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A TIEBOUT THEOREM

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Tiebout (1956) has conjectured that when public goods are local rather than pure, competitive forces tend to make local governments provide the public goods in a near-optimal manner. He hypothesized that 'the greater the number of communities and the greater the variety among them, the closer the consumer will come to fully realizing his preference position'.

In this paper, a model with potential 'entry' of jurisdictions is considered. In equilibrium, no firm could enter, form a new jurisdiction, and charging possibly discriminatory admittance fees, earn a positive profit.

A Tiebout-type entry equilibrium with lump-sum taxes is shown to exist for all sufficiently large economies. An equilibrium jurisdiction structure and the associated lump-sum taxes are constrained only by the property that entry is unprofitable. The equilibrium is of the type suggested by Tiebout in that equilibrium states of the economy are approximately optimal and, the larger the economy and the smaller the costs of forming a jurisdiction, the closer an equilibrium is to an optimum.

To characterize the Tiebout equilibrium, we develop another specific equilibrium concept, called a competitive equilibrium. In this equilibrium, agents pay Lindahl prices for the local public goods, and receive payments (possibly negative) for their effects on the production and/or consumption possibilities of the other agents. The competitive equilibrium states of the economy are Pareto-optimal.

It is shown that the Tiebout equilibrium states of the economy converge to the competitive states as the economy grows large and jurisdiction formation costs, small.

Key words: Tiebout equilibrium; Lindahl price; Pareto-optimal.

1. Introduction

Tiebout (1956) conjectured that when public goods are local rather than pure, competitive forces tend to make local governments provide the public goods in a near-optimal manner. He also hypothesized (p. 418) that 'the greater the number of communities and the greater the variety among them, the closer the consumer will come to fully realizing his preference position'.

Given the diversity of interpretations in the literature of Tiebout's conjectures, there are several aspects of his paper that we wish to stress. First, as the above quotation indicates, Tiebout was making conjectures about 'large' economies with 'many' jurisdictions and about convergence properties. Second, he conjectured that

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a *near*-optimal, 'market-type' equilibrium would arise in sufficiently large economies. Third, economies with local public goods 'have mobility as a cost of registering demand. The higher the cost, *ceteris paribus*, the less optimal the allocation of resources' (Tiebout, 1956, p. 422).

In this paper, a model with 'entry' of jurisdictions is considered. In equilibrium, no firm could enter, form a new jurisdiction and, charging possibly discriminatory admittance fees, earn a positive profit. Potential entry will guarantee that there are sufficiently many jurisdictions of sufficient variety for near-optimality to obtain.

A Tiebout-type entry equilibrium with lump-sum taxes is shown to exist for all sufficiently large economies. The equilibrium jurisdiction structures and the lump-sum taxes are constrained only by the property that entry is unprofitable. The equilibrium is of the type suggested by Tiebout in that it is shown to have the following properties: the equilibrium jurisdiction structures and associated assignment of lump-sum taxes are approximately optimal; and, the larger the economy and the smaller the costs of forming a jurisdiction, the closer the equilibrium is to an optimum.

Our approach enables us to finesse the questions of the motivation of goals of local governments and the methods of determination of lump-sum taxes. Instead of postulating how taxes are determined, we characterize tax systems that are sustainable in the face of entry. In particular, we show that if local public goods are provided by governments, and the entry of profit-maximizing firms is allowed (but, in equilibrium, does not occur) then in a Tiebout equilibrium the jurisdiction structure and taxation scheme must be approximately optimal. In other words, most jurisdictions producing the local public goods must behave approximately 'competitively' and the number and variety of jurisdictions must be 'near-optimal'. A feature which is important in the interpretation of this result is that it is for 'large' economies; given jurisdiction formation costs, for all *sufficiently large economies* the equilibrium exists and is approximately optimal. Also, the proofs reveal that what is 'sufficiently large' depends on the sizes of 'optimal' (or near optimal) jurisdictions; the economy must be sufficiently large relative to the sizes of optimal jurisdictions.

To characterize the Tiebout equilibrium, we develop another highly specific equilibrium concept, called a competitive equilibrium. In this equilibrium, agents pay Lindahl prices for the local public goods, and receive payments (possibly negative) for their effects on the production and/or consumption possibilities of the other agents. The competitive equilibrium states of the economy are Pareto-optimal. Conditions are demonstrated under which the equilibrium states of the economy are in the core and, if all replications of a state of the economy are in the cores of the corresponding replicated economies, then that state is an equilibruim state – an equivalence theorem. We show conditions under which such states exist and thus obtain an existence theorem for the equilibrium.

For the case on one private good (with possibly more than one public good) the Tiebout equilibrium states are equivalent to the core. From our equivalence theorem for the competitive equilibrium and the core, we then have an equivalence theorem for the Tiebout equilibrium and the competitive equilibrium; the Tiebout equilibrium approaches a near-optimal, 'market-type' equilibrium – our Tiebout Theorem.

We remark that jurisdiction-formation costs (in terms of inputs of private goods) are taken as given. These costs depend on a parameter, ε, and go to zero as ε goes to zero. To show existence of the Tiebout equilibrium for all sufficienctly large economies, it is required that ε be positive. For the competitive equilibrium, existence is obtained for a subsequence of economies when jurisdiction formation costs are greater than or equal to zero.

One can reinterpret the equilibria, when jurisdiction formation is costly, as approximate equilibria. Using the results herein and in other related papers on cores of games and economies, it is clear that approximate competitive equilibria which are approximately feasible exist for all sufficiently large economies. (See, especially, Wooders, 1988.)

There are a number of features of the model that could be altered while the results remain unchanged. In particular, we allow the private goods to be traded among jurisdictions while all public goods produced for a jurisdiction must be consumed by the members of that jurisdiction only. We could have the agents belong to different jurisdictions providing different public goods. Moreover, we allow an agent to belong to one and only one jurisdiction – this could be relaxed in a number of ways as long as the 'local' public goods aspects are retained (for examples of other possible restrictions rather than simply partitions, see Shubik and Wooders, 1983, 1986).

This paper consists of seven sections. In the next section, the model is presented. The third section develops the competitive equilibrium and the fourth, the Tiebout equilibrium. The proofs are collected in the fifth section. The sixth section relates this paper to the literature and the seventh concludes the paper.

2. The model

The following notation and terminology will be used: R^n_+, the nonnegative orthant of R^n; R^n_{++}, the positive orthant of R^n; given a set S; $|S|$ denotes the cardinal number of S. The unit vector in R^n is denoted by $I=(1,1,\dots,1)\in R^n$.

We follow the convention that given x and y in R^n, $x\geq y$ means $x_i\geq y_i$ for all i; $x>y$ means $x\geq y$ and, for at least one i, $x_i>y_i$; and $x\gg y$ means $x_i>y_i$ for all i.

Given $x\in R^n$, $\|x\|=\max|x_i|$ where $|x_i|$ is the absolute value of the ith coordinate of x.

2.1. Agents

The set of agents of the rth replica economy is denoted by $N_r=\{(1,1),\dots,(t,q),\dots,(T,r)\}$ where (t,q) is called the qth *agent of type t*. Given N_r and $t\in$

$\{1,\dots,T\}$, let $[t]_r=\{(t,q)\colon q\in\{1,\dots,r\}\}$; the set $[t]_r$ is the set of *agents of type t of the rth replica economy*. When $r=1$, we denote N_r simply by N.

Given $S\subset N_r$, let s be the vector whose tth coordinate is defined by $s_t=|S\cap[t]_r|$; s is called the *profile of S* and is simply a list of the numbers of agents of each type in S. When S has profile s, we define $\varrho(S)=s$. Let I denote the T-fold Cartesian product of the nonnegative integers excluding the zero vector; then for every r and every nonempty subset S of N_r, we have $s\in I$ where $s=\varrho(S)$. We denote the set of elements of I whose tth coordinate is nonzero by $I(t)$; a member of $I(t)$ is the profile of a subset containing an agent of type t. The set of profiles of nonempty subsets of N_r is denoted by $I_r=\{s\in I\colon s\le rI\}$ and the set of profiles of subsets of N_r containing an agent of type t is denoted by $I_r(t)=\{s\in I\colon s\in I(t)\cap I_r\}$.

2.2. Goods

The economy has L private goods and M public goods. A vector of the public goods is denoted by $x=(x_1,\dots,x_m,\dots,x_M)\in R^M$ and a vector of private goods by $y=(y_1,\dots,y_l,\dots,y_L)\in R^L$.

2.3. Endowments and preferences

It is assumed that each agent has a positive initial endowment of each private good and that there are no initial endowments of the public goods. Write w_l^{tq} *for the endowment of the* $(t,q)^{\text{th}}$ *agent of the* l^{th} *private good* and write $w^{tq}=(w_1^{tq},\dots,w_l^{tq},\dots,w_L^{tq})\in R^L$ for the *endowment of the* $(t,q)^{\text{th}}$ *agent*.

We also assume that $w^{tq}=w^{t'q'}$ whenever $t=t'$, i.e., agents of the same type have the same endowment.

The preferences of an agent of type t, say (t,q), are described by a complete preordering $\succsim_{tq}$ on $X^t\times I(t)$ where X^t is a subset of R_+^{M+L} called the (goods) *consumption set* (for agents of type t). The symbols $\succ_{tq}$, $\prec_{tq}$ and $\sim_{tq}$ have the usual interpretation. An element of $X^t\times I(t)$ is called a *total consumption for* (t,q) and is denoted by $(x,y;s)$ where $x\in R_+^M$, $y\in R_+^L$, and s is a profile with s_t positive. For each (t,q) it is assumed that X^t is closed, convex, and has a lower bound for $\le$. Define X'^t by $X'^t=\{y\in R^L\colon (0,y)\in X^t\}$; we assume that $X'^t=R_{++}^L$.

Given any $s\in I(t)$, the preference preordering satisfies the usual properties. More specifically, given any $s\in I(t)$, preferences satisfy:

(a) Monotonicity: given any $(x,y)\in X^t$, if $(x',y')\in R^{M+L}$ and $(x,y)<(x',y')$ then $(x',y')\in X^t$ and $(x,y;s)\prec_{tq}(x',y';s)$ for each $s\in I(t)$;

(b) Continuity: for any $(x',y')\in X^t$, the two sets $\{(x,y)\colon (x,y;s)\succsim_{tq}(x',y';s)\}$ and $\{(x,y)\colon (x,y;s)\precsim_{tq}(x',y';s)\}$ are closed in X^t for each $s\in I(t)$;

(c) Convexity: Let (x',y') and (x'',y'') be two consumptions for (t,q) where $(x',y';s)\succsim_{tq}(x'',y'',s)$. Let λ be an arbitrary number such that $0\le\lambda\le1$ and let $(x,y)=\lambda(x',y')+(1-\lambda)(x'',y'')$. Then $(x,y;s)\succsim_{tq}(x'',y'';s)$.

We assume that all agents of the same type have the same endowments, consumption sets, and preferences. We do not, however, assume that agents of different types have necessarily different endowments, consumption sets, or preferences, so the assumption that N has one agent of each type is not as restrictive as it might at first seem.

Given two subsets, say S and S', (t,q) in S and S', and (x,y), (x',y') in X^t, we write $(x,y;S)\succsim_{tq}(x',y';S')$ if and only if $(x,y;s)\succsim_{tq}(x',y';s')$ where $\varrho(S)=s$ and $\varrho(S')=s'$. Informally, the crowding effect experienced by an agent depends only on the profile of the subset containing that agent.

We say that crowding is *nondiscriminatory in consumption* if, for each $(t,q)\in N_r$, for all $(x,y)\in X^t$, and for all $s,s'\in I_r(t)$, we have $(x,y;s)\sim_{tq}(x,y;s')$ whenever $\sum_{t=1}^{T}s_t=\sum_{t=1}^{T}s'_t$; nondiscriminatory crowding in consumption depends only on the total number of agents in a jurisdiction. In this paper, we do *not* assume nondiscriminatory crowding. Instead, crowding can depend on the entire profile of the set of agents in a jurisdiction, i.e., crowding is possibly *discriminatory*.

2.4. Jurisdiction structures

A *jurisdiction structure of* $S\subset N_r$ is a partition of S, denoted by $\mathbf{S}=\{S_1,\dots,S_k,\dots,S_K\}$. A jurisdiction structure of N_r is called simply a *jurisdiction structure* and denoted by $\eta_r=\{J_1,\dots,J_g,\dots,J_G\}$.

Given a subset of agents S, a jurisdiction structure $\mathbf{S}$ of S, $(t,q)\in S$, and $(x,y)\in X^t$, write $(x,y;\mathbf{S})$ for $(x,y;S')$ where $(t,q)\in S'$ and $S'\in\mathbf{S}$.

2.5. Allocations

Given a nonempty subset S of N_r and $\mathbf{S}$, a jurisdiction structure of S, and *allocation for S relative to* $\mathbf{S}$, or simply *an allocation for S*, denoted by $\alpha(\mathbf{S})$, is a pair (x^S,y^S) where $x^S\in R^{MS}$ and $y^S\in R^{LS}$ such that:

(a) for each $(t,q)\in S$, $(x^{tq},y^{tq})\in X^t$;

(b) for each $S'\in\mathbf{S}$ and for all $(t,q),(t',q')\in S'$, we have $x^{tq}=x^{t'q'}$ (all agents in each jurisdiction are allocated the same amounts of the public goods).

Given an allocation for S, $\alpha(\mathbf{S})=(x^S,y^S)$, the total consumption of the agent (t,q) is $(x^{tq},y^{tq};\mathbf{S})$.

Given two allocations for $S\subset N_r$, say $\alpha(\mathbf{S})=(x,y)$ and $\alpha'(\mathbf{S}')=(x',y')$, we write $\alpha(\mathbf{S})\succ_S\alpha'(\mathbf{S}')$ if for all $(t,q)\in S$, we have

$$(x^{tq},y^{tq};\mathbf{S})\succ_{tq}(x'^{tq},y'^{tq};\mathbf{S}')$$

and say $\alpha(\mathbf{S})$ *is preferred to* $\alpha'(\mathbf{S}')$.

Given $S'\subset S$ and $\alpha(\mathbf{S})=(x^S,y^S)$, let $\alpha(\mathbf{S})|_{S'}=(x^{S'},y^{S'})$ where $x^{S'}$ and $y^{S'}$ are the projections of x^S and y^S onto the subspaces associated with S'. We interpret $\alpha(\mathbf{S})|_{S'}$ as the total consumptions $(x^{tq},y^{tq};\mathbf{S})$ for each $(t,q)\in S'$.

2.6. Production

The production possibility set for public goods available to a jurisdiction depends on the profile of that jurisdiction. We take as given a correspondence, Y_0, from the set of profiles I to $R^M_+ \times -R^L_+$ where $Y_0[s]$, for all $s \in I$, is a nonempty, closed, convex cone with vertex 0. An element of $Y_0[s]$ is denoted by (x, z) where x represents output of the public goods and z represents inputs of private goods. Given any r and any $S \subset N_r$ where $\varrho(S) = s$, we define $Y_0[S]$ by $Y_0[S] = Y_0[s]$.

The production possibility set for the public goods relative to a jurisdiction structure $\mathbf{S} = \{S_1, \dots, S_k, \dots, S_K\}$ of S will be denoted by $Y_0[\mathbf{S}]$. We assume that $Y_0[\mathbf{S}] = \prod_{k=1}^K Y_0[S_k]$. An element of $Y_0[\mathbf{S}]$ is denoted by $\beta(\mathbf{S})$. Note that, given $\beta(\mathbf{S})$, for some $(x_k, z_k) \in Y_0[S_k]$ for each k, $\beta(\mathbf{S}) = \prod_{k=1}^K (x_k, z_k)$. Given $\mathbf{S}' \subset \mathbf{S}$, and $\beta(\mathbf{S}) \in Y_0[\mathbf{S}]$, we write $\beta(\mathbf{S})|_{\mathbf{S}'}$ to denote the restriction of $\beta(\mathbf{S})$ to those jurisdictions contained in $\mathbf{S}'$; i.e., $\beta(\mathbf{S})|_{\mathbf{S}'} = \prod_{\{k: S_k \in \mathbf{S}'\}} (x_k, z_k)$.

We remark that some special cases of the class of production possibilities for the public goods which we have described are:

Case 1. $Y_0[S] = Y_0[S']$ for all S and S' in N_r;
Case 2. $Y_0[S] = \{(x, z) : (x, z/|S|) \in Y_0\}$ where Y_0 is some given production possibility set and $|S|$ denotes the number of agents in the set S;
Case 3. $Y_0[S] = Y_0[S']$ whenever $|S| = |S'|$.

In Case 1, all coalitions have access to the same production possibility set so there is no crowding in the production of the public goods. In Case 2, the public goods resemble private goods in that it takes $|S|$ times as many resources to produce x for $|S|$ agents as it does to produce x for one agent (this is the case considered in Bewley, 1981). When Case 3 holds, we say that crowding in production is *non-discriminatory*.

In this paper, we do not restrict crowding in production to any of these three cases (although we will require assumptions ensuring that the advantages of large jurisdictions are eventually outweighed by the costs of congestion phenomena). When crowding in production can depend on the entire profile of the set of agents in a jurisdiction rather than simply the number of agents, we have *discriminatory crowding*; this is the case in which we are particularly interested.

The production possibility set for private goods is not dependent upon the jurisdiction structure. We denote this production possibility set by Y_1 and an element of Y_1 is denoted by $z \in R^L$. We assume that $Y_1 \cap R^L_+ = \{0\}$ and that Y_1 is a nonempty, closed convex cone with vertex 0.

The (entire) production possibilities for S relative to the jurisdiction structure $\mathbf{S}$ is denoted by $Y[\mathbf{S}]$ and it is assumed that $Y[\mathbf{S}] = Y_0[\mathbf{S}] \times Y_1$. An element of $Y[\mathbf{S}]$ is called a *production for S relative to* $\mathbf{S}$ and is denoted by $(\beta(\mathbf{S}), z)$ where $\beta(\mathbf{S}) \in Y_0[\mathbf{S}]$ and $z \in Y_1$.

2.7. Jurisdiction formation costs

We take as given a mapping c from $I \times R^1_+$ to R^L. Given $s \in I$ and $\varepsilon \in R^1_+$, $c(s; \varepsilon)$ represents the vector of inputs of private goods required by a subset S with profile

s to form a jurisdiction consisting of the members of S. For simplicity, we assume that for some $\bar{z} \in R^L$ where $\bar{z} < 0$, we have $c(s;\varepsilon) = \varepsilon\,|S|\,\bar{z}$. What is essential is that $\|c(s;\varepsilon)\| \to 0$ as $\varepsilon \to 0$ and $\|c(s;\varepsilon)\| \to \infty$ as $|s| \to \infty$.

Given a subset S of N_r for any r, define $c(S;\varepsilon) = c(s;\varepsilon)$ where $s = \varrho(S)$.

2.8. *States of the economy*

Given $S \subset N_r$ and $\mathcal{S} = \{S_1, \ldots, S_k, \ldots, S_K\}$, a jurisdiction structure of S, a *state of the economy for S relative to $\mathcal{S}$* is an ordered pair $\psi(\mathcal{S}) = (\alpha(\mathcal{S}), (\beta(\mathcal{S}), z))$ where $\alpha(\mathcal{S}) = (x^S, y^S)$ is an allocation for S relative to $\mathcal{S}$ and $(\beta(\mathcal{S}), z)$ is a production for S relative to $\mathcal{S}$ such that, given $\beta(\mathcal{S}) = \prod_{k=1}^{K} (x_k, z_k)$, for each S_k we have $x_k = x^{tq}$ for all $(t,q) \in S_k$ (the consumption of the public goods by a member of a jurisdiction must equal production of public goods by that jurisdiction). The state of the economy for S relative to $\mathcal{S}$ is *$c(\varepsilon)$-feasible for S* if

$$\sum_{tq \in S} (y^{tq} - w^{tq}) \le z + \sum_{k=1}^{K} z_k + \sum_{k=1}^{K} c(S_k;\varepsilon).$$

It is *feasible* for S if

$$\sum_{tq \in S} (y^{tq} - w^{tq}) \le z + \sum_{k=1}^{K} z_k.$$

Given a state of the economy for η_r of N_r, say $\psi(\eta_r) = (\alpha(\eta_r), (\beta(\eta_r), z))$, a coalition can *$c(\varepsilon)$-improve upon* $\psi(\eta_r)$ if there is a jurisdiction structure of S, say $\mathcal{S}$, and a state of the economy for S of $\mathcal{S}$, say $\psi'(\mathcal{S}) = (\alpha'(\mathcal{S}), (\beta'(\mathcal{S}), z'))$, such that $\psi'(\mathcal{S})$ is $c(\varepsilon)$-feasible for S and $\alpha'(\mathcal{S}) \succ_S \alpha(\eta_r)|_S$.

A state of the economy, $\psi(\eta_r)$, is in the *$c(\varepsilon)$-core* (or in the *core relative to coalition formation costs*) if it cannot be $c(\varepsilon)$-improved upon by any subset S of N_r.

We remark that the notation of the $c(\varepsilon)$-core for economies with local public goods was introduced in Wooders (1986), where under less restrictive conditions than used in this paper, it is shown that the $c(\varepsilon)$-core is nonempty for all sufficiently large replications.

Given that resources are used up in jurisdiction and coalition formation, a state of the economy in the $c(\varepsilon)$-core is optimal. Alternatively, one could interpret the $c(\varepsilon)$-core as an approximate core concept and state of the economy in the $c(\varepsilon)$-core as 'approximately' optimal for 'small' ε.

3. The competitive equilibrium

In this section we introduce and discuss our competitive equilibrium notion. Since the equilibrium depends on jurisdiction formation costs which, in turn, depend on the value of a parameter, ε, we call the equilibrium a competitive ε-equilibrium.

A *price system for private goods* is a vector $p \in R^L_+$ where $p \gg 0$. A *net dis-*

crimination price system is a vector $\pi=(\pi^{11},\dots,\pi^{tq},\dots,\pi^{Tr})\in R^{Tr}$. A *personalized price system for public goods* is a vector $\gamma=(\gamma^{11},\dots,\gamma^{tq},\dots,\gamma^{Tr})$ where $\gamma^{tq}\in R^M_+$. The mth coordinate of $\gamma^{tq}=(\gamma^{tq}_1,\dots,\gamma^{tq}_m,\dots,\gamma^{tq}_M)$ is interpreted as the price paid by the (t,q)th agent for each unit he consumes of the mth public good.

The equilibrium is defined for the general case of discriminatory crowding. We then consider an alternative notion of equilibrium which serves to clarify the role of the net discrimination prices.

A *competitive ε-equilibrium* or simply an *equilibrium* is an ordered quadruple $e=(\psi(\eta_r),p,\pi,\gamma)$ consisting of a state of the economy $\psi(\eta_r)=(\alpha(\eta_r),(\beta(\eta_r),z))$; a price system for private goods, p; a net discrimination price system, π; and a personalized price system γ, such that

(i) $\sum_{tq\in N_r}(y^{tq}-w^{tq})\le\sum_{g=1}^{G} z_g+z$ ($\psi(\eta_r)$ is feasible);
(ii) $p\cdot z\ge p\cdot z'$ for all $z'\in Y_1$ (profit maximization in private goods production);
(iii) for each $J_g\in\eta_r$,

$$\sum_{tq\in J_g}\gamma^{tq}\cdot x_g+p\cdot z_g\ge\sum_{tq\in J_g}\gamma^{tq}\cdot x'+p\cdot z' \text{ for all } (x',z')\in Y_0[J_g]$$

(profit maximization in public goods production);
(iv) For each $(t,q)\in N_r$, $p\cdot(y^{tq}-w^{tq})+\gamma^{tq}\cdot x^{tq}=\pi^{tq}$ and if $(x',y';J_g)\succ_{tq}(x^{tq},y^{tq};J_g)$, then $p\cdot(y'-w^{tq})+\gamma^{tq}\cdot x'>\pi^{tq}$ (given the jurisdiction structure and prices, agents satisfy their budget constraints and optimize);
(v) If, for some $S\subset N_r$, $\alpha'(\{S\})=(x'^S,y'^S)$ is an allocation for S relative to $\{S\}$ and $\alpha'(\{S\})\succ_S\alpha(\eta_r)|_S$, then for all z' such that $(x',z')\in Y_0[S]$, where $x'=x'^{tq}$ for (any) $(t,q)\in S$, $p\cdot\sum_{tq\in S}(y'^{tq}-w^{tq})-p\cdot c(S;\varepsilon)-p\cdot z'>0$ (no subset of agents can afford a preferred allocation in another jurisdiction consisting of members of that subset, after paying jurisdiction-formation costs of $p\cdot c(S;\varepsilon)$).

The rationale for calling the net discrimination prices for agents by that name is that (1) given these prices, no potential jurisdiction would be able to pay a price to an agent sufficiently large to attract that agent to that jurisdiction and (2) no jurisdiction in the equilibrium jurisdiction structure would prefer to do without an agent rather than pay that agent's price.

Condition (v) is similar to the 'no improvement upon' aspect of the core. However, (v) takes prices of private goods as given, unlike the core. Also, 'improvement' *for the core* typically will require a coalition consisting of several jurisdictions while (v) only involves single jurisdictions. Moreover, for our existence and convergence theorems, we need only consider jurisdictions S (in (v)) bounded in size, $|S|$, by some 'minimum efficient scale'. In large economies, it is reasonable to suppose small groups are price-takers.

We note that it is easy to see that (v) holds if and only if no firm can enter into local goods production and earn a positive profit. A firm could enter (charging possibly different prices to different buyers of its output) and sell its output, say x', to a set of buyers, say S, so that all these buyers would be at least as well off as in the equilibrium state if an only if (v) is not satisfied. To view the equilibrium concept an entry equilibrium, one could interpret $c(S;\varepsilon)$ as set-up costs, marketing costs, or a fixed cost.

To describe our alternative competitive equilibrium concept, we require an additional definition. A *complete personalized price system* is a set $\Gamma=\{\gamma^{tq}(S)\in R^M_+ : (t,q)\in N_r \text{ and } (t,q)\in S\subset N_r\}$. The set Γ consists of a price for each public good for each agent for each possible coalition containing that agent. The equilibrium is required to satisfy (i) through (iv) above. In addition, it must satisfy:

(vi) for each nonempty subset S of N_r,

$$\max\left\{\sum_{tq\in S}\gamma^{tq}(S)\cdot x'+p\cdot z' : (x',z')\in Y_0[S]\right\}$$

is nonpositive;

(vii) given all prices, there does not exist a coalition S, a set of side-payments $\{b^{tq}: (tq)\in S\}$ with $\sum_{tq\in S} b^{tq}=0$, and, for each (t',q') in S, and (x',y') in $X^{t'}$ such that $(x',y';S)\succ_{t'q'}(x^{t'q'},y^{t'q'};\eta_r)$, and $p\cdot(y'-w^{t'q'})+\gamma^{t'q'}(S)\cdot x'=b^{t'q'}$ (there does not exist a coalition S and a set of 'income transfers' among the members of S which would enable each member of the coalition to be 'better off').

It is easy to see that a state of the economy which is an equilibrium state for the above type of equilibrium is also a competitive ε-equilibrium state and in section 5 we will discuss this relationship further.

The intuitive idea of the alternative equilibrium is as follows. Given prices for goods we can associate an amount of 'income' with each coalition, say $v(S)$ for the coalition S. This amount, $v(S)$, can be interpreted as the maximum amount of income which the coalition S can forego while keeping its members at least as well off as they are in the equilibrium state. The pair (N,v) is a game with side-payments with the property that $v(S)\leq 0$ for all coalitions S and, for all coalitions S in the equilibrium jurisdiction structure η_r, we have $v(S)=0$. The net discrimination prices are in the core of this game. These properties are demonstrated in the final section.

3.1. Core-equilibrium equivalence

Since our existence theorem for the competitive ε-equilibrium follows from our equivalence theorem, in this subsection we consider the equivalence results and, in the next, the existence results.

We first, however, show that the equilibrium states of the economy are in the $c(\varepsilon)$-core.

Theorem 1. *Assume that $e=(\psi(\eta_r),p,\pi,\gamma)$ is a competitive ε-equilibrium. Then $\psi(\eta_r)$ is in the $c(\varepsilon)$-core.*

Proof. Assume that $e=(\psi(\eta_r), p, \pi, \gamma)$ is a $c(\varepsilon)$-equilibrium and suppose $\psi(\eta_r)=(\alpha(\eta_r), (\beta(\eta_r), z))$ is not in the $c(\varepsilon)$-core. Then for some $S \subset N_r$ and some jurisdiction structure $S=\{S_1, \ldots, S_k, \ldots, S_K\}$ of S, there is a state for S of S, say $\psi'(S)=(\alpha'(S), (\beta'(S), z'))$, such that $\psi'(S)$ is $c(\varepsilon)$-feasible for S of S and $\alpha'(S) \succ_S \alpha(\eta_r)|_S$.

Since $\psi'(S)$ is $c(\varepsilon)$-feasible,

$$\sum_{tq \in S} (y'^{tq} - w^{tq}) \leq \sum_{k=1}^{K} z'_k + z' + \sum_{k=1}^{K} c(S_k; \varepsilon).$$

Since Y_1 is a closed convex cone, we have $p \cdot z' \leq 0$. Multiplying the $c(\varepsilon)$-feasibility inequality for $\psi'(S)$ by p and dropping the term $p \cdot z'$, we obtain

$$p \cdot \sum_{tq \in S} (y'^{tq} - w^{tq}) \leq p \cdot \sum_{k=1}^{K} z'_k + p \cdot \sum_{k=1}^{K} c(S_k; \varepsilon).$$

In particular, for some $S_{k'} \in S$, we have

$$p \cdot \sum_{tq \in S_{k'}} (y'^{tq} - w^{tq}) - p \cdot c(S_k; \varepsilon) \leq p \cdot z'_{k'}.$$

However, this contradicts (v) of the definition of equilibrium. □

Before stating our next theorem, we need to define 'replicas' of a state of an economy. First, we define replicas of a jurisdiction structure.

Given r', a jurisdiction structure $\eta_{r'}=\{J_1, \ldots, J_g, \ldots, J_G\}$ of $N_{r'}$, and a positive integer n, let $r=nr'$. Let $n\eta_{r'}$ be a jurisdiction structure of N_r containing nG jurisdictions, say $n\eta_{r'}=\{J_{gj}: g=1, \ldots, G, j=1, \ldots, n\}$ where $J_{gj}=\{(t,q)$: for some $(t,q') \in J_g, q=(j-1)r'+q'\}$ for all $g \in \{1, \ldots, G\}$ and $j \in \{1, \ldots, n\}$. Informally $n\eta_{r'}$ consists of n 'copies' of $\eta_{r'}$. Note that for each j, the profile of J_{gj} equals the profile of J_g. We call $n\eta_{r'}$ the nth *replica of* $\eta_{r'}$. This definition of a replica of a jurisdiction structure is more restrictive than is actually required. Essentially, we need only that the nth replica of $\eta_{r'}$ contains n jurisdictions with the same profile as $J_g \in \eta_{r'}$ for each J_g. The additional restriction, that $(t,q) \in J_{gj}$ when $(t,q') \in J_g$ and $q=(j-1)r'+q'$, simplifies notation and subsequent definitions.

Let $\psi(\eta_{r'})=(\alpha(\eta_{r'}), (\beta(\eta_{r'}, z))$ be a state of the rth economy where $\eta_{r'}=\{J_1, \ldots, J_g, \ldots, J_G\}$. Given a positive integer n, let $r=nr'$ and let $n\eta_{r'}=\{J_{11}, \ldots, J_{gj}, \ldots, J_{Gn}\}$ denote the nth replica of $\eta_{r'}$. Define S_j by $S_j=\{J_{1j}, \ldots, J_{gj}, \ldots, J_{Gj}\}$ and S_j by $S_j=\bigcup_{g=1}^{G} J_{gj}$. Observe that S_j has the same profile as $N_{r'}$, and that S_j is a 'copy' of $\eta_{r'}$. Let $\psi'(\eta_r)=(\alpha'(\eta_r), (\beta'(\eta_r), z'))$ denote a state of the rth economy. Then $\psi'(\eta_r)$ is the nth *replica of* $\psi(\eta_{r'})$ if:

(i) $\eta_r=n\eta_{r'}$ (η_r is the nth replica of $\eta_{r'}$);
(ii) For each $j \in \{1, \ldots, n\}, \alpha'(\eta_r)|_{S_j}=\alpha(\eta_{r'}), \beta'(\eta_r)|_{S_j}=\beta(\eta_{r'})$ and $z'=nz$ (each copy of the set of agents of $N_{r'}$ has the same allocation and production as the r'th economy).

Given $\psi(\eta_{r'})=(\alpha(\eta_{r'}),(\beta(\eta_{r'}),z))$, where $\alpha(\eta_{r'})=(x^{N_{r'}},y^{N_{r'}})$ and $\beta(\eta_{r'})=\prod_{g=1}^{G}(x_g,z_g)$, we denote the nth replica of $\psi(\eta_{r'})$ by $\psi_n(\eta_{r'})=(\alpha_n(\eta_{r'}),(\beta_n(\eta_{r'}),nz))$ where $\alpha_n(\eta_{r'})=(x^{N_r},y^{N_r})$ with $r=nr'$, $x^{tq}=x^{tq'}$ and $y^{tq}=y^{tq'}$ whenever $q=(j-1)r'+q'$ for some $q'\leq r'$ and some $j\in\{1,\dots,n\}$, and $\beta_n(\eta_{r'})=\prod_{g=1}^{G}\prod_{j=1}^{n}(x_g,z_g)$.

We say that a *state of the* rth *economy*, $\psi(\eta_r)$, *is in the* $c(\varepsilon)$*-core for all replications of the economy* if, for each n, $\psi_n(\eta_r)$ is in the $c(\varepsilon)$-core of the nrth replica economy, where $\psi_n(\eta_r)$ is the nth replica of $\psi(\eta_r)$.

Theorem 2. *Given r there is an ε^* such that, for all ε with $0\leq\varepsilon\leq\varepsilon^*$, if $\psi(\eta_r)$ is in the $c(\varepsilon)$-core for all replications of the economy, then $\psi(\eta_r)$ is a competitive ε-equilibrium state of the economy, i.e., there is a price system for private goods, p, a net discrimination price system, π, and a personalized price system for public goods, γ, such that $e=(\psi(\eta_r),p,\pi,\gamma)$ is a competitive ε-equilibrium.*

We observe that, given r, the upper bound on ε in Theorem 2 is any ε^* such that, for all ε with $0\leq\varepsilon\leq\varepsilon^*$, for all (t,q) and for all $S\subset N_r$ with $(t,q)\in S$, we have $-c(S;\varepsilon)/|S|\ll w^{tq}$. Informally, ε^* must be sufficiently small so that each agent can contribute, from the agent's endowment, the per-capita inputs required to form any jurisdiction containing that agent. From the assumptions on jurisdiction-formation costs, the selection of such an ε^* is possible.

Theorem 2 and all remaining theorems are proven in Section 5.

3.2. *Existence of the competitive ε-equilibrium*

For our remaining theorems, we require some additional assumptions, in particular, that of a 'minimum efficient scale for jurisdictions', MES, and 'overriding desirability' of private goods. Informally, the MES assumption is that for some sufficiently large economy, say N_{r^*}, agents can do as well by forming jurisdictions with profiles less than or equal to that of N_{r^*} as they can by forming large jurisdictions. For convenience, the assumption is made in a strong form. What seems essential for the results is that, given prices for private goods, the equal-treatment payoffs in the cores of the derived games are bounded. Given that the model and techniques of this paper (and those of Wooders, 1986) are novel and the model is complicated, using the assumption in a strong form, which facilitates the statements and proofs of the theorems, seems justified. The overriding desirability assumption is that private goods can substitute for public goods and crowding effects and are essential for consumption.

Formally, the sequence of economies has the *minimum efficient scale*, MES, *property* if there is an r^* such that given any $r\geq r^*$, for any $\varepsilon\geq 0$ if $\psi(\eta_r)$ is a $c(\varepsilon)$-feasible state of the economy with associated allocation (x^{N_r},y^{N_r}), then there is a $c(\varepsilon)$-feasible state of the economy, say $\psi(\eta_r')$ with associated allocation (x'^{N_r},y'^{N_r}), such that

(a) $(x'^{tq},y'^{tq},\eta_r')\succsim_{tq}(x^{tq},y^{tq},\eta_r)$ for all (t,q) in N_r and

(b) for all $S \in \eta_r'$, we have $\varrho(S) \leq \varrho(N_{r^*})$.
In this case we say r^* is an *MES bound*.

We also assume that given any t, any s and s' in $I(t)$ and any (x,y) and (x',y') in X^t, there is a y'' such that $(x,y'') \in X^t$ and $(x',y'';s') \succsim_t (x,y;s)$, the *overriding desirability property*.

Theorem 3. *Assume that the sequence of economies satisfies the MES and overriding desirability of private goods properties. Then, given any $\varepsilon \geq 0$, there is an r^0 such that the competitive ε-equilibrium exists for the* rth *economy for all $r = kr^0$ for each positive integer k.*[1]

In Wooders (1988, Theorem 3) conditions are given under which there is an r^0 such that the core of the kr^0th replica economy is nonempty for all positive integers k. Moreover, there are states in the core of the kr^0th economy for all k which are replicas of a state in the core of the r^0th economy. Theorem 2 of this paper gives conditions under which such states are competitive ε-equilibrium states (for $\varepsilon \geq 0$). Therefore our existence theorem is a corollary of Theorem 2 and Wooders (1988, Theorem 3).

4. The Tiebout theorem

As stated in the introduction, in this section we address the following question. Suppose local public goods are provided by governments charging lump-sum taxes. Then what properties must the jurisdiction structures and assignment of taxes have to ensure that no firm could profitably enter? Here, entry entails setting up a new jurisdiction.

A *tax assignment* is a vector $\tau = (\tau^{11}, \dots, \tau^{tq}, \dots, \tau^{Tr}) \in R^{N_r}$. An *admission fee* is a real number $\delta \in R_+$.

A *Tiebout ε-equilibrium* (relative to coalition-formation costs) is an ordered triple $e_T = (\psi(\eta_r), p, \tau)$ consisting of a state of the economy $\psi(\eta_r) = (\alpha(\eta_r), \beta(\eta_r), z))$; a price system for private goods, p; and a tax assignment τ with the properties that

(i) $\psi(\eta_r)$ is feasible;
(ii) $p \cdot z \geq p \cdot z'$ for all $z' \in Y_1$;
(iii) $p \cdot (y^{tq} - w^{tq}) = \tau^{tq}$ (consumer budgets are balanced);
(iv) $\sum_{tq \in N_r} \tau^{tq} - p \cdot \sum_{g=1}^{G} z_g = 0$ (the central government balances its budget);
(v) for all subsets S of N_r, there does not exist an $(x', z') \in Y_0[S]$ and a set of admission fees, $\{\delta^{tq}: (t,q) \in S\}$ such that

[1] An early version of parts of this paper, containing Theorem 2, appears in Wooders (1981). The subsequent results on cores and approximate cores in Wooders (1983) and their application to economies with local public goods (1988), enable the existence theorem. It is unclear how more standard techniques of convexity or the convexifying effects of large numbers could be used to obtain existence.

(a) for all $(t,q)\in S$, we have $\delta^{tq}=p\cdot(y'^{tq}-w^{tq})$

(b) $(x', y'^{tq}; S) \succ_{tq} (x^{tq}, y^{tq}; \eta_r)$ for all $(t,q)\in S$.

(c) $\sum_{tq\in S}\delta^{tq}-p\cdot z'-c(S;\varepsilon)\leq 0$

(no firm can find a set of consumers S, admission fees and public goods production such that each consumer in S can afford a preferred allocation and such that the firm earns positive profits).

We remark that the no-entry condition ensures that for 'most' jurisdictions J_g we have

$$\left\| \sum_{tq\in J_g} \tau^{tq} - p\cdot z_g \right\| \leq \|p\cdot c(J_g;\varepsilon)\|;$$

otherwise the no-entry condition would be violated. If we required that each jurisdiction separately satisfied its budget constraints, then we could easily obtain the existence of an approximate equilibrium, but at the expense of relaxing feasibility and substituting instead approximate feasibility (in per-capita terms). Our preference is to maintain feasibility.

Theorem 4. *Let $e_T=(\psi(\eta_r),p,\tau)$ be a Tiebout ε-equilibrium. Then $\psi(\eta_r)$ is in the $c(\varepsilon)$-core.*

It is easy to see that a competitive ε-equilibrium state of the economy is a Tiebout ε-equilibrium state (the opposite, of course, may well not hold). More formally, let $e=(\psi(\eta_r),p,\pi,\gamma)$ be a competitive ε-equilibrium. For each (t,q), define $\tau^{tq}=-\gamma^{tq}\cdot x^{tq}+\pi^{tq}$. Then $e_T=(\psi(\eta_r),p,\tau)$ is a Tiebout ε-equilibrium. Thus, we have the following corollary to Theorem 3.

Corollary. *Assume that the sequence of economies satisfies the MES and overriding desirability of private goods properties. Then, given any $\varepsilon\geq 0$ there is an r^0 such that the Tiebout ε-equilibrium exists for the* rth *economy for $r=kr^0$ for each positive integer k.*

For the one-private-good case, another existence theorem follows from the relationship of the $c(\varepsilon)$-core to the Tiebout ε-equilibrium states.

Theorem 5. *Assume $L=1$ and that the sequence of economies has the MES and overriding desirability properties. Then, given $\varepsilon>0$ there is an r^* such that the Tiebout ε-equilibrium exists for all $r>r^*$.*

We remark that under the conditions of Theorem 5, the set of Tiebout ε-equilibrium states of the economy is equivalent to the $c(\varepsilon)$-core.

Up to this point, we have introduced the notions of competitive ε-equilibrium and of Tiebout ε-equilibrium and obtained existence results for both. These results are of independent interest since, with nondiscriminatory crowding, no analogous existence results appear in the literature. Moreover, the methods used to obtain these results may well be applicable in a variety of models with endogenous group formation (clubs, or firms, for example). However, the main economic result of this paper is stated in the following theorem, our 'Tiebout Theorem'. The interpretation is, informally, that the Tiebout ε-equilibrium states converge to the 'market type' competitive ε-equilibrium states. More formally, if a state of the economy is a Tiebout ε-equilibrium state for all replications of the economy, it is a competitive ε-equilibrium state.

A Tiebout Theorem (Theorem 6). *Suppose $\psi(\eta_r)$ is a Tiebout ε-equilibrium state for all replications of the economy. Then $\psi(\eta_r)$ is a competitive ε-equilibrium state.*

5. Proofs

5.1. Proof of Theorem 2

The proof of the theorem is an adaptation of proofs of convergence of the core to equilibrium states due to Debreu and Scarf (1963) and Foley (1970), and of some techniques developed by Wooders (1983). For ease in notation, we prove the theorem for the original (unreplicated) economy. There is no loss of generality since given r the agents of N_r can be renumbered so that there is only one agent of each type.

Assume that $\psi(\eta)=(\alpha(\eta),(\beta(\eta),z))$ is a state of the economy in the $c(\varepsilon)$-core for all replications of the economy, with

$$\eta=\{J_1,\dots,J_g,\dots,J_G\}, \alpha(\eta)=(x^N,y^N), \quad \text{and} \quad \beta(\eta)=\sum_{g=1}^{G}(x_g,z_g).$$

For ease in notation, denote a member of N, $(t,1)$, simply by t.

Preliminaries. We first extend the commodity space for public goods, similarly to Foley's (1970) extension of the commodity space for pure public goods. However, we extend the space to R^{MTK} where K is the number of nonempty subsets of N so that the number of commodities in the extended space is the number of public goods, M, times the number of agents, T, times the number of nonempty subsets of N.

Let $\{S_1,\dots,S_k,\dots,S_K\}$ denote the set of all nonempty subsets of N and, for each k, let s_k be the profile of S_k. Let $a=(a_1,\dots,a_k,\dots,a_K)\in R^{MTK}$ be a vector where $a_k=(a_k^1,\dots,a_k^t,\dots,a_k^T)\in R^{MT}$ for each k and $a_k^t\in R^M$. Let A_k be the set of elements in R^{MTK} which have the property that $a_{k'}=0$ for all $k'\neq k$, and $a_k^t=0$ for all t not in S_k so $A_k=\{a\in R^{MTK}: a_{k'}^t=0 \text{ if } k'\neq k \text{ or if } t\notin S_k\}$. Note that we have not required that $a_k^t=a_k^{t'}$ for members of A_k.

As above, let $a_k=(a_k^1,\ldots,a_k^t,\ldots,a_k^T)$ denote an element of R^{MT} where $a_k^t\in R^M$ for each t. Define $Y_0'[S_k]$ as the set of elements $(a_k,z)\in R^{MT+L}$ having the properties that: $a_k^t=a_k^{t'}$ for all t and t' in S_k; $a_k^t=0$ if t is not in S_k; and for (any) t in S_k, $(a_k^t,z)\in Y_0[S_k]$.

Define Y as the set of elements $(a,b)\in R^{MTK+L}$ such that for some $z'\in Y_1$, and, for each k, for some $(a_k,z_k')\in Y_0'[S_k]$, we have $a=\prod_{k=1}^K a_k$ and $b=\sum_{k=1}^K z_k'+z'$. The set Y is the 'aggregate production set' in an 'extended' commodity space. Since $Y_0[S_k]$ is a closed convex cone with vertex zero for each S_k and Y_1 is also a closed convex cone with vertex zero, Y is a closed convex cone with vertex zero.

Select ε^* such that for all ε with $0\le\varepsilon\le\varepsilon^*$, for all t and for all $S\subseteq N$ with $t\in S$, we have $-c(S;\varepsilon)/|S|\ll w^t$; from the assumptions on jurisdiction-formation costs and since $w^t\gg 0$, this is possible.

Given $\varepsilon\in[0,\varepsilon^*]$, for each S_k such that $S_k\in\eta$ define $c'(S_k)=0\in R^L$ and, if $S_k\notin\eta$ define $c'(S_k)=c(S_k;\varepsilon)$.

The proof now proceeds through several steps.

Step 1. The sets Γ_k. Let Γ_k denote the set of members (a,b) in $A_k\times R^L$ which have the properties that for each $t\in S_k$, there is a $y'^t\in R^L$ such that:
(a) $(a_k^t,y'^t;S_k)\succ_t(x^t,y^t;\eta)$, and
(b) $b=\sum_{t\in S_k}(y'^t-w^t)-c'(S_k)$.

Note that if $(a,b)\in\Gamma_k$, then $a\in A_k$ so we have $a_{k'}=0\in R^{MT}$ for all $k'\neq k$. However, we have not required that $a_k^t=a_k^{t'}$ for all t and t' in S_k, only that $a_k^t=0$ if t is not in S_k.

It is a standard argument to show that Γ_k is convex.

Let $(a,b)\in\Gamma_k$ and suppose (λ^n) is a sequence of positive numbers with $\lambda^n\le 1$ for all n and where (λ^n) converges to 1. We now show that for all n sufficiently large, $\lambda^n(a,b)\in\Gamma_k$. From the definition of Γ_k, for some $y'^t\in R^L$ for each $t\in S_k$ such that

$$\sum_{t\in S_k}(y'^t-w^t)-c'(S_k)=b,$$

we have

$$(a_k^t,y'^t;S_k)\succ_t(x^t,y^t;\eta).$$

Also, we have

$$(0,w^t-c'(S_k)/|S_k|;S_k)\in X^t\times I(t)\quad\text{for each } t\in S_k.$$

Given n,

$$\text{let } y_n'^t=\lambda^n y'^t+(1-\lambda^n)w^t+(1-\lambda^n)c'(S_k)/|S_k|\quad\text{for each } t\in S_k.$$

From the convexity of X^t we have $(\lambda^n a_k^t,y_n'^t)\in X^t$. From the continuity assumption on preferences, it follows that for all n sufficiently large, $(\lambda^n a_k^t,y_n'^t;S_k)\succ_t(x^t,y^t;\eta)$. Also, we have

$$\sum_{t\in S_k}(y_n'^t-w^t)-c'(S_k)=\lambda^n\left(\sum_{t\in S_k}y'^t-w^t\right)-\lambda^n c'(S_k)=\lambda^n b,$$

so we have established the claim.

Step 2. The set Γ. Let Γ denote the convex hull of the union of the Γ_k's. Since Γ_k is convex for each k, Γ is the set of all vectors which can be written

$$\sum_{k=1}^{K} \lambda_k(a^k, b^k)$$

with $\lambda_k \geq 0$, $\sum_{k=1}^{K} \lambda_k = 1$ and $(a^k, b^k) \in \Gamma_k$.

We now show, in the remainder of Step 2, that $\Gamma \cap Y = \emptyset$. Suppose, on the contrary, that $(a,b) \in \Gamma \cap Y$, where $(a,b) = \sum_{k=1}^{K} \lambda_k(a^k, b^k)$ with $\lambda_k \geq 0$, $\sum_{k=1}^{K} \lambda_k = 1$ and $(a^k, b^k) \in \Gamma_k$ for each k. From the definition of Y, there is a $z'_k \in R^L$ for each k and a $z' \in Y_1$ so that $(a,b) = (a, \sum_{k=1}^{K} z'_k + z')$ where $(a_k, z'_k) \in Y'_0[S_k]$ for each k. Let K' be the set of k's for which $\lambda_k > 0$.

Our first problem not of a type encountered by Debreu and Scarf (1963) is that if $\lambda_k = 0$, we cannot simply say that $(a,b) = \sum_{k \in K'} \lambda_k(a^k, b^k)$. It is the case that if $\lambda_k = 0$, then the coordinate of a associated with the subset S_k is zero but it is not necessarily the case that z'_k is zero. However, we will show that without loss of generality, we can suppose $z'_k = 0$ if $\lambda_k = 0$. Observe that if $(a_{k'}, z'_{k'}) \in Y'_0[S_{k'}]$ and $a_{k'} = 0$, then $(a_{k'}, 0) \in Y'_0[S_{k'}]$ so, when $a_{k'} = 0$, we have $(a, \sum_{k \neq k'} z'_k + z') \in Y$. Also, since $z'_k \leq 0$ for all k and from monotonicity of preferences, $\sum_{k=1}^{K} \lambda_k(a^k, b^k - z'_{k'}) \in \Gamma$. (We have added $-\lambda_k z'_{k'}$ to each term in the sum so, in total, we've added $-z'_{k'}$ to the expression.) Consequently, we can assume without loss of generality that $z'_k = 0$ for all $k \notin K'$, and that $(a, \sum_{k \in K'} z'_k + z') = \sum_{k \in K'} \lambda_k(a^k, b^k)$.

We now show that we can form a blocking coalition for some sufficiently large replication. Select a positive integer n which will eventually tend to infinity and, for each k in K', let λ_k^n be the smallest integer which is greater than or equal to $n\lambda_k$. Define $(a^{kn}, b^{kn}) = n\lambda_k(a^k, b^k)/\lambda_k^n$. From the concluding paragraph of the last section, for all n sufficiently large $(a^{kn}, b^{kn}) \in \Gamma_k$; let n satisfy the property that $(a^{kn}, b^{kn}) \in \Gamma_k$ for all $k \in K'$. Observe that λ_k^n/n is a rational number. Let r' be a replication number such that $r'\lambda_k^n/n$ is an integer for all k in K'. Let $\delta_k = r'\lambda_k^n/n$. We now have

$$r'\left(a, \sum_{k \in K'} z'_k + z'\right) = \sum_{k \in K'} \delta_k(a^{kn}, b^{kn}).$$

Let s_k denote the profile of S_k for each S_k such that $k \in K'$ and recall that $\mathbf{1}$ is the profile of N_1. Let r'' be sufficiently large so that $\sum_{k \in K'} \delta_k s_k \leq r''\mathbf{1}$. Consequently $N_{r''}$ contains a subset, say S with profile s, such that $s = \sum_{k \in K'} \delta_k s_k$; i.e., S contains δ_k subsets with profile s_k for each $k \in K'$. From the equation

$$r'\left(a, \sum_{k \in K'} z'_k + z'\right) = \sum_{k \in K'} \delta_k(a^{kn}, b^{kn}),$$

since $a^{kn} \in A_k$ (i.e., $a_{k'}^{kn} = 0$ for all $k' \neq k$), we have $r'a_k = \delta_k a_k^{kn}$ and $a_k^{kn} = (r'/\delta_k)a_k$. Since $Y_0[S_k]$ is a cone with vertex zero,

$$\frac{r'}{\delta_k}(a_k, z'_k) \in Y'_0[S_k] \quad \text{for each } k \in K'$$

i.e., the production set for each jurisdiction S_k, where $k \in K'$, contains $(r'/\delta_k)(a_k, z_k')$. Since

$$b^{kn} = \sum_{t \in S_k} (y'^t - w^t) - c'(S_k),$$

where, for each t, y'^t is some vector satisfying the properties required in the definition of Γ_k, we have

$$\sum_{k \in K'} r' z_k' + r' z' = \sum_{k \in K'} \delta_k \left(\sum_{t \in S_k} (y'^t - w^t) - c'(S_k) \right).$$

Consequently, S can ε-improve upon the r''th replica of $\psi(\eta)$, which is a contradiction. Therefore, $\Gamma \cap Y = \varnothing$.

Step 3. Prices. From the Minkowski separating hyperplane theorem, there is a hyperplane with normal $(\tilde{\gamma}, p)$ and a constant C so that $\tilde{\gamma} \cdot a + p \cdot b \geq C$ for all $(a, b) \in \Gamma$ and $\tilde{\gamma} \cdot a + p \cdot b \leq b \leq C$ for all $(a, b) \in Y$. Since Y is a closed convex cone with vertex 0, we have $C \geq 0$.

Given J_g, let $\tilde{x}_g = (\tilde{x}_{g1}, \ldots, \tilde{x}_{gk}, \ldots, \tilde{x}_{gK})$ denote the vector in R^{MTK} where $\tilde{x}_{gk} = (\tilde{x}_{gk}^1, \ldots, \tilde{x}_{gk}^T) \in R^{MT}$, $\tilde{x}_{gk}^t = 0$ if $S_k \neq J_g$ or if $t \notin J_g$, and $\tilde{x}_{gk}^t = x^t$ otherwise. Let $\tilde{x} = \sum_{g=1}^{G} \tilde{x}_g$; the vector $\tilde{x}$ is simply a representation of the equilibrium public goods allocation x^N in the extended public goods space. We then have $(\tilde{x}, \sum_{g=1}^{G} z_g + z) \geq (\tilde{x}, \sum_{t \in N} (y^t - w^t))$ since $\psi(\eta)$ is feasible. Since, for each g, $(\tilde{x}_g, \sum_{t \in J_g} (y^t - w^t))$ is in the closure of Γ, we have $\tilde{\gamma} \cdot \tilde{x}_g + p \cdot z_g \geq 0$ for each g. However, $(\tilde{x}_g, z_g)$ is in Y so $\tilde{\gamma} \cdot \tilde{x}_g + p \cdot z_g \leq 0$ for each g. Since $(0, z) \in Y$, we also have $p \cdot z \leq 0$. It follows that for each g,

$$\tilde{\gamma} \cdot \tilde{x}_g + p \cdot z_g = 0, \quad \tilde{\gamma} \cdot \tilde{x}_g + p \cdot \sum_{t \in J_g} (y^t - w^t) = 0, \quad \text{and} \quad p \cdot z = 0.$$

We now show that z maximizes profit, $p \cdot z$, in the production of private goods. Suppose there is a $z' \in Y_1$ so that $p \cdot z' > 0$. But then

$$\tilde{\gamma} \cdot \tilde{x} + p \cdot \left(\sum_{g=1}^{G} z_g + z' \right) > 0,$$

which is a contradiction.

For each $S_k \in \eta$ and for each $t \in S_k$ let $\gamma^t = \tilde{\gamma}_k^t$. Define γ by $\gamma = (\gamma^1, \ldots, \gamma^t, \ldots, \gamma^T)$.

To show profit maximization in the production of public goods, assume that for some J_g we have $(x', z') \in Y_0[J_g]$ and

$$\sum_{t \in J_g} \gamma^t \cdot x' + p \cdot z' > \sum_{t \in J_g} \gamma^t \cdot x^t + p \cdot z_g.$$

As in profit maximization in the production of private goods, this contradicts the fact that $(\tilde{\gamma}, p)$ separates Γ and Y.

From monotonicity it follows that $p \gg 0$ and that $\gamma \gg 0$.

For each $t \in N$, define π^t by $\pi^t = p \cdot (y^t - w^t) + \gamma^t \cdot x^t$. Note that $\sum_{t \in J_g} \pi^t = 0$ for each J_g.

We have shown the existence of prices p, π, and γ satisfying properties (ii) and (iii) of the definition of equilibrium. Also, since if for some S and an allocation for S, say $\alpha'(\{S\})$, such that $\alpha'(\{S\}) \succ_S \alpha(\eta)|_S$. we have

$$p \cdot \sum_{t \in S} (y'^t - w^t) - p \cdot c'(S) - p \cdot z' \leq 0,$$

we then obtain a contradiction to the fact that $(\bar{\gamma}, p)$ separates Γ and Y. Therefore, p, π, and γ also satisfy property (v).

We now show that these prices satisfy (iv), the individual maximization condition of the ε-equilibrium. Suppose for some $t' \in J_g \in \eta$ where $J_g = S_k$ and some (x', y') such that $(x', y'; J_g) \succ_t (x^{t'}, y^{t'}; J_g)$ we have $p \cdot (y' - w^{t'}) + \gamma^{t'} \cdot x' \leq \pi^{t'}$. Since $w^{t'}$ is in the interior of $X''^{t'}$, the projection of the goods consumption set on the space of private goods, there is an $(x^0, y^0) \in X^{t'}$ such that $p \cdot (y^0 - w^{t'}) + \gamma^{t'} \cdot x^0 < \pi^{t'}$. It follows that for some (x'', y'') in the segment $[(x^0, y^0), (x', y')]$, we have $(x'', y''; J_g) \succ_{t'} (x^{t'}, y^{t'}; J_g)$ and $p \cdot (y'' - w^{t'}) + \gamma^{t'} \cdot x'' < \pi^{t'}$. However, then

$$\sum_{\substack{t \in S_k \\ t \neq t'}} (p \cdot (y^t - w^t) + \gamma^t \cdot x^t) + p \cdot (y'' - w^{t'}) + \gamma^{t'} \cdot x'' < 0.$$

From monotonicity, it follows that there is y'^t for each $t \in S_k$ with $t \neq t'$ so that $(x^t, y'^t; S_k) \succ_t (x^t, y^t; S_k)$ and so that

$$\sum_{\substack{t \in S_k \\ t \neq t'}} (p \cdot (y'^t - w^t) + \gamma^t \cdot x^t) + p \cdot (y'' - w^{t'}) + \gamma^{t'} \cdot x'' \leq 0.$$

This, however, contradicts the fact that $(\bar{\gamma}, p)$ separates Γ and Y. Therefore, for each J_g and for all $t \in J_g$, for any (x', y') such that $(x', y'; J_g) \succ_t (x^t, y^t; J_g)$, we have $p \cdot (y' - w^t) - \gamma^t \cdot x' > \pi^t$. □

We remark that even though in the above proof we ignored the coordinates of $\bar{\gamma}$ associated with 'jurisdictions' not in η (i.e., quasi-jurisdictions), $\bar{\gamma}$ gives us prices for the public goods for each agent in each possible jurisdiction – both those in η and the quasi-jurisdictions.

5.2. *The alternative defintion of equilibrium*

We now verify our claims concerning the alternative definition of the equilibrium. In particular, under the conditions of Theorem 2 we show that the state of the economy $\psi(\eta_r)$ satisfies conditions (vi) and (vii) of the definition of the alternative competitive equilibrium concept. To do this, we extend our proof of Theorem 2. First, it is obvious that (vi) is satisfied by the complete personalized price system defined (in the obvious manner) from $\bar{\gamma}$.

Suppose (vii) does not hold. In particular, suppose for a coalition S_k there is a set of side payments $\{\beta^t : t \in S_k\}$ with $\sum_{t \in S_k} \beta^t = 0$ such that for each $t \in S_k$, for some (x'^t, y'^t) in X^t, we have

$$(x'^t, y'^t; S_k) \succ_t (x^t, y^t; \eta) \quad \text{and} \quad p \cdot (y^t - w^t) + \gamma^t(S_k) \cdot x^t = \beta^t.$$

Let (a, b) represent this allocation in Γ_k in the obvious manner. It is clear that (a, b) is in the interior of Γ_k; therefore $\tilde{\gamma} \cdot a + p \cdot b > 0$. However, this contradicts the assumption that $\sum_{t \in S_k} \beta^t = 0$. Therefore, (vii) holds.

To show that the net discrimination prices are in the core of a certain game, first, for each S_k, define

$$v(S_k) = -\inf\{(\tilde{\gamma}, p) \cdot (a, b) : (a, b) \in \Gamma_k\}.$$

Clearly, $v(S_k) \leq 0$ and represents the maximum amount of income which the coalition S can forego while keeping its members at least as well off as they are in the equilibrium state. Now suppose π is not in the core of the game (N, v). Therefore there must be some coalition S_k, which can improve upon π; i.e., $\sum_{t \in S_{k'}} \pi_t < v(S_k)$. But this implies that for some (a, b) in the interior of $\Gamma_{k'}$, we have $p \cdot b < 0$, i.e., there is an (a, b) in the interior of $\Gamma_{k'}$ which is affordable by the members of $S_{k'}$, which is a contradiction.

5.3. *The remaining proofs*

Proof of Theorem 4. The general strategy of the proof of this type of theorem is well-known and is used herein in the proof of Theorem 1. Consequently, the proof of this theorem is omitted.

Proof of Theorem 5. When $L = 1$, under our assumptions the set of Tiebout ε-equilibrium states of the economy is equivalent to the $c(\varepsilon)$-core. Thus, this theorem is simply a restatement of Wooders (1986, Theorem 2).

Proof of Theorem 6. The proof of Theorem 6 follows immediately from the fact that a Tiebout ε-equilibrium state is in the $c(\varepsilon)$-core.

6. Relationships to the literature

In this subsection we will consider primarily the relationship of the work in this paper to Bewley's (1981) critique of 'Tiebout's Theory' and also recent related research of Schweizer (1983) and Scotchmer (1984). Discussion of other related literature appears in Bewley's paper. First, however, it is convenient to discuss Tiebout's (1956) conjecture and to compare briefly the model and results in this paper to some previous work of the author.

Tiebout (1956) conjectured that when public goods are local rather than pure, competitive forces tend to make local governments provide the public goods in a near-optimal manner. He also hypothesized (p. 418) that 'the greater the number of communities and the greater the variety among them, the closer the consumer will come to fully realizing his preference position'. Given the diversity of interpretations in the literature of Tiebout's conjectures, there are several aspects of his paper that we wish to stress. First, as the above quotation indicates, Tiebout was making conjectures about 'large' economies with 'many' jurisdictions and about convergence properties. Second, he conjectured that a *near*-optimal, 'market-type' equilibrium would arise in sufficiently large economies. Third, economies with local public goods 'have mobility as a cost of registering demand. The higher the cost, *ceteris paribus*, the less optimal the allocation of resources' (Tiebout, 1956, p. 422). Motivated by the core-equilibrium equivalence theorems justifying the hypothesis of perfect competition in exchange economies, we have adopted a particular approach to Tiebout's conjectures. While our approach may not be exactly that imagined by Tiebout, we believe it is just to call our equivalence result a 'Tiebout Theorem'.

The model and results in this paper are related to those in Wooders (1980) but the model in the current work is significantly more general. In particular, in the previous work, preferences and/or production possibilities depend only on the number of agents jointly consuming and producing the local public goods, with the result that most jurisdictions associated with equilibrium states are nearly homogeneous and agents in the same jurisdiction pay the same taxes. In this paper, because preferences and/or production possibilities can depend on the size and composition of the jurisdiction, equilibrium jurisdiction structures may well have only heterogeneous jurisdictions. In Wooders (1980), as in this paper, a not-necessarily-optimal equilibrium with lump-sum taxation is shown to exist for all sufficiently large economies and to 'converge' to an optimal, competitive equilibrium.

In a recent paper, Bewley (1981) points out three major problems with the current state of Tiebout-type models, the major one being the existence and optimality of equilibrium. Also, he questions the motivation for the assumption of profit maximization by local governments (a problem we avoid) and the realism of models with homogeneous equilibrium jurisdictions.

Our results concerning existence and optimality of equilibrium may seem at odds with the conclusions of Bewley. Bewley argues that to obtain existence of Pareto-optimal equilibria in economies with local public goods, 'one is obliged to strip the problem of all its distinguishing characteristics and reduce it to a problem already solved in general equilibrium theory' (Bewley, 1981, p. 736).

There is in fact no contradiction between our results and Bewley's examples. In particular, while we can demonstrate conditions under which the competitive equilibrium relative to zero coalition formation costs exists for a *subsequence* of economies, without further assumptions it appears that usually exact competitive equilibrium may well not exist. Moreover, except for considering some examples with a continuum of agents, Bewley does not investigate large economies. In our model, *both* the number of agents and the number of jurisdictions may become large (our model in this respect is similar to Tiebout's special case).

Bewley's example also serves to illustrate that, in addition to 'convexifying' assumptions, additional assumptions are required to obtain the existence of Pareto-optimal equilibrium in economies with 'essential' coalition structures (ones with local public goods in particular). In the coalition production literature, where again there are essential coalition structures, 'balancedness' assumptions are standard; these assumptions ensure that the games derived from the economies have nonempty cores. Also, number-theoretic assumptions can be made to ensure that agents can be partitioned into 'optimal' jurisdictions; these are also balancedness assumptions. This suggests that to obtain the existence of approximate equilibrium in large economies with essential coalition structures, one needs to show that, in some sense, the economies become approximately 'balanced'. (This, in fact, is done for a more restrictive class of economies with local public goods in Wooders (1980).)

The existence results in this paper and Wooders (1986) are partially based on results in Wooders (1983) showing that large games are approximately balanced. It appears that the 'balancifying' effects of large numbers plays an analogous role, in economies with essential coalition structures, to the 'convexifying' effect of large numbers in private goods exchange economies.[2] It is when one attempts to solve Tiebout's problem by using only convexifying techniques from private good exchange economy analysis that one seems forced to reduce the problem to one already solved in private goods exchange economy equilibrium theory.

One other point which Bewley stresses is that for the existence of Pareto-optimal equilibrium one must have homogeneous communities – i.e., each community must contain only identical individuals. This arises in Bewley's model. It also occurs in Wooders (1980) because preferences and/or production possibilities depend only on the number of agents in jurisdictions (nondiscriminatory crowding). As Bewley remarks, homogeneous communities are contrary to our experience. The author sees two separate issues here. First, the situations we are attempting to model may be ones where heterogeneity is desirable (and clearly observed) because of interactions of the characteristics of agents with production and/or consumption (discriminatory crowding); this situation is permitted in our model. Second, although we do not see *exactly* homogeneous jurisdictions, we do see jurisdictions consisting primarily of nearly identical agents (or, at least, nearly-identical in some respects[3]). For example, many suburbs mainly consist of similar residences. The question then becomes whether or not an economy where all agents may differ can be approximated by one with 'types'; it appears that in diverse economic situations, the answer is in the affirmative (see Kaneko–Wooders (1985) where, in the proofs, large games are approximated by ones with types).

The conclusion of this author, however, is similar to that of Bewley; Tiebout's theory is applicable only when local public goods are essentially 'private'. However,

[2] To see that convexity, including divisibility of goods *and* agents, does not suffice for balancedness to obtain, see Shubik and Wooders (1983).

[3] As shown in Wooders (1978, Theorem 3), the theory only requires that individuals in the same jurisdiction have the same *equilibrium demands* for public goods.

the view of this author is that the sense in which the local public goods must be essentially private is that: 'optimal' jurisdictions must be 'small'; jurisdiction formation must be 'easy'; and the local public goods can be provided as well by profit-maximizing private firms as by governments (we must permit *entry*). We view the model herein as more closely related to a competitive model of an economy with differentiated products and entry (with entry costs) rather than an Arrow–Debreu private goods economy.

The spirit of this conclusion is consistent with and reinforced by recent results by Scotchmer (1984). She considers noncooperative (Nash) equilibrium with entry and shows that the noncooperative solution approaches the cooperative solution, in the context of a model similar to that in Wooders (1980).[4]

This model and the competitive equilibrium results of this paper are also related to those of Schweizer (1983). Schweizer considers a situation where there is some given supply of public goods for a club, and the composition of the club (in terms of the numbers of members of each type of agent) is determined endogenously. He shows that his 'optimal' club has a dual characterization in terms of prices for private goods. In several respects our model generalizes that of Schweizer and 'closes' the model by determining the jurisdiction structure and public good provision endogenously, by taking into account the total number of agents of each type, and by considering the efficiency of the equilibrium state of the economy (rather than of one club in isolation). A similarly arises in our methods of proof of the existence of supporting prices in that we both use (different) variations on the technique of Debreu and Scarf (1963). To obtain our existence of equilibrium results however, since we have endogenized jurisdiction structures, we use in addition results on nonemptiness of approximate cores of large games (and economies).

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