HOW RAMANUJAN MAY HAVE DISCOVERED THE MOCK THETA FUNCTIONS

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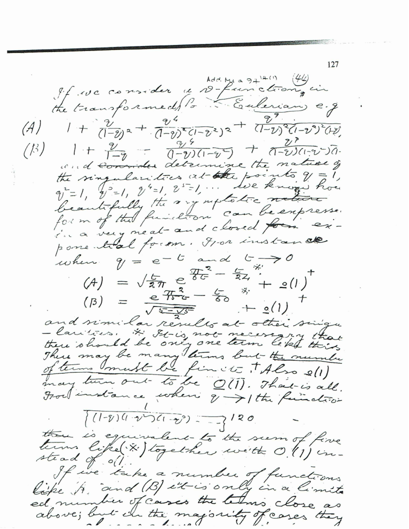
100 Years of Mock Theta Functions

May 22, 2022



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If we consider a ϑ-function in the transformed Eulerian form e.g.

(A) $1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + \frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2} + \cdots$

(B) $1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots$

and determine the nature of the singularities at the points $q=1$, $q^2=1$, $q^3=1$, $q^4=1$, $q^5=1$, ... we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance when $q = e^{-t}$ and $t \to 0$

$$(A) = \sqrt{\frac{t}{2\pi}}\, e^{\frac{\pi^2}{6t} - \frac{t}{24}} + o(1)^{*\,\dagger}$$

$$(B) = \frac{e^{\frac{\pi^2}{15t} - \frac{t}{60}}}{\sqrt{\frac{5-\sqrt{5}}{2}}} + o(1)^{*\,\dagger}$$

and similar results at other singularities. * It is not necessary that there should be only one term like this. There may be many terms but the number of terms must be finite. † Also $o(1)$ may turn out to be $O(1)$. That is all. For instance when $q \to 1$ the function

$$\frac{1}{\{(1-q)(1-q^2)(1-q^3)\cdots\}^{120}}$$

is equivalent to the sum of five terms like (*) together with $O(1)$ instead of $o(1)$.

If we take a number of functions like (A) and (B) it is only in a limited number of cases the terms close as above; but in the majority of cases they

Figure: Page 127 of Ramanujan's Lost Notebook

IF WE CONSIDER A θ-FUNCTION IN THE TRANSFORMED EULERIAN FORM, E.G.

(A)

$$1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\cdots$$

(B) $1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots$

AND DETERMINE THE NATURE OF THE SINGULARITIES AT THE POINTS

$$q=1, q^2=1, q^3=1, q^4=1, q^5=1, \ldots$$

WE KNOW HOW BEAUTIFULLY THE ASYMPTOTIC FORM OF THE FUNCTION CAN BE EXPRESSED IN A VERY NEAT AND CLOSED EXPONENTIAL FORM.

OF COURSE, THE NICE BEHAVIOR OF (A) AND (B) RESULTS FROM

(A) $$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}$$

(B) $$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

WHERE

$$(a;q)_N = (1-A)(1-Aq)\cdots(1-Aq^{N-1}).$$

RAMANUJAN THEN PROCEEDS TO SPECULATE ABOUT THE POSSIBILITY OF FUNCTIONS THAT (1) BEHAVE BEAUTIFULLY NEAR THE UNIT CIRCLE, AND (2) ARE NOT THETA FUNCTIONS (I.E. MODULAR FORMS).

WHILE RAMANUJAN SAYS HE HAS NOT PROVED THE EXISTENCE OF SUCH FUNCTIONS, HE NONETHELESS LISTS 17 FUNCTIONS WHICH HE FIRMLY BELIEVES ARE THESE NEW FUNCTIONS WHICH HE TERMS MOCK THETA FUNCTIONS.

THE FIRST THREE FUNCTIONS HE LISTS ARE:

$$f_3(q) = 1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + \cdots + \frac{q^{n^2}}{\prod_{k=1}^{n}(1-q^k)^2} + \cdots$$

$$\phi_3(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots + \frac{q^{n^2}}{\prod_{k=1}^{n}(1+q^{2k})} + \cdots$$

$$\psi_3(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \cdots + \frac{q^{n^2}}{\prod_{k=1}^{n}(1-q^{2k-1})} + \cdots$$

AND HE ASSERTS THAT

$$\begin{aligned} 2\phi_3(-q) - f_3(q) &= f_3(q) + 4\psi_3(-q) \\ &= (1 - 2q + 2q^4 - 2q^9 + \cdots)(1+q)(1+q^2)(1+q^3)\cdots \end{aligned}$$

BUT WHAT WOULD HAVE GIVEN RAMANUJAN THE IDEA THAT SUCH FUNCTIONS MIGHT BE POSSIBLE?
IN ADDITION, WHY LATCH ONTO $F_3(Q), \phi_3(Q)$, AND $\psi_3(Q)$?

I BELIEVE THAT RAMANUJAN'S STUDY OF IDENTITIES LIKE THE HEINE TRANSFORMATION

$$\sum_{n\geq 0}\frac{(a;q)_n(b;q)_n t^n}{(q;q)_n(c;q)_n}=\frac{(b;q)_\infty(at;q)_\infty}{(c;q)_\infty(t;q)_\infty}\sum_{n\geq 0}\frac{(c/b;q)_n(t;q)_n b^n}{(q;q)_n(at;q)_n}$$

LED HIM TO THE DISCOVERY OF THE MOCK THETA FUNCTIONS.

THE PROOF OF HEINE'S TRANSFORMATION (AND SUBSEQUENT IDENTITIES CONSIDERED BY RAMANUJAN) RELIES ON THE Q-BINOMIAL SERIES

$$\sum_{n\geq 0} \frac{(A;q)_n z^n}{(q;q)_n} = \frac{(Az;q)_\infty}{(z;q)_\infty}$$

AND

$$(A;q)_n = \frac{(A;q)_\infty}{(Aq^N;q)_\infty}.$$

HOW DOES THE PROOF GO?

$$\sum_{n\geq 0} \frac{(a;q)_n (b;q)_n t^n}{(q;q)_n (c;q)_n} = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n\geq 0} \frac{(a;q)_n (cq^n;q)_\infty t^n}{(q;q)_n (bq^n;q)_\infty}$$

HOW DOES THE PROOF GO?

$$\sum_{n\geq 0} \frac{(a;q)_n(b;q)_n t^n}{(q;q)_n(c;q)_n} = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n\geq 0} \frac{(a;q)_n(cq^n;q)_\infty t^n}{(q;q)_n(bq^n;q)_\infty}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n\geq 0} \frac{(a;q)_n t^n}{(q;q)_n} \sum_{m\geq 0} \frac{(c/b;q)_m(bq^n)^m}{(q;q)_m}$$

HOW DOES THE PROOF GO?

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$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n\geq 0} \frac{(a;q)_n t^n}{(q;q)_n} \sum_{m\geq 0} \frac{(c/b;q)_m (bq^n)^m}{(q;q)_m}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m\geq 0} \frac{(c/b;q)_m b^m}{(q;q)_m} \sum_{n\geq 0} \frac{(a;q)_n (tq^m)^n}{(q;q)_n}$$

HOW DOES THE PROOF GO?

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$$= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m\geq 0}\frac{(c/b;q)_m b^m}{(q;q)_m}\frac{(atq^m;q)_\infty}{(tq^m;q)_\infty}$$

HOW DOES THE PROOF GO?

$$\sum_{n\geq 0} \frac{(a;q)_n(b;q)_n t^n}{(q;q)_n(c;q)_n} = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n\geq 0} \frac{(a;q)_n(cq^n;q)_\infty t^n}{(q;q)_n(bq^n;q)_\infty}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n\geq 0} \frac{(a;q)_n t^n}{(q;q)_n} \sum_{m\geq 0} \frac{(c/b;q)_m(bq^n)^m}{(q;q)_m}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m\geq 0} \frac{(c/b;q)_m b^m}{(q;q)_m} \sum_{n\geq 0} \frac{(a;q)_n(tq^m)^n}{(q;q)_n}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m\geq 0} \frac{(c/b;q)_m b^m}{(q;q)_m} \frac{(atq^m;q)_\infty}{(tq^m;q)_\infty}$$

$$= \frac{(b;q)_\infty(at;q)_\infty}{(c;q)_\infty(t;q)_\infty} \sum_{n\geq 0} \frac{(c/b;q)_n(t;q)_n b^n}{(q;q)_n(at;q)_n}$$

BUT MUCH MORE IS POSSIBLE WITH THIS METHOD AS RAMANUJAN CLEARLY REVEALS IN THE LOST NOTEBOOK. E.G.

$$\frac{(aq;q)_\infty (cq;q^2)_\infty}{(-bq;q)_\infty (kq^2;q^2)_\infty} \times \sum_{n\geq 0} \frac{(kq^2;q^2)_n(-bq/a;q)_n a^n q^n}{(cq;q^2)_{n+1}(q;q)_n} \ (\dagger)$$

$$(\dagger) = \sum_{n\geq 0} \frac{(cq/k;q^2)_n (aq/a;q)_{2n} k^n q^{2n}}{(q^2;q^2)_n(-bq;q)_{2n+1}}$$

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$$(\dagger) = \sum_{n\geq 0} \frac{(cq/k;q^2)_n (aq/a;q)_{2n} k^n q^{2n}}{(q^2;q^2)_n(-bq;q)_{2n+1}}$$

$$\sum_{n\geq 0} \frac{a^n q^n}{(q;q)_n(bq;q^2)_n} = \frac{1}{(aq;q)_\infty (bq;q^2)_\infty} \sum_{n\geq 0} \frac{(-1)^n (aq;q)_{2n} b^n q^{n^2}}{(q^2;q^2)_n}$$

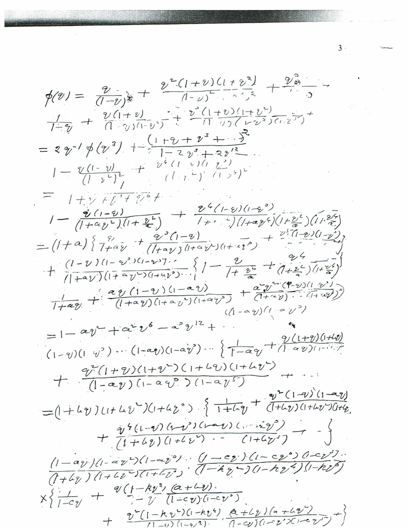

Figure: Page 3 of Ramanujan's Lost Notebook

$$\sum_{n\geq 0} \frac{a^n q^{2n}}{(q^2;q^2)_n (bq;q)_{2n}} = \frac{\sum_{n\geq 0} \frac{(-1)^n (aq^2;q^2)_n b^n q^{n(n+1)/2}}{(q;q)_n}}{(aq^2;q^2)_\infty (bq;q)_\infty}$$

$$\sum_{n\geq 0} \frac{a^n q^{2n}}{(q^2;q^2)_n (bq;q)_{2n}} = \frac{\sum_{n\geq 0} \frac{(-1)^n (aq^2;q^2)_n b^n q^{n(n+1)/2}}{(q;q)_n}}{(aq^2;q^2)_\infty (bq;q)_\infty}$$

$$\sum_{n\geq 0} (-aq;q)_n (-q;q)_n = (-q;q)_\infty (-aq;q)_\infty \sum_{n\geq 0} \frac{(q;q^2)_n q^{2n}}{(-aq;q)_{2n+1}}$$

$$\sum_{n\geq 0} \frac{a^n q^{2n}}{(q^2;q^2)_n (bq;q)_{2n}} = \frac{\sum_{n\geq 0} \frac{(-1)^n (aq^2;q^2)_n b^n q^{n(n+1)/2}}{(q;q)_n}}{(aq^2;q^2)_\infty (bq;q)_\infty}$$

$$\sum_{n\geq 0} (-aq;q)_n (-q;q)_n = (-q;q)_\infty (-aq;q)_\infty \sum_{n\geq 0} \frac{(q;q^2)_n q^{2n}}{(-aq;q)_{2n+1}}$$

$$\frac{(-aq;q)_\infty}{(-q;q)_\infty} \sum_{n\geq 0} \frac{(aq;q)_n q^{n^2}}{(q^2;q^2)_n} = \sum_{n\geq 0} \left(\frac{(a^2q^2;q^2)_n q^{2n^2}}{(q^4;q^4)_n} \right) - a \sum_{n\geq 1} \left(\frac{(a^2q^2;q^2)_{n-1} q^{n(n+1)/2}}{(-q;-q)_{n-1}} \right)$$

THE LAST IDENTITY HAS 2 TERMS ON THE RIGHT-HAND SIDE. HERE'S WHY:

$$\sum_{n\geq 0} \frac{(aq;q)_n q^{n^2}}{(q^2;q^2)_n} = (aq;q)_\infty \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n (aq^{n+1};q)_\infty}$$

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$$= (aq;q)_\infty \sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n}\sum_{m\geq 0}\frac{a^m(q^{(n+1)})^m}{(q;q)_m}$$

THE LAST IDENTITY HAS 2 TERMS ON THE RIGHT-HAND SIDE. HERE'S WHY:

$$\begin{aligned}\sum_{n\geq 0}\frac{(aq;q)_n q^{n^2}}{(q^2;q^2)_n} &= (aq;q)_\infty \sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n (aq^{n+1};q)_\infty}\\ &= (aq;q)_\infty \sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n}\sum_{m\geq 0}\frac{a^m (q^{(n+1)})^m}{(q;q)_m}\\ &= (aq;q)_\infty \sum_{m\geq 0}\frac{a^m q^m}{(q;q)_m}(-q^{1+m};q^2)_\infty\end{aligned}$$

TO PROCEED, WE MUST TAKE 2 SUMS: M EVEN AND M ODD:

$$\sum_{n\geq 0}\frac{(aq;q)_n q^{n^2}}{(q^2;q^2)_n} = (aq;q)_\infty \left\{ \begin{array}{c} \displaystyle\sum_{m\geq 0}\frac{a^{2m}q^{2m}}{(q;q)_{2m}}(-q^{2m+1};q^2)_\infty \\ \\ \displaystyle +\sum_{m\geq 0}\frac{a^{2m+1}q^{2m+1}}{(q;q)_{2m+1}}(-q^{2m+2};q^2)_\infty \end{array} \right\}$$

$$\sum_{n\geq 0}\frac{(aq;q)_n q^{n^2}}{(q^2;q^2)_n} = (aq;q)_\infty \left\{ \begin{array}{c} \displaystyle\sum_{m\geq 0}\frac{a^{2m}q^{2m}}{(q;q)_{2m}}(-q^{2m+1};q^2)_\infty \\ \displaystyle +\sum_{m\geq 0}\frac{a^{2m+1}q^{2m+1}}{(q;q)_{2m+1}}(-q^{2m+2};q^2)_\infty \end{array} \right\}$$

$$= (aq;q)_\infty(-q;q^2)_\infty \sum_{m\geq 0}\frac{a^{2m}q^{2m}}{(q;q)_{2m}(-q;q^2)_m}$$

$$+ (aq;q)_\infty(-q^2;q^2)_\infty \sum_{m\geq 0}\frac{a^{2m+1}q^{2m+1}}{(q;q)_{2m+1}(-q^2;q^2)_m}$$

$$\sum_{n\geq 0}\frac{(aq;q)_n q^{n^2}}{(q^2;q^2)_n} = (aq;q)_\infty \left\{ \begin{array}{c} \displaystyle\sum_{m\geq 0}\frac{a^{2m}q^{2m}}{(q;q)_{2m}}(-q^{2m+1};q^2)_\infty \\ \displaystyle + \sum_{m\geq 0}\frac{a^{2m+1}q^{2m+1}}{(q;q)_{2m+1}}(-q^{2m+2};q^2)_\infty \end{array} \right\}$$

$$= (aq;q)_\infty(-q;q^2)_\infty \sum_{m\geq 0}\frac{a^{2m}q^{2m}}{(q;q)_{2m}(-q;q^2)_m}$$

$$+ (aq;q)_\infty(-q^2;q^2)_\infty \sum_{m\geq 0}\frac{a^{2m+1}q^{2m+1}}{(q;q)_{2m+1}(-q^2;q^2)_m}$$

$$= (aq;q)_\infty(-q;q^2)_\infty \sum_{m\geq 0}\frac{a^{2m}q^{2m}}{(q;q)_{2m}(-q;q^2)_m}$$

$$+ (aq;q)_\infty(-q^2;q^2)_\infty \sum_{m\geq 0}\frac{a^{2m+1}q^{2m+1}}{(q;q)_{2m+1}(-q^2;q^2)_m}$$

WE RETURN NOW TO THE SERIES (A) THAT BEGAN RAMANUJAN'S CONSIDERATIONS:

(A)

$$1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\cdots$$

$$=\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n^2}$$

WHAT HAPPENS IF WE WISH TO APPLY HEINE-TYPE TRANSFORMATIONS TO THIS SERIES? WITH q^{n^2} IN THE NUMERATOR, WE NEED

$$(q^2; q^2)_n$$

IN THE DENOMINATOR

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2} = \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n} \frac{(-q;q)_n}{(q;q)_n}$$

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2} = \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n} \frac{(-q;q)_n}{(q;q)_n}$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n} \frac{(q^{n+1};q)_\infty}{(-q^{n+1};q)_\infty}$$

$$
\begin{aligned}
\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2} &= \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n} \frac{(-q;q)_n}{(q;q)_n} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n} \frac{(q^{n+1};q)_\infty}{(-q^{n+1};q)_\infty} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n} \sum_{m\geq 0} \frac{(-1;q)_m (-q^{n+1})^m}{(q;q)_m}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2} &= \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n} \frac{(-q;q)_n}{(q;q)_n} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n} \frac{(q^{n+1};q)_\infty}{(-q^{n+1};q)_\infty} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{q^{n^2}}{(q^2;q^2)_n} \sum_{m\geq 0} \frac{(-1;q)_m(-q^{n+1})^m}{(q;q)_m} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{m\geq 0} \frac{(-1;q)_m q^m}{(q;q)_m} (-q^{m+1};q^2)_\infty
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n^2} &= \sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n}\frac{(-q;q)_n}{(q;q)_n}\\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n}\frac{(q^{n+1};q)_\infty}{(-q^{n+1};q)_\infty}\\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n}\sum_{m\geq 0}\frac{(-1;q)_m(-q^{n+1})^m}{(q;q)_m}\\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{m\geq 0}\frac{(-1;q)_m q^m}{(q;q)_m}(-q^{m+1};q^2)_\infty\\
(\dagger) &= \frac{(-q;q)_\infty}{(q;q)_\infty}\left\{\begin{array}{c} \displaystyle\sum_{m\geq 0}\frac{(-1;q)_{2m}q^{2m}}{(q;q)_{2m}}(-q^{2m+1};q^2)_\infty \\ \displaystyle +\sum_{m\geq 0}\frac{(-1;q)_{2m+1}q^{2m+1}}{(q;q)_{2m+1}}(-q^{2m+2};q^2)_\infty \end{array}\right\}
\end{aligned}
$$

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n^2} \stackrel{\dagger}{=} \frac{(-q;q)_\infty}{(q;q)_\infty} \left\{ \begin{array}{c} \displaystyle\sum_{m\geq 0} \frac{(-1;q)_{2m} q^{2m}}{(q;q)_{2m}} (-q^{2m+1};q^2)_\infty \\ \displaystyle + \sum_{m\geq 0} \frac{(-1;q)_{2m+1} q^{2m+1}}{(q;q)_{2m+1}} (-q^{2m+2};q^2)_\infty \end{array} \right\}$$

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n^2} \overset{\dagger}{=} \frac{(-q;q)_\infty}{(q;q)_\infty}\left\{\begin{array}{c}\displaystyle\sum_{m\geq 0}\frac{(-1;q)_{2m}q^{2m}}{(q;q)_{2m}}(-q^{2m+1};q^2)_\infty \\ \displaystyle +\sum_{m\geq 0}\frac{(-1;q)_{2m+1}q^{2m+1}}{(q;q)_{2m+1}}(-q^{2m+2};q^2)_\infty\end{array}\right\}$$

$$= \frac{(-q;q)_\infty(-q;q^2)_\infty}{(q;q)_\infty}\sum_{m\geq 0}\frac{(-1;q^2)_m q^{2m}}{(q;q)_{2m}}$$

$$-\frac{q(-q;q)_\infty(-1;q^2)_\infty}{(q;q)_\infty}\sum_{m\geq 0}\frac{(-q;q^2)_m q^{2m}}{(q;q)_{2m+1}}$$

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n^2} \overset{\dagger}{=} \frac{(-q;q)_\infty}{(q;q)_\infty}\left\{\begin{array}{c}\displaystyle\sum_{m\geq 0}\frac{(-1;q)_{2m}q^{2m}}{(q;q)_{2m}}(-q^{2m+1};q^2)_\infty \\ \displaystyle+\sum_{m\geq 0}\frac{(-1;q)_{2m+1}q^{2m+1}}{(q;q)_{2m+1}}(-q^{2m+2};q^2)_\infty\end{array}\right\}$$

$$= \frac{(-q;q)_\infty(-q;q^2)_\infty}{(q;q)_\infty}\sum_{m\geq 0}\frac{(-1;q^2)_m q^{2m}}{(q;q)_{2m}}$$

$$- \frac{q(-q;q)_\infty(-1;q^2)_\infty}{(q;q)_\infty}\sum_{m\geq 0}\frac{(-q;q^2)_m q^{2m}}{(q;q)_{2m+1}}$$

$$= \frac{(-q;q)_\infty(-q;q^2)_\infty}{(q;q)_\infty}\sum_{m\geq 0}\frac{(-1;q^2)_m(0;q^2)_m q^{2m}}{(q^2;q^2)_m(q;q^2)_m}$$

$$- \frac{q(-q;q)_\infty(-1;q^2)_\infty}{(q;q)_\infty(1-q)}\sum_{m\geq 0}\frac{(-q;q^2)_m(0;q^2)_m q^{2m}}{(q^2;q^2)_m(q^3;q^2)_m}$$

NOW WE HAVE ALREADY SEEN HEINE'S TRANSFORMATION

$$\sum_{n\geq 0}\frac{(a;q)_n(b;q)_n t^n}{(q;q)_n(c;q)_n}=\frac{(b;q)_\infty(at;q)_\infty}{(c;q)_\infty(t;q)_\infty}\sum_{n\geq 0}\frac{(c/b;q)_n(t;q)_n b^n}{(q;q)_n(at;q)_n}$$

AND EACH OF THE TWO FINAL SERIES IS A SERIES OF THIS FORM.

ONCE HEINE HAS BEEN APPLIED AND SIMPLIFICATIONS HAVE BEEN MADE, IT ALL BOILS DOWN TO

$$\frac{1}{(q;q)_\infty} = \frac{(-q;q^2)_\infty}{(q;q)_\infty^2} \times \left\{ \sum_{n\geq 0} \frac{(-1)^n q^{n^2}}{(-q^2;q^2)_n} + 2 \sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(-q;q^2)_n} \right\}$$

OR, RECALLING

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty},$$

WE OBTAIN

$$(q;q^2)_\infty \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \phi_3(-q) + 2\psi_3(-q)$$

WHERE

$$\phi_3(q) = \sum_{n\geq 0}\frac{q^{n^2}}{(-q^2;q^2)_n}$$

$$\psi_3(q) = \sum_{n\geq 1}\frac{q^{n^2}}{(q;q^2)_n}.$$

I CLAIM THAT THIS MAY WELL HAVE BEEN THE

"AHA"

MOMENT. FOR IT IS INTUITIVELY CLEAR THAT $\sum_{n\geq 0} \frac{(-1)^n q^{n^2}}{(-q^2;q^2)_n}$ IS BOUNDED WITH $q \to e^{2\pi ih/k}$ IF $4 \nmid k$, AND $= \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q^2)_n}$ BOUNDED WITH $q \to e^{2\pi ih/k}$ PROVIDED k IS ODD.

THUS AT CERTAIN ROOTS OF UNITY $\rho, \phi_3(-q)$ REMAINS BOUNDED AS $q \to \rho$ RADIALLY, AND AT OTHERS IT BEHAVES LIKE

$$(q;q^2)_\infty \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty^2}.$$

THESE BOUNDED LIMITS AS $q \to e^{2\pi ih/k}$ ARE EASILY PROVED. E.G.

$$\lim_{q\to 1^-} \sum_{n\geq 1} \frac{(-1)^n q^{n^2}}{(-q;q^2)_n} = -1,$$

BECAUSE

$$\sum_{n\geq 1} = \sum_{n=1}^{N} + \sum_{n=N+1}^{\infty},$$

AND THE 1^{ST} SUM AS $q \to 1^-$ IS THE GEOMETRIC SERIES $= -1 + \frac{(-1)^N}{2^N}$. IN THE 2^{ND} SUM EACH TERM CAN BE BOUNDED BY $2^- n$.

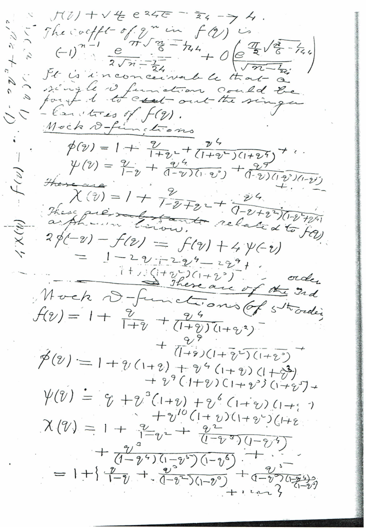
A LITTLE REFLECTION REVEALS THAT IF WE HAD STARTED WITH

$$f_3(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2},$$

WE WOULD HAVE WOUND UP WITH

$$f_3(q) = \phi_3(-q) - 2\psi_3(-q).$$

FROM HERE THE RELEVANT IDENTITIES IN THE LAST LETTER FOLLOW.



$f(q) + \sqrt{\frac{\pi}{t}}\, e^{\frac{\pi^2}{24t} - \frac{t}{24}} \to 4.$

The coeff. of q^n in $f(q)$ is

$$(-1)^{n-1} \frac{e^{\pi\sqrt{\frac{n}{6}-\frac{1}{144}}}}{2\sqrt{n-\frac{1}{24}}} + O\left(\frac{e^{\frac{\pi}{2}\sqrt{\frac{n}{6}-\frac{1}{144}}}}{\sqrt{n-\frac{1}{24}}}\right)$$

It is inconceivable that a single ϑ function could be found to cut out the singularities of $f(q)$.

Mock ϑ-functions

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \cdots$$

$$\chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)} + \cdots$$

These are related to $f(q)$.

$$2\phi(-q) - f(q) = f(q) + 4\psi(-q) = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1+q)(1+q^2)(1+q^3)\cdots}$$

Mock ϑ-functions (of 5th order)

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q^2)(1+q^3)} + \cdots$$

$$\phi(q) = 1 + q(1+q) + q^4(1+q)(1+q^3) + q^9(1+q)(1+q^3)(1+q^5) + \cdots$$

$$\psi(q) = q + q^3(1+q) + q^6(1+q)(1+q^2) + q^{10}(1+q)(1+q^2)(1+q^3) + \cdots$$

$$\chi(q) = 1 + \frac{q}{1-q^2} + \frac{q^2}{(1-q^3)(1-q^4)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)} + \cdots$$

$$= 1 + \left\{\frac{q}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^3}{(1-q^3)(1-q^4)(1-q^5)} + \cdots\right\}$$

Figure: Ramanujan's Last Letter

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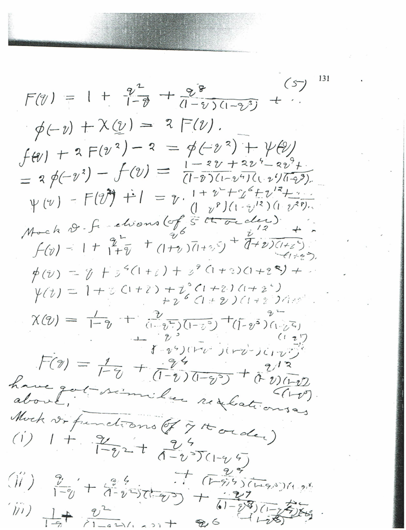

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$$F(q) = 1 + \frac{q^2}{1-q} + \frac{q^8}{(1-q)(1-q^3)} + \cdots$$

$$\phi(-q) + \chi(q) = 2F(q).$$

$$f(q) + 2F(q^2) - 2 = \phi(-q^2) + \psi(q)$$

$$= 2\phi(-q^2) - f(q) = \frac{1-2q+2q^4-2q^9+\cdots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots}$$

$$\psi(q) - F(q^4) + 1 = q\,\frac{1+q^2+q^6+q^{12}+\cdots}{(1-q^8)(1-q^{12})(1-q^{28})\cdots}$$

Mock θ-functions (of 5th order)

$$f(q) = 1 + \frac{q^2}{1+q} + \frac{q^6}{(1+q)(1+q^2)} + \frac{q^{12}}{(1+q)(1+q^2)(1+q^3)} + \cdots$$

$$\phi(q) = q + q^4(1+q) + q^9(1+q)(1+q^3) + \cdots$$

$$\psi(q) = 1 + q(1+q) + q^3(1+q)(1+q^2) + q^6(1+q)(1+q^2)(1+q^3) + \cdots$$

$$\chi(q) = \frac{1}{1-q} + \frac{q}{(1-q^2)(1-q^3)} + \frac{q^2}{(1-q^3)(1-q^4)(1-q^5)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)(1-q^7)}$$

$$F(q) = \frac{1}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^{12}}{(1-q)(1-q^3)(1-q^5)}$$

have got similar relations as above.

Mock θ-functions (of 7th order)

(i) $1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}$

(ii) $\frac{q}{1-q} + \frac{q^4}{(1-q^2)(1-q^3)} + \frac{q^9}{(1-q^3)(1-q^4)(1-q^5)}$

(iii) $\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + q^6 \ldots$

Figure: Ramanujan's Last Letter

WHAT ABOUT THE FIFTH ORDER MOCK THETA FUNCTIONS?
CONSIDER AN IDENTITY FROM THE LOST NOTEBOOK THAT WE PROVED EARLIER:

$$\frac{(-aq;q)_\infty}{(-q;q)_\infty}\sum_{n\geq 0}\frac{(aq;q)_n q^{n^2}}{(q^2;q^2)_n} = \begin{cases} \displaystyle\sum_{n\geq 0}\left(\frac{(a^2q^2;q^2)_n q^{2n^2}}{(q^4;q^4)_n}\right) \\ \displaystyle -a\sum_{n\geq 1}\left(\frac{(a^2q^2;q^2)_{n-1} q^{n(n+1)/2}}{(-q;-q)_{n-1}}\right) \end{cases}$$

SET $a = -1$:

$$\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n} = \left\{ \begin{array}{c} \displaystyle\sum_{n\geq 0}\left(\frac{(-1)^n q^{2n^2}}{(-q^2;q^2)_n}\right) \\ \displaystyle +\sum_{n\geq 1}\left((q;-q)_{n-1}(-q)^{n(n+1)/2}\right) \end{array} \right.$$

SET $a = 1$:

$$\sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n} = \begin{cases} \displaystyle\sum_{n\geq 0} \left(\frac{(-1)^n q^{2n^2}}{(-q^2;q^2)_n} \right) \\ \displaystyle -\sum_{n\geq 1} \left((q;-q)_{n-1} (-q)^{n(n+1)/2} \right) \end{cases}$$

WHAT ABOUT THE SEVENTH ORDER MOCK THETA FUNCTIONS?

WHAT ABOUT THE SEVENTH ORDER MOCK THETA FUNCTIONS?

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THANK YOU!