{3}} + \frac{\alpha(T)}{\pi t}\right)}.$$

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$$\sum_{n=1}^{\infty} \frac{t^2 n(T-1)}{T-1+e^{tn\alpha}} = -\frac{\mathrm{Li}_2(1-T)}{\alpha^2} + O_\eta(1), \tag{2}$$

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+ Connect to Han's Gen. Fcns + Technical "saddle point" calculations.

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(2) Prove convergence and recognize as normal and shifted Gamma respectively.

□

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Theorem (G, Ono, Rolen, Tsai (2021))

If $C := \pi\sqrt{2/3}$ *and* $k_n = k_n(x) := \frac{1}{AC}\sqrt{n}\log n + \sqrt{n}x$, *then*

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Remarks

(1) These are Gumbel distributions.

(2) The mean and variance are:

$$\text{Mean} := \frac{2}{AC}\left(\log\left(\frac{2}{AC}\right) + \gamma_{Euler}\right),$$
$$\text{Variance} := 1/A^2.$$

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(2) If $k_{t,n}(x) := \mu_t(n) + \sigma_t(n)x$, *then we have*

$$\lim_{n\to+\infty} D_t(k_{t,n}(x); n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy =: E(x).$$

Problem 3: Hooks in $t\mathbb{N}$

Theorem (G, Ono, Tsai (2022))

(1) If $t \geq 4$, *then*

$$\widehat{Y}_t(n) \sim \frac{n}{t} - \frac{\sqrt{3(t-1)n}}{\pi t} \cdot X_{\frac{t-1}{2},\sqrt{\frac{2}{t-1}}},$$

and has mean $\widehat{\mu}_t(n) \sim \frac{n}{t} - \frac{(t-1)\sqrt{6n}}{2\pi t}$ *and variance* $\widehat{\sigma}_t^2(n) \sim \frac{3(t-1)n}{\pi^2 t^2}$.

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(2) If $\widehat{k}_{t,n}(x) := \widehat{\mu}_t(n) + \widehat{\sigma}_t(n)x$, *then in the lower incomplete* γ*-function*

$$\lim_{n \to +\infty} \widehat{D}_t(\widehat{k}_{t,n}(x); n) = \frac{\gamma\left(\frac{t-1}{2}; \sqrt{\frac{t-1}{2}}x + \frac{t-1}{2}\right)}{\Gamma\left(\frac{t-1}{2}\right)}.$$

Executive Summary

- Parts in $A\mathbb{N}$ correspond to **Gumbel Distributions**.
- t-hooks correspond to **Normal Distributions.**
- Hooks in $t\mathbb{N}$ correpond to **shifted Gamma distributions.**