Wronskians of graded dimensions

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Rational conformal field theory and modular forms

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- A vertex operator algebra V gives rise to *characters* of the irreducible modules of V.
- Usually, the characters of a rational vertex operator algebras span a modular invariant vector space. Then we study the quotient

$$\mathcal{F}_V(\tau) = \frac{\mathcal{W}'_V(\tau)}{\mathcal{W}_V(\tau)},$$

where $\mathcal{W}_V$ and $\mathcal{W}'_V$ are defined using Wronskians.

Wronskians

Given a collection of q-series $f_1, \ldots, f_m$, we consider the Wronskian determinant with respect to Ramanujan's derivative $q\frac{q}{dq}$

$$W(f_1, \ldots, f_m) := \begin{vmatrix} f_1 & f_2 & \cdots & f_m \\ f_1' & f_2' & \cdots & f_m' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \cdots & f_m^{(m-1)} \end{vmatrix}$$

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$$\mathcal{W}(f_1, \ldots, f_m) := \alpha W(f_1, \ldots, f_m)$$

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If $\{f_1, \ldots, f_m\}$ is a basis of the modular invariant vector space coming from a VOA V, we set

$$\mathcal{F}_V(\tau) := \frac{\mathcal{W}'(f_1, \ldots, f_m)}{\mathcal{W}(f_1, \ldots, f_m)}$$

Results of Milas-Mortenson-Ono (2008)

For example, for irreducible characters of $\mathcal{M}(2, 2k+1)$ Virasoro minimal models, we have characters

$$\mathrm{ch}_{i,k}(q) = q^{(h_{i,k} - c_k/24)} \prod_{1 \leq n \not\equiv 0, \pm i \pmod{2k+1}} \frac{1}{1-q^n}.$$

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For each k, we study the Wronskians for

$$\{\mathrm{ch}_{1,k}, \mathrm{ch}_{2,k}, \ldots, \mathrm{ch}_{k,k}\}$$

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For example, when $k = 2$ we have

$$\mathrm{ch}_{1,2} = q^{11/60} \prod_{n \geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

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$$W(\mathrm{ch}_{1,k}, \mathrm{ch}_{2,k}) = \frac{-1}{5} \left(q^{1/6} - 4q^{7/6} + 2q^{13/6} + \cdots \right)$$

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Note: $\eta := q^{1/24} \prod_{n \geq 1} (1 - q^n)$

Theorem (Milas, Milas-Mortenson-Ono)

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4. *If* $p = 2k + 1$ *is prime then* $\mathcal{F}_k$ *has p-integral coefficients and satisfies*
$$\mathcal{F}_k(z) \equiv 1 \pmod{p}.$$

Switching to a different VOA

Now consider $L_{\widehat{sl_2}}(k\Lambda_0)$. Here we have

$$\mathrm{ch}_{i,k}(q) = \frac{\displaystyle\sum_{n\equiv i \pmod{2k+2}} nq^{n^2/4(k+2)}}{\eta(\tau)^3}$$

for $k \geq 1$ and $i = 1, \ldots, k+1$.

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for $k \geq 1$ and $i = 1, \ldots, k+1$. We define $\mathcal{W}_k, \mathcal{W}'_k$ and $\mathcal{F}_k$ as before.

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Conjecture (Milas)

If $p = 2k+3 \geq 5$ *is prime then*

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Theorem

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Computing the quotient of the Wronskians for

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is painless, because when $p = 2k+3$ we have

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&\equiv \left(\frac{2}{p}\right) \theta_{i,k} \pmod{p}.
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$$\begin{vmatrix} \theta_{1,k} & \theta_{2,k} & \cdots & \theta_{k+1,k} \\ \theta'_{1,k} & \theta'_{2,k} & \cdots & \theta'_{k+1,k} \\ \vdots & \vdots & \vdots & \vdots \\ \theta^{(k)}_{1,k} & \theta^{(k)}_{2,k} & \cdots & \theta^{(k)}_{k+1,k} \end{vmatrix} \equiv \pm \begin{vmatrix} \theta'_{1,k} & \theta'_{2,k} & \cdots & \theta'_{k+1,k} \\ \theta''_{1,k} & \theta''_{2,k} & \cdots & \theta''_{k+1,k} \\ \vdots & \vdots & \vdots & \vdots \\ \theta^{(k+1)}_{1,k} & \theta^{(k+1)}_{2,k} & \cdots & \theta^{(k+1)}_{k+1,k} \end{vmatrix} \pmod{p}.$$

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$$\mathcal{W}(\theta_{1,k}, \ldots, \theta_{k+1,k}) \equiv \mathcal{W}(\theta'_{1,k}, \ldots, \theta'_{k+1,k}) \pmod{p}.$$

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$$\mathcal{W}(\theta_{1,k}, \ldots, \theta_{k+1,k}) \equiv \mathcal{W}(\theta'_{1,k}, \ldots, \theta'_{k+1,k}) \pmod{p}.$$

But what changes when we replace $\theta_{i,k}$ with $\mathrm{ch}_{i,k} = \dfrac{\theta_{i,k}}{\eta^3}$?

Relating $\mathcal{W}(\mathrm{ch}_{1,k}, \ldots, \mathrm{ch}_{k+1,k})$ and $\mathcal{W}(\theta_{1,k}, \ldots, \theta_{k+1,k})$

The standard fact

$$W(f \cdot f_1, \ldots, f \cdot f_m) = f^m W(f_1, \ldots, f_m)$$

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tells us that

$$\begin{aligned}\mathcal{W}(\mathrm{ch}_{1,k}, \ldots, \mathrm{ch}_{k+1,k}) &= \mathcal{W}\left(\frac{\theta_{1,k}}{\eta^3}, \ldots, \frac{\theta_{k+1,k}}{\eta^3}\right) \\ &= \eta^{-3k(k+1)} \mathcal{W}\left(\theta_{1,k}, \ldots, \theta_{k+1,k}\right).\end{aligned}$$

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On the other hand,

$$\mathcal{W}(\mathrm{ch}'_{1,k}, \ldots, \mathrm{ch}'_{k+1,k}) = \mathcal{W}\left(\left(\frac{\theta_{1,k}}{\eta^3}\right)', \ldots, \left(\frac{\theta_{k+1,k}}{\eta^3}\right)'\right)$$

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since $\left(\eta^3\right)' = \frac{1}{8}\eta^3 E_2$.

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since $\left(\eta^3\right)' = \frac{1}{8}\eta^3 E_2$. So we need to understand

$$\begin{vmatrix} \theta'_{1,k} - \frac{1}{8}E_2\theta_{1,k} & \cdots & \theta'_{k+1,k} - \frac{1}{8}E_2\theta_{k+1,k} \\ \theta''_{1,k} - \frac{1}{8}E_2\theta'_{1,k} - \frac{1}{8}E'_2\theta_{1,k} & \cdots & \theta''_{k+1,k} - \frac{1}{8}E_2\theta'_{k+1,k} - \frac{1}{8}E'_2\theta_{k+1,k} \\ \vdots & \ddots & \vdots \end{vmatrix}.$$

Relating $\mathcal{W}(\mathrm{ch}'_{1,k}, \ldots, \mathrm{ch}'_{k+1,k})$ and $\mathcal{W}\left(\theta'_{1,k}, \ldots, \theta'_{k+1,k}\right)$

After some elementary row operations, this becomes

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where $f_1 = \frac{1}{8}E_2$ and $f_n = f'_{n-1} + \frac{1}{8}E_2 f_{n-1}$ for $1 < n \leq k+1$.

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$f_{k+1} \equiv \frac{1}{8^{k+1}} \pmod{p}$

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So: $\mathcal{W}(\mathrm{ch}'_{1,k}, \ldots, \mathrm{ch}'_{k+1,k}) \equiv \eta^{-3k(k+1)}\mathcal{W}\left(\theta'_{1,k}, \ldots, \theta'_{k+1,k}\right)$.

Putting it all together

Thus we have

$$\begin{aligned}\mathcal{W}(\mathrm{ch}_{1,k},\ldots,\mathrm{ch}_{k+1,k}) &= \eta^{-3k(k+1)}\mathcal{W}\left(\theta_{1,k},\ldots,\theta_{k+1,k}\right)\\ &\equiv \eta^{-3k(k+1)}\mathcal{W}(\theta'_{1,k},\ldots,\theta'_{k+1,k}) \pmod{p}\\ &\equiv \mathcal{W}(\mathrm{ch}'_{1,k},\ldots,\mathrm{ch}'_{k+1,k}) \pmod{p}\end{aligned}$$

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and thus

$$\mathcal{F}_k = \frac{\mathcal{W}(\mathrm{ch}'_{1,k},\ldots,\mathrm{ch}'_{k+1,k})}{\mathcal{W}(\mathrm{ch}_{1,k},\ldots,\mathrm{ch}_{k+1,k})} \equiv 1 \pmod{p}.$$

□

Proof of lemma

Lemma

Define $f_1, f_2, \ldots, f_{k+1}$ *by* $f_1 = \frac{1}{8}E_2$ *and* $f_n = f'_{n-1} + \frac{1}{8}E_2 f_{n-1}$ *for* $1 < n \leq k+1$. *Then* $f_{k+1} \equiv \frac{1}{8^{k+1}} \pmod{p}$.

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Note that since $\left(\eta^3\right)' = \frac{1}{8}\eta^3 E_2$, one can describe $f_1, f_2, \ldots$ by

$$f_0 = 1 \qquad f_n = \eta^{-3}\left(\eta^3 f_{n-1}\right)'.$$

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or, equivalently,

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Conclusions

Theorem

If $p = 2k + 3 \geq 5$ is prime then

$$\mathcal{F}_k(z) \equiv 1 \pmod{p}.$$

Thank you!