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A necessary and sufficient condition for the compactness of individually rational and feasible outcomes and the existence of an equilibrium*

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Abstract

In an economy with arbitrary closed and convex consumption sets we show that Page's (*Journal of Economic Theory*, 1987, 41, 392–404) condition of no unbounded arbitrage is necessary and sufficient for compactness of the set of individually rational and feasible allocations. In strictly reconcilable economies, in which at most one agent may have half-lines in his indifference surfaces, we show that Page's condition is necessary and sufficient for compactness of the set of individually rational utility possibilities, for the existence of an equilibrium, for non-emptiness of the core, and for the existence of a Pareto-optimal allocation. These results allow one agent to be risk-neutral, not permitted by Werner's (*Econometrica*, 1987, 55, 1403–1418) assumption of no half-lines in indifference surfaces. Our proofs that no unbounded arbitrage is necessary and sufficient for the existence of an equilibrium and for a Pareto-optimal allocation both rely on the result that non-emptiness of the core implies no unbounded arbitrage.

Keywords: Arbitrage; General equilibrium; Core; Pareto-optimality; Necessary and sufficient conditions

JEL classification: C62; C71; D51; D61; G12

1. Introduction

Werner (1987) shows that a form of 'no unbounded arbitrage' is sufficient and, when there are no half-lines in indifference surfaces, necessary and sufficient for the existence of an equilibrium. In a broader context, Nielsen (1989) shows that Page's (1987) condition is sufficient for the existence of an equilibrium and notes that it is also sufficient for compactness

* For the case of concave utility functions, this paper extends and subsumes Page and Wooders (1993). A previous version of the results of this paper was contained in Page and Wooders (1994b).

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of the set of individually rational and feasible outcomes. Page and Wooders (1993) provide a proof of Werner's claim that no unbounded arbitrage is necessary for the existence of an equilibrium and, in addition, show that no unbounded arbitrage is necessary and sufficient for the compactness of the Pareto set and non-emptiness of the core. Chichilnisky (1994, 1995) are related; see Montiero et al. (1995) for a discussion.

In this paper we show that when preferences are representable by continuous, concave utility functions, with no other restrictions on preferences, Page's (1987) condition of no unbounded arbitrage is necessary and sufficient for compactness of the set of individually rational and feasible allocations. We also provide a Second Welfare Theorem; such a result has not previously been demonstrated for unbounded economies.

When, at most, one agent has half-lines in his indifference surfaces we show that Page's no unbounded arbitrage is necessary and sufficient for the existence of a competitive equilibrium, extending Werner's (1987) result. Under the same assumptions on the economy we show that Page's condition is necessary and sufficient for compactness of the utility possibilities set and for non-emptiness of the core, extending the results of Page and Wooders (1993). We provide a new result: no unbounded arbitrage is necessary and sufficient for the existence of a Pareto-optimal allocation. The result that no unbounded arbitrage is necessary for the existence of a Pareto-optimal allocation is the most striking of the necessity results, since the existence of a Pareto-optimal allocation is weaker than the existence of an equilibrium or non-emptiness of the core.

For the case where the arbitrage cones – those directions in which arbitrarily large trades are utility non-decreasing – are pointed, Page's condition limiting arbitrage is equivalent to the condition used in Werner (1987). Werner's condition, however, does not hold for economies where even one agent has straight-line indifference surfaces, while our condition and results apply to such economies. Thus, in a financial market setting, for example, our results permit one agent to have risk-neutral preferences.

A discussion of related literature appears in Section 4 and a detailed discussion appears in Page and Wooders (1994a). All proofs appear in Section 5.

2. An economy with unbounded consumption sets and non-monotonicities

Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ denote an *exchange economy*. Each agent j has *consumption set* $X_j \subset \Re^L$ and *endowment* $\omega_i \in \text{int}\, X_j$. The jth agent's preferences over X_j are specified via a *utility function* $u_j(\cdot): X_j \to \Re$.[1] It is assumed that for each $j \in N := \{1, \ldots, n\}$, X_j is closed and convex and $\omega_j \in \text{int}\, X_j$, where 'int' denotes 'interior'.

The set of *individually rational allocations* is given by

$$A = \left\{ (x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n : \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \omega_j \text{ and } u_j(x_j) \geq u_j(\omega_j), \text{ for all } j \right\}.$$

[1] Our assumptions are chosen for brevity and clarity; in other research we relax several of the assumptions of this paper, including the representation of preferences by concave utility functions. See especially Page and Wooders (1994a, 1996).

Given any individually rational allocation $x = (x_1, \ldots, x_n) \in A$, let $pr_j(x) = x_j$. For each $x_j \in X_j$ the *preferred set* is $P_j(x_j) := \{x' \in X_j : u_j(x') > u_j(x_j)\}$. We assume that utility functions satisfy *non-satiation at rational allocations*; that is, for each $j \in N$, $P_j(x_j) \neq \emptyset$ for all $x_j \in pr_j(A)$.[2]

The problem of the non-existence of a solution – the competitive equilibrium, the core, or even a Pareto-optimal allocation – in unbounded economies is that the preferences of agents may be too dissimilar to be reconciled by a price. There is no price vector at which gains from trade can be exhausted by finite trades. We say that an economy is *reconcilable* if, for each $j \in N$, $u_j(\cdot)$ is continuous, concave, and satisfies non-satiation at rational allocations. For such economies, no unbounded arbitrage, which limits the diversity of agents, ensures the existence of solutions.

Given prices $p \in \mathscr{B} := \{p' \in \Re^L : \|p'\| \leq 1\}$ the *budget set* for the jth agent is given by $B(p, \omega_j) = \{x \in X_j : \langle p, x \rangle \leq \langle p, \omega_j \rangle\}$, where $\langle p, x \rangle = \Sigma_{\ell=1}^{L} p_\ell \cdot x_\ell$. An *equilibrium* for the economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ is an $(n+1)$-tuple of vectors $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ such that: (i) $(\bar{x}_1, \ldots, \bar{x}_n) \in A$; (ii) $\bar{p} \in \mathscr{B} \backslash \{0\}$; and (iii) for each j, $\langle \bar{p}, \bar{x}_j \rangle = \langle \bar{p}, \omega_j \rangle$ and $P_j(\bar{x}_j) \cap B(\bar{p}, \omega_j) = \emptyset$.

An allocation $x = (x_1, \ldots, x_n)$ is *Pareto optimal* if it is feasible and if there does not exist another feasible allocation $x' = (x'_1, \ldots, x'_n)$ such that $u_j(x'_j) \geq u_j(x_j)$ for all $j \in N$ and, for at least one j, $u_j(x'_j) > u_j(x_j)$.

Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ denote an economy and let S be a coalition – a subset of N. Define the set of *S-allocations* by $A(S) = \{(x_j)_{j \in S} : \Sigma_{j \in S} x_j = \Sigma_{j \in S} \omega_j$ and for ech $j \in S$, $x_j \in X_j$ and $u_j(x_j) \geq u_j(\omega_j)\}$. An allocation $x = (x_1, \ldots, x_n) \in A$ is in the *core* of the economy if there does not exist an S-allocation $(x'_j)_{j \in S}$ such that $u_j(x'_j) \geq u_j(x_j)$ for all $j \in S$ and for at least one $j \in S$, $u_j(x'_j) > u_j(x_j)$.

2.1. No unbounded arbitrage

Page's condition of no unbounded arbitrage (see Page, 1987; Nielsen, 1989; and other papers by Page and his co-authors) is stated in terms of 'arbitrage cones'. The jth *agent's arbitrage cone* is the closed convex cone[3] containing the origin given by

$$\mathscr{C}P_j = \{y \in \Re^L : \text{for some } x \in X_j,\ x + \lambda y \in X_j \text{ and } u_j(x + \lambda y) \text{ is non-decreasing in } \lambda \text{ for } \lambda \geq 0\}\ .$$

An economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ satisfies *no unbounded arbitrage* if:

[2] For concave utility functions the following implication holds:

$$z_j \in P_j(x_j) \text{ implies that } tz_j + (1-t)x_j \in P_j(x_j)\ , \quad \text{for all } t \in (0, 1]\ .$$

Thus, concavity, together with non-satiation at rational allocations, implies *local* non-satiation at rational allocations: given any $\epsilon > 0$, $B_\epsilon(x_j) \cap P_j(x_j) \neq \emptyset$ for all j and for all $x_j \in pr_j(A)$, where $B_\epsilon(x_j)$ denotes the open ball of radius ϵ centered at x_j.

[3] The arbitrage cone is the recession cone of the preferred set; this, and the fact that the arbitrage cone is closed and convex, follows from results in Rockafellar (1970, section 8).

(2.1) whenever $\Sigma_{j=1}^{n} y_j = 0$ and $y_j \in \mathscr{C}P_j$ for all agents j, it holds that $y_j = 0$ for all agents j.

For $x \in X_j$, define the *increasing cone for the* jth *agent*[4] by

$$I_j(x) = \{y \in \mathscr{C}P_j : \text{for all } \lambda \geq 0 \text{ there exists } \lambda' > \lambda \text{ such that } u_j(x + \lambda' y) > u_j(x + \lambda y)\} .$$

The jth agent satisfies *extreme desirability* if for any $x \in X_j$ it holds that $I_j(x) = \mathscr{C}P_j \backslash \{0\}$. The economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^{n}$ satisfies *extreme desirability* if at least $n-1$ agents' preferences satisfy extreme desirability. An economy is *strictly reconcilable* if it is reconcilable and satisfies extreme desirability. In a strictly reconcilable economy the existence of an equilibrium implies that arbitrage opportunities can be eliminated by finite trades.

Remark. As Page (1987) remarks, no unbounded arbitrage dictates that no agent or group of agents can find a mutually compatible trading partner or group of trading partners with whom to engage in unbounded and utility increasing trades. This is suggestive of the core. In fact, in our proof of the necessity of no unbounded arbitrage for the existence of an equilibrium (Theorem 2) we appeal to the Debreu and Scarf (1963) result that an equilibrium is in the core, and then show that non-emptiness of the core implies no unbounded arbitrage. In the context of our current model, it is not clear how else the theorem may be proven; it is precisely the core-like property of no unbounded arbitrage that enables us to prove the result.

3. Arbitrage and boundedness

To state the results of this section, we require the following definition. Given a positive integer k, a *k-bounded economy* is denoted by $(X_{kj}, \omega_j, u_j(\cdot))_{j=1}^{n}$, where $X_{kj} := B_k(\omega_j) \cap X_j$ and where $B_k(\omega_j)$ is the closed ball of radius k centered at the jth agent's endowment ω_j. The set of *individually rational S-allocations for the k-bounded economy* is given by $A_k(S) = \{(x_j)_{j \in S} \in A(S) : \text{for each } j \in S, x_j \in X_{kj}\}$. Define $A_k := A_k(N)$ and, for each $S \subset N$, define $U(A_k(S)) := \{(u_j)_{j \in S} : \text{for some } (x_j)_{j \in S} \text{ in } A_k(S), u_j = u_j(x_j) \text{ for each } j \in S\}$.

Theorem 1. No unbounded arbitrage is necessary and sufficient for the compactness of the set of individually rational allocations. Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^{n}$ *be a reconcilable economy. The following statements are equivalent*:

(3.1) $(X_j, \omega_j, u_j(\cdot))_{j=1}^{n}$ *satisfies no unbounded arbitrage.*

[4] The increasing cone was introduced in Page (1982) to obtain necessary and sufficient conditions for the existence of an equilibrium in an asset market model, a special case of the model herein, and has since been used in several papers, including Page and Wooders (1993), where it was given its name. Here we broaden the definition to accommodate thick indifference curves. Since we have assumed concavity, thick indifference curves may appear in only limited ways. See, however, Page and Wooders (1996).

(3.2) *There exists an integer* $\bar{k}$ *such that for any coalition* $S \subset N$, $A(S) = A_k(S)$ *for all* $k \geq \bar{k}$ *and* $A(S)$ *is compact.*

Nielsen (1989) states that, in a reconcilable economy, Page's (1987) no unbounded arbitrage is sufficient for compactness of A, the set of individually rational and feasible allocations. Koutsougeras (1995) provides a proof that Page's condition implies that A is bounded. Theorem 1 shows that no unbounded arbitrage is necessary and sufficient for compactness of A. Without further restrictions on the economic model, no unbounded arbitrage is *not* necessary and sufficient for the compactness of the utility possibility set. The following corollary is, however, now immediate.

Corollary 1. Compactness of the utility possibility set. Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ *be a reconcilable economy satisfying no unbounded arbitrage. Then, for any coalition* $S \subset N$, *the set of utility possibilities* $U(A(S))$ *is compact.*

The following result appears in Nielsen (1989). We provide a very short proof based on Theorem 1.

Theorem 2. No unbounded arbitrage is sufficient for the existence of an equilibrium. Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ *be a reconcilable economy. Assume that no unbounded arbitrage* (2.1) *holds. Then the economy has an equilibrium.*

Our next result is a Second Welfare Theorem.[5]

Theorem 3. A Second Welfare Theorem. Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ *be a reconcilable economy satisfying no unbounded arbitrage. Then, for every individually rational and Pareto-optimal allocation* $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ *where* $\bar{x}_j \in \operatorname{int} X_j$ *for all j there is a price vector* $\bar{p}$ *such that* $\langle \bar{x}, \bar{p} \rangle$ *is an equilibrium relative to some redistribution of the initial endowment.*

Our final result shows that in economies where at most one agent's indifference surfaces may include half-lines, Page's (1987) condition of no unbounded arbitrage is necessary and sufficient for the existence of an equilibrium, non-emptiness of the core, and the existence of a Pareto-optimal point.

Theorem 4. In a strictly reconcilable economy, no unbounded arbitrage is necessary and sufficient for compactness of the sets of utility possibilities, the existence of an equilibrium, non-emptiness of the core, and the existence of an individually rational and Pareto-optimal allocation. Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ *be a strictly reconcilable economy. The following statements are equivalent:*

(3.3) $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ *satisfies no unbounded arbitrage.*

[5] With a stronger non-satiation assumption, the following result holds for all Parto-optimal allocations.

(3.4) *There exists an integer* $\bar{k}$ *such that for any coalition* S, $U(A(S)) = U(A_k(S))$ *for all* $k \geqslant \bar{k}$.

(3.5) *The economy has a competitive equilibrium.*

(3.6) *The economy has a non-empty core.*

(3.7) *The economy has an individually rational and Pareto-optimal allocation.*

4. Further relationships to the literature

As discussed in detail in Page and Wooders (1994a), since Hart (1974) a number of authors have studied both necessary and sufficient conditions for the existence of an equilibrium. Werner (1987) uses the condition that the intersection of the strict duals of the arbitrage cones is non-empty and shows that this condition is sufficient for the existence of an equilibrium.[6] To describe the condition, we first define the *strict dual* of the set $\mathscr{C}P_j$, denoted by $\mathscr{D}(\mathscr{C}P_j)$:

(4.1) $\mathscr{D}(\mathscr{C}P_j) := \{p \in \Re^L : \langle p, y \rangle > 0 \text{ for all } y \in \mathscr{C}P_j \backslash \{0\}\}$.

Any price vector p contained in $\mathscr{D}(\mathscr{C}P_j)$ assigns a *positive* value to any *non-zero* net trade vector y contained in $\mathscr{C}P_j$. The economy satisfies *the intersection form of no unbounded arbitrage* if

(4.2) $\cap_j \mathscr{D}_j(\mathscr{C}P_j) \neq \varnothing$.

It follows from the Dubovitskii–Milyutin Theorem (1965) that if arbitrage cones are pointed, then no unbounded arbitrage (2.1) and its intersection form (4.2) are equivalent.[7]

Werner (1987) notes that when there are no half-lines in indifference surfaces the intersection form of no unbounded arbitrage is necessary for the existence of an equilibrium.[8] Page and Wooders (1993) provide details of the proof that the intersection form of no unbounded arbitrage is necessary for the existence of an equilibrium and also give a short proof of sufficiency. Since the condition of no unbounded arbitrage used in this paper is less restrictive than that of Werner (1987), our results hold for a broader class of models – one agent may have risk-neutral preferences.

To treat economies with short sales, Chichilnisky (1995) introduces a new cone concept, the

[6] This condition, for general equilibrium models of asset markets, also appears in Hammond (1983) and Page (1982).

[7] Page and Wooders (1993) prove this result from the Hahn–Banach Theorem.

[8] In a general equilibrium asset market setting with price dependent preferences, both Hammond (1983) and Page (1982) show that if all agents' expected utility functions are concave and satisfy extreme desirability, then the intersection form of no unbounded arbitrage is necessary and sufficient for the existence of an equilibrium. Monteiro et al. (1995) note that an agent satisfies the no half-line condition if and only if the agent satisfies extreme desirability.

global cone, consisting of the set of directions in which utility approaches its supremum or, in some cases, the closure of this set. The *global cone* is defined by

(4.3) $A(u_j, \omega_j) = \{y \in R^L : \text{for all } z \in \Re^L, \text{ there exists } \lambda > 0 \text{ such that } (\omega_j + \lambda y) >_j z\}$

or

(4.4) $\overline{A(u_j, \omega_j)}$, the closure of $A(u_j, \omega_j)$.

Note that the global cone is, in general, smaller than the arbitrage cone, since the arbitrage cone may include rays with the property that unbounded trades in the directions of these rays are utility increasing, but not necessarily increasing to the supremum or to infinity. See Monteiro et al. (1995) for a discussion of Chichilnisky (1994, 1995) and the relationship of her papers to those of other authors, including Werner (1987) and Nielsen (1989).

Page (1989) stresses that it not boundedness of feasible trades that is required for the existence of an equilibrium but rather the boundedness of utility-increasing trades. In fact, in Page (1989, 1996), no unbounded arbitrage is defined in terms of a *strict* increasing cone. The (strict) increasing cone is defined in Page and Wooders (1993) as

$$I_j^s(x) := \{y \in R^L : u_j(x + \lambda y) \text{ is strictly increasing in } \lambda \text{ for } \lambda \in [0, \infty)\} .$$

The reader can verify that under the assumptions of local non-satiation (everywhere) and concavity, it holds that $I_j^s(x) = I_j(x)$. In Page and Wooders (1996) a broader class of models, where this equivalence no longer holds, is considered.

5. Proofs

We first prove Theorem 1, using the following lemmas. The first lemma is an easy consequence of results on recession cones in Rockafellar (1970); in particular, Theorem 8.2. (See also Theorems 8.3 and 8.7 in Rockafellar, 1970).

Lemma 1. (Theorem 8.2, Rockafellar, 1970). The following statements are equivalent: (a) $y \in \mathscr{C}P_j$ and (b) for some $x \in X_j$, y is a cluster point of some sequence $\{\lambda_\nu x_\nu\}_\nu$ where $\{x_\nu\}_\nu \subset \{x' \in X_j : u_j(x') \geq u_j(x)\}$ and $\{\lambda_\nu\}$ is a sequence of positive real numbers converging to zero.

The following lemma can be proven using elementary facts concerning sequences.

Lemma 2. Let $\{x^k\}_k = \{(x_1^k, \ldots, x_n^k)\}_k \subset A$ be a sequence of individually rational allocations such that $\Sigma_{j=1}^n \|x_j^k\| \to \infty$ as $k \to \infty$. Then, for any cluster point $(y_1, \ldots, y_n)$ of the sequence $\{\lambda^k x^k\}_k$ where $\lambda^k = (\Sigma_{j=1}^n \|x_j^k\|)^{-1}$, it holds that $\Sigma_{j=1}^n y_j = 0$ and $\Sigma_{j=1}^n \|y_j\| = 1$.

Proof of Theorem 1. That (3.2) implies (3.1) is obvious. To see that (3.1) implies (3.2), consider the following. Let $\{\bar{x}^k\}_k = \{(\bar{x}_1^k, \ldots, \bar{x}_n^k)\}_k \subset A$ be a sequence of individually rational allocations such that for each k, $\bar{x}^k \not\in A_k$. This implies that $\Sigma_{j=1}^n \|\bar{x}_j^k\| \to \infty$ as $k \to \infty$. Let $(\bar{y}_1, \ldots, \bar{y}_n) \in \Re^L \times \ldots \times \Re^L$ be a cluster point of the sequence $\{(\lambda^k \bar{x}_1^k, \ldots, \lambda^k \bar{x}_n^k)\}$ where $\lambda^k = [\Sigma_{j=1}^n \|\bar{x}_j^k\|]^{-1}$. Since $\bar{x}_j^k \in \{x \in X_j : u_j(x) \geq u_j(\omega_j)\}$ for all k, it follows from Lemma 1 that $\bar{y}_j \in \mathscr{C}P_j$ for each j. By Lemma 2, $\Sigma_{j=1}^n \bar{y}_j = 0$ and $\Sigma_{j=1}^n \|\bar{y}_j\| = 1$. Thus, for some j, $\bar{y}_j \neq 0$, contradicting (3.1). Compactness of A follows from the fact that, for k sufficiently large, $A = A_k$ and A_k is compact. □

The proof of Theorem 2 follows directly from Theorem 1 and Theorem 4 in Debreu (1982). (See also footnote 2.) Here we restate Debreu's Theorem to fit our model.

Theorem (Debreu, 1982). *Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ denote an economy satisfying the properties that for all $j \in N$, (a) X_j is compact and convex; (b) $u_j(\cdot)$ is continuous and non-satiated at rational allocations; (c) $x_j \in P_j(x_j)$ implies that $tz_j + (1-t)x_j \in P_j(x_j)$ for all $t \in (0, 1]$; (d) $\omega_j \in \operatorname{int} X_j$.*

Proof of Theorem 2. Consider the k-bounded economy $(X_{kj}, \omega_j, u_j(\cdot))_{j=1}^n$ where, as before, $X_{kj} = X_j \cap B_k(\omega_j)$. By Theorem 1, if the unbounded economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ satisfies no unbounded arbitrage, then k can be chosen sufficiently large so that for any equilibrium $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ for the k-bounded economy, $\bar{x}_j \in \operatorname{int} X_{kj}$ for all j. Given the concavity of agents' utility functions and the consequent convexity of preferred sets, such an equilibrium for the k-bounded economy is an equilibrium for the unbounded economy. By Theorem 4 of Debreu (1982), for each k, the bounded economy $(X_{kj}, \omega_j, u_j(\cdot))_{j=1}^n$ has an equilibrium. Thus, the unbounded economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ has an equilibrium. □

Proof of Theorem 3. Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ be a reconcilable economy satisfying no unbounded arbitrage. Let $x' = (x_1', \ldots, x_n')$ be an individually rational and Pareto-optimal allocation. For each agent j define $\omega_j' = \bar{x}_j$ and $\omega' = (\omega_1', \ldots, \omega_n')$. Consider the unbounded economy $(X_j, \omega_j', u_j(\cdot))_{j=1}^n$. Since no unbounded arbitrage is satisfied and since, from concavity, the arbitrage cones are independent of endowments, the economy $(X_j, \omega_j', u_j(\cdot))_{j=1}^n$ satisfies no unbounded arbitrage. Thus, the economy $(X_j, \omega_j', u_j(\cdot))_{j=1}^n$ has an equilibrium, say $(\bar{x}, \bar{p})$, where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$. Since $(\bar{x}, \bar{p})$ is an equilibrium, for each agent j it holds that $u_j(\bar{x}_j) \geq u_j(\omega_j')$. Suppose that for one (or more) agents it holds that $u_j(\bar{x}_j) > u_j(\omega_j')$. We then have a contradiction to the Pareto-optimality of x'. Thus, $u_j(\bar{x}_j) = u_j(\omega_j')$. Since $(\bar{x}, \bar{p})$ is an equilibrium, for any x_j'' such that $u_j(x_j'') > u_j(\omega_j')$, it holds that $\bar{p} \cdot x_j'' > \bar{p} \cdot \omega_j'$. Thus, for any x_j'' such that $u_j(x_j'') > u_j(\omega_j')$, it holds that $\bar{p} \cdot x_j'' > \bar{p} \cdot \omega_j'$. Therefore $(\omega', \bar{p})$ is an equilibrium. □

Proof of Theorem 4. Since a strictly reconcilable economy is reconcilable, that (3.3) implies (3.4) is immediate from the corollary to Theorem 1. To prove the other direction, assume that (3.4) holds but that (3.3) does not hold. Then there exists $(y_1, \ldots, y_n) \neq (0, \ldots, 0)$ in $\mathscr{C}P_1 \times \ldots \times \mathscr{C}P_n$ such that $\Sigma_j y_j = 0$. Thus, given any $(x_1, \ldots, x_n) \in A$, $(x_1 + \lambda y_1, \ldots, x_n + \lambda y_n) \in A$ for all $\lambda \geq 0$. Let $u' \in U(A)$ be such that: (*) $\Sigma_j u_j' = \max\{\Sigma_j u_j : u = (u_1, \ldots, u_n) \in U(A)\}$. Since $U(A)$ is compact there is such a vector u' in $U(A)$. Now let $(x_1', \ldots, x_n') \in A$ be such that $u_j' = u_j(x_j')$ for each j and k. Since $\Sigma_j y_j = 0$ and for at least one agent j' it holds that

$y_{j'} \neq 0$, there is at least one other agent, say j'', for whom $y_{j''} \neq 0$. One of these agents, say j'', must satisfy extreme desirability and for this agent there exists a positive real number λ' such that $u_{j''}(x'_{j''} + \lambda' y_{j''}) > u_{j''}(x'_{j''})$. Therefore, $\Sigma_j u_j(x'_j + \lambda' y_j) > \Sigma_j u_j(x'_j) = \Sigma_j u'_j = \max\{\Sigma_j u_j : u = (u_1, \ldots, u_n) \in U(A)\}$. But since $(u_1(x'_1 + \lambda y_1), \ldots, u_n(x'_n + \lambda y_n)) \in U(A)$ for all $\lambda > 0$, we have a contradiction.

That (3.3) implies (3.5) is immediate from Theorem 2. To show that (3.5) implies (3.3), let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ be a strictly reconcilable economy and let $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ be an equilibrium. Since $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is an equilibrium, $(\bar{x}_1, \ldots, \bar{x}_n)$ is in the core of the economy (exactly as shown in Debreu and Scarf, 1963). Suppose that no unbounded arbitrage is not satisfied. Then there is an n-tuple of net trades $(y_1, \ldots, y_n) \neq (0, \ldots, 0)$ in $\mathcal{C}P_1 \times \ldots \times \mathcal{C}P_n$ such that $\Sigma_j y_j = 0$. It follows from strict reconcilability that there is at least one agent j and positive number λ such that $u_j(\bar{x}_j + \lambda y_j) > u_j(\bar{x}_j)$. It follows that the set of agents N can improve upon the allocation $(\bar{x}_1, \ldots, \bar{x}_n)$ with the allocation $(\bar{x}_1 + \lambda y_1, \ldots, \bar{x}_n + \lambda y_n)$, contradicting the fact that $\bar{x}$ is in the core of the economy.

The argument above also shows that the non-emptiness of the core (3.6) implies no unbounded arbitrage (3.3). Since the competitive equilibrium is in the core, (3.3) implies (3.5), which implies (3.6).

It is immediate that (3.6) implies (3.7). To show that (3.7) implies (3.6), let $x' = (x'_1, \ldots, x'_n)$ be an individually rational and Pareto-optimal allocation. Now for each agent j let $\omega'_j = x'_j$ and consider the economy $(X_j, \omega'_j, u_j(\cdot))_{j=1}^n$. We claim that x' is in the core of $(X_j, \omega'_j, u_j(\cdot))_{j=1}^n$. For suppose there is a coalition S that can improve. But since the consumptions of the members of the complementary coalition can remain unchanged, this contradicts the assumption that x' is Pareto optimal. Since x' is in the core, the economy $(X_j, \omega'_j, u_j(\cdot))_{j=1}^n$ satisfies no unbounded arbitrage. But since the arbitrage cones are the same for the economies $(X_j, \omega'_j, u_j(\cdot))_{j=1}^n$ and $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$, the economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ satisfies no unbounded arbitrage. □

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