Theta-type congruences for partitions and colored partitions

Olivia Beckwith, Tulane

Joint works with S. Ahlgren, M. Raum;
A. Caione, J. Chen, M. Diluia, O. Gonzalez, J. Su

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Partitions

Definition

A partition of n is a nonincreasing sequence of positive integers whose sum is n.

The number of partitions of n is $p(n)$.

Example

The partitions of 4:

$$4$$
$$3 + 1$$
$$2 + 2$$
$$2 + 1 + 1$$
$$1 + 1 + 1 + 1$$

Therefore $p(4) = 5$.

Colored Partitions

Definition

An r-colored partition of n is a partition in which each term is assigned one of r colors. The number of r-colored partitions of n is $p_r(n)$.

Example

The two colored partitions of 2:

1. 2
2. 2
3. 1 + 1
4. 1 + 1
5. 1 + 1

So $p_2(2) = 5$.

Congruences

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for all integers n, where A, t are fixed integers.

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In this talk ℓ is prime.

The Ramanujan Congruences

Ramanujan (1921)

For all n,

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

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- Proved by Ramanujan using the Eisenstein series E_2, E_4, E_6 and their derivatives.
- Later explained combinatorially via the “rank" function of Dyson (1944) and the “crank" function of Andrews and Garvan (1988).

Other congruences?

Question

Other than Ramanujan's three congruences, are there any primes ℓ and integers t such that $p_r(\ell n + t) \equiv 0 \pmod{\ell}$ for all n?

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- $r > 1$: Many examples are known.

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Other than Ramanujan's three congruences, are there any primes ℓ and integers t such that $p_r(\ell n + t) \equiv 0 \pmod{\ell}$ for all n?

- $r = 1$: No, by work of Ahlgren and Boylan (2003).
- $r > 1$: Many examples are known.
- We call a congruence of this form a Ramanujan-type congruence.

Ramanujan-type congruences

- **Kiming-Olsson (1992):** There are Ramanujan-type congruences when $\ell | r$, $r \equiv -1 \pmod{\ell}$, or $r \equiv -3 \pmod{\ell}$.

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- **Boylan (2000) and Dawsey-Wagner (2016):** Found more Ramanujan-type congruences using CM modular forms.
- **Rolen-Tripp-Wagner (2022):** Generalized the crank function to combinatorially explain many of these congruences.

Non-Ramanujan-type congruences?

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- These works used the Shimura correspondence and Galois representations associated to modular forms of integral weight.
- **Ahlgren-Allen-Tang (2022):** provide examples similar to Atkin's using different properties of these Galois representations.

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- We have two useful restrictions that were conjectured by Ahlgren and Ono for $p(n)$ (2001), proved by Radu (2013), and generalized to other eta-quotients by Andersen (2014):
  - $\ell | A$.
  - $\left(\frac{r(r-24t)}{\ell}\right) \neq 1$.

Theta-type congruences

- In all known non-Ramanujan-type congruences with a maximal arithmetic progression $\{An + t\}$, one has $A = \ell \cdot Q^n$, where Q is prime and $n \geq 3$.

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- We know that (1) holds trivially if $p_r(\ell n + t) \equiv 0 \pmod{\ell}$ is a Ramanujan-type congruence.
- We call congruences of the form of (1) that do not follow trivially from Ramanujan-type congruences theta-type congruences.

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- $r = 1$: Work of Ahlgren-B-Raum implies that the answer is probably not, based on numerical data and a result that such congruences are expected to be “scarce". These results will be described in the "Scarcity" section of the talk.
- $r > 1$: Work of BCCDGS: infinitely many theta-type congruences exist. This will be described in the "Examples" section.

The Dedekind Eta Function

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Transformation Law

For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$\eta\left(\frac{az+b}{cz+d}\right) = \nu_\eta(\gamma)(cz+d)^{1/2}\eta(z).$$

Here

$$\nu_\eta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \left(\frac{d}{|c|}\right) e\left(\frac{1}{24}\left((a+d)c - bd(c^2-1) - 3c\right)\right), & 2 \nmid c, \\ \left(\frac{c}{d}\right) e\left(\frac{1}{24}\left((a+d)c - bd(c^2-1) + 3d - 3 - 3cd\right)\right), & 2 | c, \end{cases}$$

with $e(x) = e^{2\pi i x}$.

Generating function

$$\sum_{n=0}^{\infty} p_r(n) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-r}$$
$$= q^{r/24} \eta^{-r}(z)$$

Modular forms with the ν_η multiplier system

Let $k \in \frac{1}{2}\mathbb{Z}$.

$M_k(\nu_\eta^n)$ is the space of $f : \mathbb{H} \to \mathbb{C}$ such that

1. f is holomorphic,
2. $f(z)$ is bounded as $\mathrm{Im}(z) \to \infty$, and
3. for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\left(f\middle|\gamma\right)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \nu_\eta^n(\gamma) f(z).$$

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The $f \in M_k(\nu_\eta^n)$ are *weight k modular forms with respect to ν_η^n*.

Cusp forms with multiplier system

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Fourier series

Any $f \in M_k(\nu_\eta^n)$ has a Fourier expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a(n) q^{n/24}.$$

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The following linear maps act nicely on modular forms:

$$\left(\sum_n a(n) q^{\frac{n}{24}}\right) \big| U_m := \sum_n a(mn) q^{\frac{n}{24}} \quad \text{and} \quad \left(\sum_n a(n) q^{\frac{n}{24}}\right) \big| V_m := \sum_n a(n) q$$

$$\left(\sum_n a(n) q^{\frac{n}{24}}\right) \otimes \chi := \sum \chi(n) a(n) q^{\frac{n}{24}}.$$

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- These are useful for restricting to n in an arithmetic progression.

Hecke Operators

For $Q \geq 5$ prime and $(r, 24) = 1$, we have the index Q^2 Hecke operator

$$T_{Q^2} : M_{k/2}(\nu_\eta^r) \to M_{k/2}(\nu_\eta^r).$$

given by

$$\left(\sum_n a(n) q^{n/24}\right)\Big| T_{Q^2} =$$

$$\sum_n \left(a(Q^2 n) + Q^{k-\frac{3}{2}} \left(\frac{(-1)}{Q}\right)^{k-\frac{1}{2}} \left(\frac{12n}{Q}\right) a(n) + Q^{2k-2} a\left(\frac{n}{Q^2}\right)\right) q^{\frac{n}{24}}$$

$$= f\big| U_{Q^2} + Q^{k-\frac{3}{2}} \left(\frac{-1}{Q}\right)^{k-\frac{1}{2}} \left(\frac{12}{Q}\right) f \otimes \chi_Q + Q^{2k-2} f\big| V_{Q^2}$$

Modular forms modulo ℓ

Let $\ell \geq 5$ be prime and let $k \in \mathbb{Z}$.

$$M_k(\mathbb{F}_\ell) := M_k(1) \cap \mathbb{F}_\ell[[q]]$$
$$= \{\text{reduction modulo } \ell \text{ of all } f \in M_k(1) \text{ with } \ell\text{-integral coefficients. }\}$$

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Swinnerton-Dyer (1973)

$$\begin{aligned} \mathbb{F}_\ell[E_4, E_6]/(E_{\ell-1} - 1) &= \sum_{k \in \mathbb{Z}} M_k(\mathbb{F}_\ell) \\ &= \oplus_{a \in \mathbb{Z}/(\ell-1)\mathbb{Z}} \left(\cup_{k \equiv a \pmod{\ell-1}} M_k(\mathbb{F}_\ell) \right). \end{aligned}$$

Filtrations

Definition (Serre)

Let $\ell \geq 5$ be prime, $k \in \mathbb{Z}$. For $f \in M_k(\mathbb{F}_\ell)$, the filtration of f is given by

$$w(f) = \inf\{r \in \mathbb{Z} : f \in M_r(\mathbb{F}_\ell)\}.$$

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- **Example:** Since $E_{\ell-1} \equiv 1 \pmod{\ell}$, we have $w(E_{\ell-1}) = 0$.

Filtrations and linear maps

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Filtrations and linear maps

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Serre (1972)

For $\ell \geq 5$,

- $w(f|U_\ell) \leq \ell + \frac{w(f)-1}{\ell}$,
- Let Θ be the Ramanujan Theta operator, i.e. $\Theta = q\frac{d}{dq}$. Then

$$w(f|\Theta) \leq w(f) + \ell + 1$$

with equality if and only if $\ell \nmid w(f)$.

The q-expansion principle

Deligne and Rapoport (1973)

Let ℓ be prime and $k, N \in \mathbb{N}$. Let π be a prime ideal above ℓ in a number field $\mathbb{F}$ which contains all Nth roots of unity.
Suppose that $f \in M_k(\Gamma(N))$ has π-integral coefficients and $\gamma \in \Gamma_0(\ell^m)$, where ℓ^m is the highest power of ℓ dividing N.
Then $f|\gamma$ has π-integral coefficients, and

$$f \equiv 0 \pmod{\pi} \iff f|\gamma \equiv 0 \pmod{\pi}.$$

Generating functions

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and

$$f_{r,\ell,\delta,-1} := \sum_{\left(\frac{r(r-24n)}{\ell}\right)=-1} p_r(n) q^{\frac{24n-r}{24}}.$$

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- By aforementioned work of Andersen, we don't need to consider a generating function with $\left(\frac{r(r-24n)}{\ell}\right) = 1$.
- $f_{r,\ell,0} \equiv 0 \pmod{\ell}$ when we have a Ramanujan-type congruence $p_r(\ell n - (\frac{\ell^2-1}{24})) \equiv 0 \pmod{\ell}$.

Holomorphic generating functions

Fact (Ahlgren, B, Raum)

Let $\ell \geq 5$ be prime, $\delta \in \{0, -1\}$.

There is a modular form $\sum a_{\ell,\delta}(n) q^{n/24} \in S_{k_{1,\ell,\delta}}(\nu_\eta^{-1})$ such that

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- This is generalized to $r > 1$ for $f_{r,\ell,\delta}$ with a modified weight for $\ell < r$.

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2. Fix $\delta \in \{0, -1\}$ and $\epsilon \in \{\pm 1\}$. If there is a theta-type congruence with

$$\left(\frac{1-24t}{\ell}\right) = \delta \text{ and } \left(\frac{24t-1}{Q}\right) = \epsilon,$$

then

$$f_{1,\ell,\delta}\big|U_Q \equiv -\epsilon \left(\frac{-12}{Q}\right) Q^{-1} f_{\ell,\delta}\big|V_Q \pmod{\ell}. \tag{2}$$

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- (2) holds for theta functions.
- We use a q-expansion formula of Radu (2013) at the cusp $\frac{1}{Q}$.

Scarcity Result

Theorem C (Ahlgren, B, Raum)

Suppose that $\ell \geq 5$ is prime, and fix $\delta \in \{0, -1\}$. Let S be the set of primes Q for which we have a theta-type congruence with $\left(\frac{1-24t}{\ell}\right) = \delta$.

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Suppose that $\ell \geq 5$ is prime, and fix $\delta \in \{0, -1\}$. Let S be the set of primes Q for which we have a theta-type congruence with $\left(\frac{1-24t}{\ell}\right) = \delta$. One of the following is true.

1. S has density zero, or
2. we have

$$\#\{n \leq X \ : a_{\ell,\delta}(n) \not\equiv 0 \pmod{\ell}\} \ll \sqrt{X} \log X \tag{3}$$

and

$$f_{1,\ell,\delta}\big| T_{Q^2} \equiv 0 \pmod{\ell} \quad \text{for all primes } Q \equiv -1 \pmod{\ell}. \tag{4}$$

Scarcity Result

Theorem C (Ahlgren, B, Raum)

Suppose that $\ell \geq 5$ is prime, and fix $\delta \in \{0, -1\}$. Let S be the set of primes Q for which we have a theta-type congruence with $\left(\frac{1-24t}{\ell}\right) = \delta$. One of the following is true.

1. S has density zero, or
2. we have

$$\#\{n \leq X \ : a_{\ell,\delta}(n) \not\equiv 0 \pmod{\ell}\} \ll \sqrt{X} \log X \qquad (3)$$

and

$$f_{1,\ell,\delta}\big|T_{Q^2} \equiv 0 \pmod{\ell} \quad \text{for all primes } Q \equiv -1 \pmod{\ell}. \qquad (4)$$

- The LHS of (3) is $\sim \sqrt{X}$ if $f_{1,\ell,\delta}$ is a theta function.

Some cases

We use (4) to rule out the second possibility for specific Q:

Theorem (Ahlgren, B, Raum)

For $17 \leq \ell \leq 10,000$, S has density 0.

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$\ell = 13$**?**

Barrier:
work of Atkin $\implies f_{1,13,-1}|T_{Q^2} \equiv 0 \pmod{13}$ for $Q \equiv -1 \pmod{13}$.

Arithmetic Large Sieve

Montgomery (1968)

Let R be a nonempty set of Z positive integers in $[1, N+1]$. Let $w(p)$ be the number of residue classes mod p which contain no element of R.

For $X \geq 1$,

$$Z \leq \frac{(N^{1/2} + X)^2}{T},$$

where

$$T = \sum_{q \leq X} \mu^2(q) \prod_{p|q} \frac{w(p)}{(p - w(p))}.$$

Square class structure

Radu (2012)

Suppose $p_r(An + t) \equiv 0 \pmod{\ell}$ for all n, where $(A, 24) = 1$.
If $1 - 24t' \equiv (1 - 24t) \cdot h^2 \pmod{A}$ where $(h, A) = 1$, then
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Consequence: given a theta-type congruence $p_r(\ell Qn + t) \equiv 0 \pmod{\ell}$, we must have

$$p_r(n) \equiv 0 \pmod{\ell}$$

for any n such that $\left(\frac{r-24n}{\ell}\right) = \left(\frac{r-24t}{\ell}\right)$ and $\left(\frac{r-24n}{Q}\right) = \left(\frac{r-24t}{Q}\right)$.

Sketch of proof of Theorem C

Let $f_{1,\ell,\delta} \equiv \sum a_{\ell,\delta}(n)q^{n/24} \pmod{\ell}$ be as above, $Q \in S$, $\epsilon_Q = \left(\frac{1-24t}{Q}\right)$.

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$$f_{\ell,\delta} \equiv \sum_{\left(\frac{n}{Q}\right)=-\epsilon_Q} a_{\ell,\delta}(n) q^{n/24} + \sum a_{\ell,\delta}(Q^2 n) q^{n/24}.$$

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Each $Q \in S$ imposes a quadratic condition on the $n \in \mathbb{Z}$ with $\ell \nmid a_{\ell,\delta}(n)$:

$$\left(\frac{n}{Q}\right) = \epsilon \text{ or } Q^2 | n.$$

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$$\left(\frac{n}{Q}\right) = \epsilon \text{ or } Q^2|n.$$

If S has positive density, the arithmetic large sieve bounds the number of n that satisfy all the quadratic conditions, establishing (3).

Sketch of proof of Theorem C

For every prime Q, there is an n_Q that produces a square class restriction on the Q in S, or a strong version of the $U_Q - V_Q$ relation.

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From the strong $U_Q - V_Q$ relation, a q-expansion principle calculation shows $f_{1,\ell,\delta}\big|T_{Q^2} \equiv 0 \pmod{\ell}$ if $Q \equiv -1 \pmod{\ell}$.

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The theory of Galois representations associated to modular forms in $\mathrm{Sh}_t(f_{1,r,\ell})$ implies that $f_{1,\ell,\delta}\big|T_{Q^2} \equiv 0 \pmod{\ell}$ for every $Q \equiv -1 \pmod{\ell}$. This establishes (4).

Numerical data for $r = 1$

Ahlgren, B, Raum

Apart from the Ramanujan Congruences, there are no theta-type congruences for $\ell < 10^3$ and $Q < 10^{13}$ or $\ell < 10^4$ and $Q < 10^9$.

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It seems like there are no theta-type congruences when $r = 1$, but a barrier to proving is this is that all the conditions on $f_{1,r,\delta}$ that we derive are satisfied by theta functions.

$r = 3$, $\ell = 7$

In this case there is a theta-type congruence for every Q. The table below shows the t-values for several values of Q.

Q	t
5	15,29
11	15,36,50,57,64
13	29,36,50,64,78,85
17	36,50,57,64,85,92,99,113
19	29,36,57,78,85,92,99,113,127
23	15,29,50,57,78,85,99,113,120,127,134
29	15,36,64,78,85,92,99,120,134,155,162,169,176,190
31	15,50,57,64,78,92,120,127,134,141,155,162,176,183,211
37	15,50,78,85,92,99,113,134,141,155,169,183,190,211,225,232,239,246
41	15,57,78,85,113,120,127,134,141,155,169,190,204,218,225,232,239,246,274,281
43	29,57,64,78,92,99,120,141,155,176,183,190,204,211,218,225,232,260,274,281,288
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31	15,50,57,64,78,92,120,127,134,141,155,162,176,183,211
37	15,50,78,85,92,99,113,134,141,155,169,183,190,211,225,232,239,246
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- The number of t's is $\frac{Q-1}{2}$ because of the square class property.

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- The number of t's is $\frac{Q-1}{2}$ because of the square class property.
- Every Q appears because $f_{3,7,0}$ is a theta function.

Theta functions

We say that $f \in M_k(\nu_\eta^r)$ is a *theta function* if the Fourier expansion of f is of the form

$$f = \sum_{n=0}^{\infty} a(n) q^{cn^2/24}$$

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$$\eta(z) = \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2/24}.$$

Conjectures

Conjecture (B, Caione, Chen, Diluia, Gonzalez, Su)

All theta-type congruences come from a congruence between $f_{r,\ell,\delta}$ and a theta function.

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All theta-type congruences come from a congruence between $f_{r,\ell,\delta}$ and a theta function.

So if you have one theta-type congruence, you have one for every Q.

Small r

r	3	9	15	17	19	21	23
(ℓ, δ)	(7,0)	(5,0), (13,0)	(19,0)	(7,0)	(5,0)	(5,-1)	(5, 0 and -1), (7,0 and -1)

Theorem (B, Caione, Chen, Diluia, Gonzalez, Su)

For odd r such that $1 \leq r < 24$, there are no theta-type congruences with ℓ and Q in the range $[5, 6133]$ such that $\ell \nmid r$ except when (r, ℓ) is in the table.

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For the (r, ℓ, δ) in the table, $f_{r,\ell,\delta}$ is congruent to η, η^3, η^ℓ, or $\eta^{\ell^2} - \eta$.

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For the (r, ℓ, δ) in the table, $f_{r,\ell,\delta}$ is congruent to η, η^3, η^ℓ, or $\eta^{\ell^2} - \eta$. This is true of all the theta-type congruences we've found.

Nonvanishing condition

$\delta = 0$: We say that Condition C is satisfied by (r, ℓ, δ) if the Fourier expansion of $f_{r,\ell,\delta}$ is supported on positive indices.

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- For $\delta = 0$, this is true whenever $\lceil \frac{r(\ell^2-1)}{24\ell} \rceil > \frac{r\ell}{24}$ or $\ell > r$.
- For $\delta = -1$, this is true if $r < 23$.

Weight bound

Set

$$b(r,\ell) := (\ell-1)\left\lfloor \frac{1}{\ell-1}\left(\ell + \frac{r(\ell^2-1)-2}{2\ell}\right)\right\rfloor - \frac{r\ell}{2}.$$

- This is the weight of $f_{r,\ell,0}$ computed by examining the filtration of $\Delta^{r(\ell^2-1)/24}|U_\ell$.

Some classes of examples

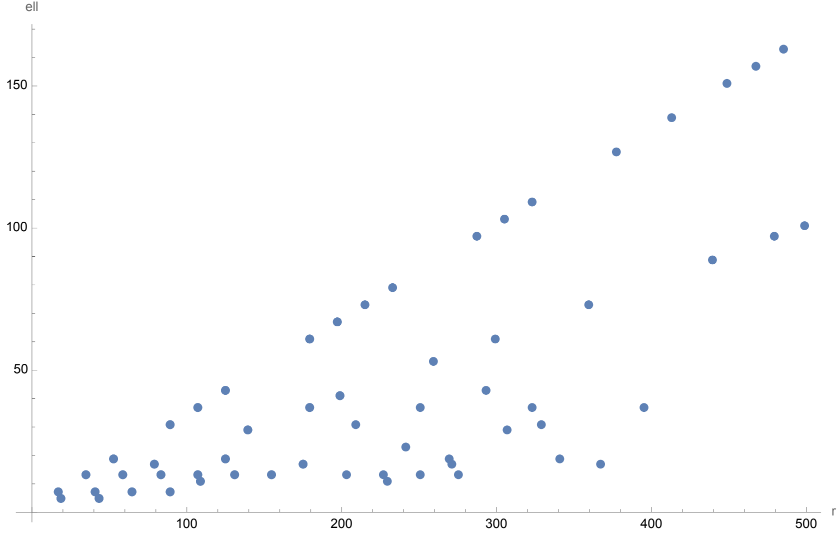
Theorem (B-C-C-D-G-S)

For r, ℓ, δ satisfying Condition A in the table below, $f_{r,\ell,\delta}$ is congruent modulo ℓ to a multiple of the corresponding function on the right. Unless Condition B holds, $f_{r,\ell,\delta} \equiv 0 \pmod{\ell}$.

Type	δ	Condition A	Condition B	Function
1a	0	$\ell = r + 4$	$\ell \equiv 1 \pmod{6}$	η^3
1b	0	Condition C $b(r,\ell) \leq \frac{3}{2}$ $r\ell \equiv -3 \pmod{2(\ell-1)}$	$r \equiv -3\ell \pmod{24}$	η^3
1c	-1	$\ell^2 = r + 4$ $f_{r,\ell,0} \equiv 0 \pmod{\ell}$ Condition C	$r \equiv -3 \pmod{24}$	η^3
2	0	Condition C $b(r,\ell) \leq 1/2$ $r \equiv \ell - 2 \pmod{2(\ell-1)}$	$r \equiv -\ell \pmod{24}$	η
3	0	Condition C $\ell \leq 53$ $r \equiv -1 \pmod{2(\ell-1)}$	$r \equiv -1 \pmod{24}$	η^{ℓ}
4	-1	$\ell^2 = r + 2$ or $r + 26$ $f_{r,\ell,0} \equiv \alpha\eta^{\ell} \pmod{\ell}$ Condition C	$r \equiv -1 \pmod{24}$	$\eta^{\ell^2} - \eta$

Type 2 Examples

For odd $r < 501$, $\ell \leq 1223$, we have 86 Type 2 examples.



Type 2 Data

$-r \pmod{\ell}$	# of Type 2 examples found with $\ell \geq 20$
4	20
6	12
8	3
10	1
12	4

Future work

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- Prove our conjecture for $r > 1$.
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- Determine whether there are theta-type congruences other than those in our table.
- Is there another way to prove these congruences?

Generalizations?

- Other eta-quotients and weakly holomorphic modular forms.

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- Other eta-quotients and weakly holomorphic modular forms.
- Mock theta functions and other mock modular forms.

The End

Thanks for listening!