Partitions associated with the Ramanujan/Watson mock theta functions $\omega(q)$ and $\nu(q)$

Ae Ja Yee

The Pennsylvania State University

100 YEARS OF MOCK THETA FUNCTIONS

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Mock theta functions

- In 1920, Ramanujan introduced the notion of a mock theta function in his last letter to Hardy.
- A mock theta function is a function f of the complex variable q, defined by a q-series of a particular type (Eulerian form), which converges for $|q| < 1$ and satisfies certain conditions.

Mock Theta Conjectures

- Partition-theoretic interpretations of various results involving mock theta functions have been the subject of intense study for many decades.
- Andrews and Garvan reduced the proofs of ten identities for the fifth order mock theta functions given in Ramanujan's Lost Notebook to proving two conjectures based on
  (a) the rank of a partition,
  (b) the number of partitions with unique smallest part and all other parts less than or equal to the double (or one plus the double) of the smallest part.
- These conjectures, known as Mock Theta Conjectures, were first proved by Dean Hickerson in 1988.

Partition-theoretic interpretations of mock theta functions

- A mock theta function itself may also admit a simple and interesting combinatorial interpretation.
- Consider

$$q\chi_1(q) := \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n-1})},$$

  where $\chi_1(q)$ is one of the fifth order mock theta functions of Ramanujan.

  It is easy to see that it is the generating function for partitions in which parts are less than twice the smallest part.

- Similarly,

$$\chi_0(q) := 1 + \sum_{n=1}^{\infty} \frac{q^n}{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n})}$$

  can be interpreted as the generating function for partitions with unique smallest part and the largest part at most twice the smallest part.

- Few other mock theta functions also have similar partition-theoretic interpretations.

Further examples

- Watson defined the following two third-order mock theta functions in 1936. They are also in Ramanujan's Lost Notebook.

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2},$$
$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1+q)(1+q^3)\cdots(1+q^{2n+1})}.$$

The function $p_\omega(n)$

- Let $p_\omega(n)$ denote the number of partitions of n in which each odd part is less than twice the smallest part.

  For example, consider the 11 partitions of 6, namely,

$$6,\quad 5+1,\quad 4+2,\quad 4+1+1,\quad 3+3,\quad 3+2+1,$$
$$3+1+1+1,\quad 2+2+2,\quad 2+2+1+1,$$
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  Then $p_\omega(11)=8$.

- The generating function of $p_\omega(n)$ is

$$\sum_{n=1}^{\infty} p_\omega(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}.$$

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Notation

$$
\begin{aligned}
(a;q)_0 &:= 1,\\
(a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}),\\
(a;q)_\infty &:= \lim_{n\to\infty}(a;q)_n.
\end{aligned}
$$

$\omega(q)$ as the generating function of $p_\omega(n)$

- The third order mock theta function $\omega(q)$ is defined by

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2}.$$

Theorem (Andrews, Dixit, Y. (2015))

$$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty} = q\,\omega(q).$$

Thus,

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Thus,

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Main ingredients in the proof

- Andrews' four-parameter q-series identity:

$$\sum_{n=0}^{\infty}\frac{(B;q)_n(-Abq;q)_nq^n}{(-aq;q)_n(-bq;q)_n}$$
$$=\frac{-a^{-1}(B;q)_\infty(-Abq;q)_\infty}{(-bq;q)_\infty(-aq;q)_\infty}\sum_{m=0}^{\infty}\frac{(A^{-1};q)_m\left(\frac{Abq}{a}\right)^m}{\left(-\frac{B}{a};q\right)_{m+1}}$$
$$+(1+b)\sum_{m=0}^{\infty}\frac{(-a^{-1};q)_{m+1}\left(-\frac{ABq}{a};q\right)_m(-b)^m}{\left(-\frac{B}{a};q\right)_{m+1}\left(\frac{Abq}{a};q\right)_{m+1}}.$$

- Ramanujan's ${}_1\psi_1$ summation formula:

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}z^n=\frac{(az;q)_\infty(q/(az);q)_\infty(q;q)_\infty(b/a;q)_\infty}{(z;q)_\infty(b/(az);q)_\infty(b;q)_\infty(q/a;q)_\infty}.$$

Main ingredients in the proof

- A relation linking $\nu(q)$ and $\omega(q)$:

$$\nu(q) + q\,\omega(q^2) = (-q^2;q^2)^3_\infty (q^2;q^2)_\infty.$$

- The result again:

$$\begin{aligned}\sum_{n=1}^{\infty} p_\omega(n) q^n &= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^{n+1};q)_n (q^{2n+2};q^2)_\infty} \\ &= q\,\omega(q).\end{aligned}$$

The function $p_\nu(n)$

- Let $p_\nu(n)$ denote the number of partitions of n in which each odd part is less than twice the smallest part and parts are distinct.

  For example, consider the 11 partitions of 6, namely,

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  Then $p_\nu(11) = 2$.

- The generating function of $p_\nu(n)$ is

$$\sum_{n=1}^{\infty} p_\nu(n) q^n = \sum_{n=1}^{\infty} q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty.$$

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$\nu(-q)$ as the generating function of $p_\nu(n)$

- The third order mock theta function $\nu(q)$ is defined by

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}.$$

Theorem (Andrews, Dixit, Y.)

$$\sum_{n=0}^{\infty} q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty = \nu(-q).$$

Thus,

$$\sum_{n=1}^{\infty} p_\nu(n)q^n + (-q^2;q^2)_\infty = \nu(-q).$$

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Further results

- Restriction on even parts:

$$\sum_{n=1}^{\infty} \frac{q^n}{(q^n;q)_{n+1}(q^{2n+1};q^2)_\infty} = -\frac{1}{2}\sigma(q) + \frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n}{1+q^n},$$

  where

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n}.$$

- Unique smallest part and restriction on even parts:

$$\sum_{n=1}^{\infty} \frac{q^n}{(q^{n+1};q)_n(q^{2n+1};q^2)_\infty} = -1 + (-q;q)_\infty.$$

- Distinct parts and restriction on even parts:

$$1 + q\sum_{n=0}^{\infty} q^n(-q^{n+1};q)_n(-q^{2n+1};q^2)_\infty = \frac{1-\phi(q)}{(-q;q^2)_\infty} + (q^2;q^2)_\infty(-q;q^2)_\infty^2,$$

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Euler's pentagonal number theorem

- $d(n)$: number of partitions of n into distinct parts,

$$(-q;q)_\infty = \sum_{n=0}^{\infty} d(n)q^n.$$

$$(q;q)_\infty = \sum_{n=0}^{\infty} (d_e(n) - d_o(n))q^n.$$

- $d_e(n)$: number of partitions of n into an even number distinct parts,
- $d_o(n)$: number of partitions of n into an odd number distinct parts,

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$

Analogues of Euler's pentagonal number theorem

Theorem (Andrews, Dixit, and Y.)

$$\sum_{n=1}^{\infty} \frac{q^n}{(-q^n;q)_{n+1}(-q^{2n+2};q^2)_\infty} = \sum_{j=0}^{\infty}(-1)^j q^{6j^2+4j+1}(1+q^{4j+2}).$$

$$\sum_{n=0}^{\infty} q^n (q^n;q)_{n+1}(q^{2n+2};q^2)_\infty = \sum_{j=0}^{\infty}(-1)^j q^{3j^2+2j}(1+q^{2j+1}).$$

Combinatorial interpretation of $\omega(q)$

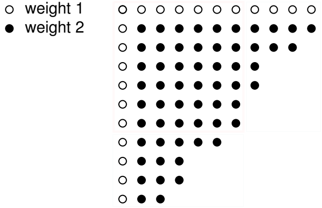

Figure: $\pi : 11 + 21 + 19 + 15 + 15 + 13 + 13 + 11 + 7 + 7 + 5.$

$$\sum_{d=1}^{\infty} \frac{q^{2d^2-2d+1}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2d-1})^2} = q\,\omega(q).$$

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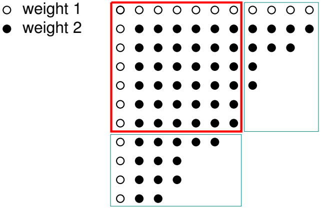

Figure: π : $11 + 21 + 19 + 15 + 15 + 13 + 13 + 11 + 7 + 7 + 5$.

$$\sum_{d=1}^{\infty} \frac{q^{2d^2-2d+1}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2d-1})^2} = \sum_{n=1}^{\infty} p_{\omega}(n)q^n.$$

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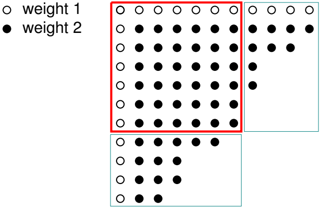

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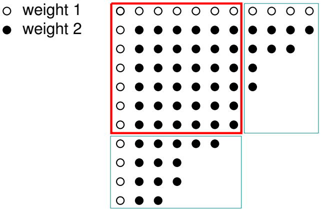

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Theorem (Andrews and Y. (2019))

$$\sum_{n=1}^{\infty} \frac{q^n}{(zq^n;q)_{n+1}(zq^{2n+2};q^2)_\infty} = \sum_{n=1}^{\infty} \frac{z^{n-1}q^n}{(q;q^2)_n} = \sum_{n\geq 0} \frac{z^n q^{2n^2+2n+1}}{(q;q^2)_{n+1}(zq;q^2)_{n+1}}. \qquad (1)$$

Set $z=-1$ in (1). Then

$$\sum_{n=1}^{\infty} \frac{q^n}{(-q^n;q)_{n+1}(-q^{2n+2};q^2)_\infty} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^n}{(q;q^2)_n}$$
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Bivariate generalizations

Theorem (Andrews and Y. (2019))

$$\sum_{n=1}^{\infty} \frac{q^n}{(zq^n;q)_{n+1}(zq^{2n+2};q^2)_\infty} = \sum_{n=1}^{\infty} \frac{z^{n-1}q^n}{(q;q^2)_n} = \sum_{n\geq 0} \frac{z^n q^{2n^2+2n+1}}{(q;q^2)_{n+1}(zq;q^2)_{n+1}}. \tag{1}$$

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Theorem (Andrews and Y.)

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$$\sum_{n\geq 0} (zq;q^2)_n(-q)^n = \sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(-q;q^2)_{n+1}}.$$

Generalizations

- Andrews (1966):

$$\omega(z,q) := \sum_{n\geq 0} \frac{z^n q^{2n^2+2n}}{(q;q^2)_{n+1}(zq;q^2)_{n+1}} = \sum_{n\geq 0} \frac{z^n q^n}{(q;q^2)_{n+1}},$$

$$\nu(z,q) := \sum_{n\geq 0} \frac{q^{n^2+n}}{(-zq;q^2)_{n+1}} = \sum_{n\geq 0} (q/z;q^2)_n(-zq)^n.$$

- Choi (2011):

$$\tilde{\omega}(\alpha,z,q) := \sum_{n\geq 0} \frac{q^{2(n-1)^2-6}\alpha^{2n}z^{4(n+1)}}{(z^2/q;q^2)_{n+1}(\alpha^2z^2/q^3;q^2)_{n+1}},$$

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Identities leading to bijective proofs

- Li–Yang:

$$\omega(y,z,q) := \sum_{n\geq 0} \frac{y^n z^n q^{2n^2+2n}}{(yq;q^2)_{n+1}(zq;q^2)_{n+1}},$$

$$\nu(y,z,q) := \sum_{n\geq 0} \frac{y^n z^n q^{n^2+n}}{(yq;q^2)_{n+1}}.$$

$$\omega(y,z,q) = z^{-2}\overline{\omega}(\sqrt{y}q/\sqrt{z}, \sqrt{z}q, q),$$

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Theorem (Li–Yang (2019))

$$\omega(y,z,q) = \sum_{n\geq 0} \frac{y^n q^n}{(zq;q^2)_{n+1}} = \sum_{n\geq 0} \frac{z^n q^n}{(yq;q^2)_{n+1}},$$

$$\nu(y,z,q) = \sum_{n\geq 0} (-zq;q^2)_n (yq)^n.$$

Identities leading to bijective proofs

- Li–Yang:

$$\omega(y,z,q) := \sum_{n\geq 0} \frac{y^n z^n q^{2n^2+2n}}{(yq;q^2)_{n+1}(zq;q^2)_{n+1}},$$

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The work of Garthwaite–Penniston, Lovejoy, and Waldherr

- Define $a_\omega(n)$ by
$$\omega(q) = \sum_{n\geq 0} a_\omega(n) q^n.$$

- Sharon Garthwaite and David Penniston showed that for any M such that $(M,6)=1$, there are infinitely many arithmetic progressions $An+B$, none of which is contained in another, so that
$$a_\omega(An+B) \equiv 0 \pmod{M}.$$

- Matthias Waldherr proved the first explicit congruences for $a_\omega(n)$ found through some computations done by Jeremy Lovejoy:
$$p_\omega(40n+28) \equiv 0 \pmod{5},$$
$$p_\omega(40n+36) \equiv 0 \pmod{5}.$$

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$$p_\omega\left(2^{2k+3}n+\frac{11\cdot 2^{2k}+1}{3}\right)\equiv 0 \pmod 4,$$
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Further congruences

- Bruinier and Ono (2011): For $p \geq 5$ prime,

$$a_\omega\left(\frac{2p^2-2}{3}\right) \equiv \begin{cases} \left(\frac{p}{3}\right) \pmod{512} & \text{if } p \equiv 1,3 \pmod 8, \\ \left(\frac{p}{3}\right)(1+2p^{255}) \pmod{512} & \text{if } p \equiv 5,7 \pmod 8. \end{cases}$$

Question: Any combinatorial explanation for the fact that $a_\omega\left(\frac{2p^2-2}{3}\right) \equiv \pm 1 \pmod{512}$ for the half of the primes which satisfy the congruence $p \equiv 1,3 \pmod 8$?

- Xia (2018): Let $p \geq 5$ be a prime and $p \equiv 3 \pmod 4$. For $\alpha, n \geq 0$ with $n \neq p$,

$$p_\omega\left(8p^{2\alpha+1}n + \frac{17p^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod{16}.$$

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Overpartitions

- An overpartition is a partition in which the first occurrence of a part may be overlined.

  Example.

$$\begin{aligned} 4 &= 4 = \overline{4} \\ &= 3+1 = \overline{3}+1 = 3+\overline{1} = \overline{3}+\overline{1} \\ &= 2+2 = \overline{2}+2 \\ &= 2+1+1 = \overline{2}+1+1 = 2+\overline{1}+1 = \overline{2}+\overline{1}+1 \\ &= 1+1+1+1 = \overline{1}+1+1+1 \end{aligned}$$

- The overpartition function $\overline{p}(n)$ counts the number of overpartitions of n.

$$\overline{p}(4) = 14.$$

The overpartition analogue $\overline{p}_\omega(n)$

- Define

$$\sum_{n=1}^{\infty} \overline{p}_\omega(n) q^n = \sum_{n=1}^{\infty} \frac{q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}.$$

- Bringmann, Jennings-Shaffer, Mahlburg (2018):

  The generating function of $\overline{p}_\omega(n)$ can be completed as a harmonic Maass form.

Theorem (Andrews, Dixit, Schultz and Y. (2017))

$$\begin{aligned} \overline{p}_\omega(4n+3) &\equiv 0 \pmod 4, \\ \overline{p}_\omega(8n+6) &\equiv 0 \pmod 4. \end{aligned}$$

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Further congruence

- Cui, Gu, and Hao: For $\alpha, n \geq 1$,

$$\overline{p}_{\omega}(2^{2\alpha+3}n + 3 \cdot 2^{2\alpha+1}) \equiv 0 \pmod 4.$$

Analogue of Euler's pentagonal number theorem for $\overline{p}_{\omega}(n)$

Theorem (Wang and Y. (2019))

$$\sum_{n=1}^{\infty} \frac{q^n (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}{(-q^n;q)_{n+1}(-q^{2n+2};q^2)_\infty} = \sum_{n=1}^{\infty} \sum_{|m|<n} (-1)^m q^{n^2+m^2}.$$

Andrews' spt-function

- In 2008, Andrews defined the smallest parts function spt(n) as the total number of appearances of the smallest parts in all of the partitions of n.
- Example: The five partitions of 4 are
$$\underline{4}, 3+\underline{1}, \underline{2}+\underline{2}, 2+\underline{1}+\underline{1}, \underline{1}+\underline{1}+\underline{1}+\underline{1}.$$
Hence spt$(4) = 10$.
- Andrews proved three surprising congruences for spt(n) mod 5, 7 and 13:
$$\begin{aligned} \text{spt}(5n+4) &\equiv 0 \pmod{5} \\ \text{spt}(7n+5) &\equiv 0 \pmod{7} \\ \text{spt}(13n+6) &\equiv 0 \pmod{13}. \end{aligned}$$

The generating function for spt(n)

- spt(n) has the following generating function:

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_\infty}.$$

- This generating function has an alternative representation coming through the second derivative of Watson's q-analogue of Whipple's theorem.

- This allowed Andrews to prove that spt(n) is related to the partition function $p(n)$ and the second Atkin–Garvan rank moment $N_2(n)$ by

$$\mathrm{spt}(n) = np(n) - \frac{1}{2} N_2(n).$$

- He then used this to prove the above spt-congruences.

The smallest parts function associated with $p_\omega(n)$

- Let $\mathrm{spt}_\omega(n)$ denote the number of smallest parts in the partitions enumerated by $p_\omega(n)$.
- Recall that $p_\omega(n)$ is the number of partitions of n such that all odd parts are less than twice the smallest part. Consider the 8 partitions of 6 enumerated by $p_\omega(n)$:

$$\underline{6},\quad 4+\underline{2},\quad 4+\underline{1}+\underline{1},\quad \underline{3}+\underline{3},\quad \underline{2}+\underline{2}+\underline{2},$$
$$2+2+\underline{1}+\underline{1},\quad 2+\underline{1}+\underline{1}+\underline{1}+\underline{1},\quad \text{and } \underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1}+\underline{1},$$

  so that $\mathrm{spt}_\omega(6) = 21$.
- This smallest parts function was also studied by Garvan and Jennings-Shaffer.

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The generating function for $\mathrm{spt}_\omega(n)$

- The generating function of $\mathrm{spt}_\omega(n)$ is given by

$$\sum_{n=1}^{\infty} \mathrm{spt}_\omega(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}.$$

- An alternate representation for this generating function, crucial for establishing the above congruences, comes from differentiating Bailey's ${}_{10}\phi_9$ transformation given below:

  As $N \to \infty$,

$${}_{10}\phi_9 \left[\begin{matrix} a, & q^2\sqrt{a}, & -q^2\sqrt{a}, & p_1, & p_1 q, & p_2, & p_2 q, & f, & q^{-2N}, & q^{-2N+1} \\ & \sqrt{a}, & -\sqrt{a}, & \frac{aq^2}{p_1}, & \frac{aq}{p_1}, & \frac{aq^2}{p_2}, & \frac{aq}{p_2}, & \frac{aq^2}{f}, & aq^{2N+2}, & aq^{2N+1} \end{matrix} ; q^2, \frac{a^3 q^{4N+3}}{p_1^2 p_2^2 f} \right]$$

$$= \frac{(aq;q)_\infty \left(\frac{aq}{p_1 p_2};q\right)_\infty}{\left(\frac{aq}{p_1};q\right)_\infty \left(\frac{aq}{p_2};q\right)_\infty} \sum_{n=0}^{\infty} \frac{(p_1;q)_n (p_2;q)_n \left(\frac{aq}{f};q^2\right)_n}{(q;q)_n (aq;q^2)_n \left(\frac{aq}{f};q\right)_n} \left(\frac{aq}{p_1 p_2}\right)^n.$$

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$$= \frac{(aq;q)_\infty \left(\frac{aq}{p_1 p_2};q\right)_\infty}{\left(\frac{aq}{p_1};q\right)_\infty \left(\frac{aq}{p_2};q\right)_\infty} \sum_{n=0}^{\infty} \frac{(p_1;q)_n (p_2;q)_n \left(\frac{aq}{f};q^2\right)_n}{(q;q)_n (aq;q^2)_n \left(\frac{aq}{f};q\right)_n} \left(\frac{aq}{p_1 p_2}\right)^n.$$

Generating function of $\mathrm{spt}_\omega(n)$ - Alternate representation

- Take second derivative (with respect to z) on both sides of the above equation and then let $z=1$ to obtain

$$\sum_{n=1}^{\infty} \mathrm{spt}_\omega(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_n (q^{2n+2};q^2)_\infty}$$
$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^{2n}) q^{n(3n+1)}}{(1-q^{2n})^2}$$
$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - \frac{1}{2} \sum_{n=1}^{\infty} N_2(n) q^{2n}.$$

- Let

$$\sum_{n=0}^{\infty} c_n q^n := \frac{1}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.$$

Then $5 \mid c_{5n+3}$ and $5 \mid c_{5n+4}$.

Theorem (Andrews, Dixit, Y. (2015))

The function $\mathrm{spt}_\omega(n)$ *satisfies the following three congruences:*

$$\begin{aligned}\mathrm{spt}_\omega(5n+3) &\equiv 0 \pmod{5},\\ \mathrm{spt}_\omega(10n+7) &\equiv 0 \pmod{5},\\ \mathrm{spt}_\omega(10n+9) &\equiv 0 \pmod{5}.\end{aligned}$$

Overpartition analogue of $\text{spt}_\omega(n)$

- Define $\overline{\text{spt}}_\omega(n)$ to be the number of of smallest parts in the overpartitions of n counted by $\overline{p}_\omega(n)$.
- The generating function for $\overline{\text{spt}}_\omega(n)$ is

$$\sum_{n=1}^{\infty} \overline{\text{spt}}_\omega(n)q^n = \sum_{n=1}^{\infty} \frac{q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty}{(1-q^n)^2(q^{n+1};q)_n(q^{2n+2};q^2)_\infty}.$$

Theorem (Andrews, Dixit, Schultz and Y.)

$$\overline{\mathsf{spt}}_\omega(3n) \equiv 0 \pmod{3},$$
$$\overline{\mathsf{spt}}_\omega(3n+2) \equiv 0 \pmod{3}.$$

The function $\text{spt}_\nu(n)$

- Let $\text{spt}_\nu(n)$ denote the number of smallest parts in the partitions enumerated by $p_\nu(n)$.
- It is trivial to see that $\text{spt}_\nu(n) = p_\nu(n)$.
- It is shown that

$$\text{spt}_\nu(10n+8) \equiv 0 \pmod 5.$$

- Proof employs the relation $\nu(-q) = q\omega(q^2) + (-q^2;q^2)_\infty \psi(q^2)$.

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Thank you!