An exact bound on epsilon for nonemptiness of epsilon cores of games.*

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Abstract

We consider collections of games with and without side payments described by certain natural parameters. Given the parameters π describing a collection of games and a lower bound n_0 on the number of players, we obtain a bound $\varepsilon_0(\pi, n_0)$ so that, for any $\varepsilon \geq \varepsilon_0(\pi, n_0)$, all games in the collection with at least n_0 players have nonempty ε-cores. Examples are provided in which the bound on ε is met. For parameter values ensuring that there are many close substitutes for most players and that relatively small groups of players can realize nearly all gains to collective activities, for games with many players the bound on ε is small.

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1 Introduction.

The core, the set of feasible outcomes of a social or economic situation that cannot be improved upon by any coalition of players, is a fundamental equilibrium concept. If, using only their own resources, the members of some group of individuals could improve upon an outcome of social or economic activities for themselves, then it seems reasonable to suppose that they would do so. Except in highly stylized situations, the core may be empty. This has motivated conditions ensuring nonemptiness of approximate cores in economies and games with many players; see Shapley and Shubik (1966) for nonemptiness of approximate cores of economies and, for games with and without side payments, Wooders (1979,1983). Kannai (1992) provides a survey of some of the subsequent literature.

In this paper we provide a new model of cooperative games, sufficiently general to encompass a variety of social and economic situations, and show that approximate cores are nonempty. The goodness of the approximation depends on certain natural parameters describing the games. Examples are provided where the closeness of the approximation cannot be improved. For choices of parameter values ensuring that there are many close substitutes for each player and that relatively small groups of players can realize nearly all gains to collective activities, for games with many players the approximation is close.

Specifically, a collection of cooperative games is parameterized by (a) a number of approximate player types and the accuracy of this approximation; (b) an upper bound on the size of near-effective groups of players and the closeness of these groups to being effective for the realization of all gains to collective activities; (c) a bound on the supremum of per capita payoffs achievable in coalitions; and (d) a measure of the extent to which boundaries of payoff sets are bounded away from being "flat." These parameters are fixed independently of the size of the game; thus the collection may contain arbitrarily large games. A simple example of a parameterized collection of games is the set of all games with transferable utility where, given some positive real number K, within each game all players are identical, only two-player coalitions are effective, and any two-player coalition can earn some payoff less than or equal to $2K$. Note that the payoff to two-player coalitions may differ between different games in the collection. Our notion of a parameterized collection of games allows games without side payments. Except for the condition that all games in the collection are described by the same parameters, all the games may differ.

Given the parameters π describing a collection of games and given a lower bound n_0 on the number of players in each game in the collection, we obtain a bound $\varepsilon_0(\pi, n_0)$ so that, for any $\varepsilon \geq \varepsilon_0(\pi, n_0)$, all games in the collection with at least n_0 players have nonempty ε-cores. Examples are provided in which the bound on ε is met. Our results also imply that given any positive real number ε_1 strictly greater than the sum of (i) the bound on the measures of the difference between players of the same approximate type and (ii) the difference of nearly effective bounded-sized groups from being fully effective, there is a lower bound n_1 on the number of players so that all games in the collection with more than n_1 players have nonempty ε_1-cores.

The motivation for this paper stems from economics, and in particular, from the study of competition in diverse economic settings, such as those with local public goods, clubs, production, location, indivisibilities, non-monotonicities, and other deviations from the classic Arrow-Debreu-McKenzie model. In the context of private goods exchange economies, bounds on measures of differences of economic outcomes from price-taking equilibrium outcomes has long been an ongoing theme in the general equilibrium literature; see Anderson (1978), for example, and references therein. Our results contribute to a line of research investigating the "market-like" properties of large games under conditions limiting returns to group size, cf. Wooders (1983,1994a,1999). In large games the core is a stand-in for the competitive equilibrium. Rather than treating properties of price-taking allocations of economies, our results provide an exact bound on the distance of approximate cores of games from satisfying the no-improvement property of the core. Like Anderson (1978), our results apply to given games rather than sequences of games and apply to any given game satisfying the conditions of the Theorems. Unlike detailed models of economies, such as those of Anderson (1978) for exchange economies or Conley (1994) for pure public goods economies, models of large games can accommodate the entire spectrum from games derived from private goods economies to games derived from economies with pure public goods.

This paper contrasts with our other research on approximate cores of large games (Kovalenkov and Wooders 1999a,b), in that in the present paper we obtain an *exact bound* on ε for games to have nonempty equal treatment ε-cores. Also in the current paper we use techniques that are completely different from our previous research, but instead related to those of Scarf (1965), an earlier unpublished version of his well-known paper, Scarf (1967), showing that balanced games have nonempty cores. (See also Billera 1970). To obtain our results we extend Scarf's techniques from simply the negative and positive orthants to recession cones and their dual negative cones. Our extension allows us to use the "sum" norm to define nearly effective groups. This broadens the class of games covered. Moreover, as we demonstrate with an example, the use of the sum norm may significantly decrease the bound on ε. Our main result, however, depends on convexity of payoff sets, not required in Kovalenkov and Wooders (1999a). In contrast to Kovalenkov and Wooders (1999b) our current results include games with unlimited side payments, for example, games with transferable utility. In fact, the results in this paper build on those for games with transferable utility. Relationships to previous research are developed further later in the paper.

The next section of this paper presents our model. Section 3 provides results and examples. Section 4 gives the sketch of the proof. Related literature, possible applications and further motivation are presented in the concluding section of the paper. We note here only that our work is related in spirit to the "least ε-core," introduced by Maschler, Peleg, and Shapley (1979), since we obtain a lower bound on ε ensuring that the ε-core is nonempty.

2 Definitions.

2.1 Cooperative games: description and notation.

Let $N = \{1, ..., n\}$ denote a *set of players.* A nonempty subset of N is called a *coalition.* For any coalition S let $\mathbf{R}^S$ denote the $|S|$-dimensional Euclidean space with coordinates indexed by elements of S. For $x \in \mathbf{R}^N, x_S$ will denote its restriction to $\mathbf{R}^S$. To order vectors in $\mathbf{R}^S$ we use the symbols $>>$, $>$ and $\geq$ with their usual interpretations. The non-negative orthant of $\mathbf{R}^S$ is denoted by $\mathbf{R}^S_+$ and the strictly positive orthant by $\mathbf{R}^S_{++}$. Let $\triangle_+$ denote the simplex in $\mathbf{R}^N_+$, that is let $\triangle_+ := \left\{\lambda \in \mathbf{R}^N_+ : \sum_{i=1}^N \lambda_i = 1\right\}$. For $A \subset \mathbf{R}^S$, $co(A)$ denotes the convex hull. We denote by $\vec{1}_S$ the vector of ones in $\mathbf{R}^S$, that is, $\vec{1}_S = (1, ..., 1) \in \mathbf{R}^S$. Each coalition S has a feasible set of payoff vectors or utilities denoted by $V_S \subset \mathbf{R}^S$. By agreement, $V_\emptyset = \{0\}$ and $V_{\{i\}}$ is nonempty, closed and bounded from above for any i. In addition, we will assume that

$$\max\left\{x : x \in V_{\{i\}}\right\} = 0 \text{ for any } i \in N;$$

this is by no means restrictive since it can always be achieved by a normalization.

It is convenient to describe the feasible utilities of a coalition as a subset of $\mathbf{R}^N$.

For each coalition S let $V(S)$, called the *payoff set for* S, be defined by

$$V(S) := \left\{x \in \mathbf{R}^N : x_S \in V_S \text{ and } x_a = 0 \text{ for } a \notin S\right\}.$$

A *game without side payments* (called also an *NTU game* or simply a *game*) is a pair (N, V) where the correspondence $V : 2^N \longrightarrow \mathbf{R}^N$ is such that $V(S) \subset \left\{x \in \mathbf{R}^N : x_a = 0 \text{ for } a \notin S\right\}$ for any $S \subset N$ and satisfies the following properties :

(2.1) $V(S)$ is nonempty and closed for all $S \subset N$.

(2.2) $V(S) \cap \mathbf{R}^N_+$ is bounded for all $S \subset N$, in the sense that there is a real number $K > 0$ such that if $x \in V(S) \cap \mathbf{R}^N_+$, then $x_i \leq K$ for all $i \in S$.

(2.3) $V(S_1) + V(S_2) \subset V(S_1 \bigcup S_2)$ for any disjoint $S_1, S_2 \subset N$ (superadditivity).

We next introduce the uniform version of strong comprehensiveness assumed for our results. Informally, if one person can be made better off (while all the others remain at least as well off), then all persons can be made better off. Roughly, this notion dictates that payoff sets are both comprehensive and uniformly bounded away from having level segments in their boundaries. Consider a set $W \subset \mathbf{R}^S$. We say that W is *comprehensive* if $x \in W$ and $y \leq x$ implies $y \in W$. The set W is *strongly comprehensive* if it is comprehensive, and whenever $x \in W$, $y \in W$, and $x < y$ there exists $z \in W$ such that $x << z$. This property has also been called "nonleveledness." Given (i) $x \in \mathbf{R}^S$, (ii) $i, j \in S$, (iii) $0 \leq q \leq 1$ and (iv) $\phi \geq 0$, define a vector

$x^q_{i,j}(\phi) \in \mathbf{R}^S$, where

$$
\begin{aligned}
(x^q_{i,j}(\phi))_i &= x_i - \phi, \\
(x^q_{i,j}(\phi))_j &= x_j + q\phi, \text{ and} \\
(x^q_{i,j}(\phi))_k &= x_k \text{ for } k \in S \setminus \{i,j\}.
\end{aligned}
$$

The set W is *q-comprehensive* if W is comprehensive and if, for any $x \in W$, it holds that $(x^q_{i,j}(\phi)) \in W$ for any $i, j \in S$ and any $\phi \geq 0$. (See Figure 1 below.) This condition places a lower bound of q on the degree of side paymentness and, for $q > 0$, uniformly bounds the slopes of the Pareto frontier of payoff sets away from zero. Note that for $q = 0$, 0-comprehensiveness is simply comprehensiveness. Also note that if a game is q-comprehensive for some $q > 0$ then the game is q'-comprehensive for all q' with $0 \leq q' \leq q$.

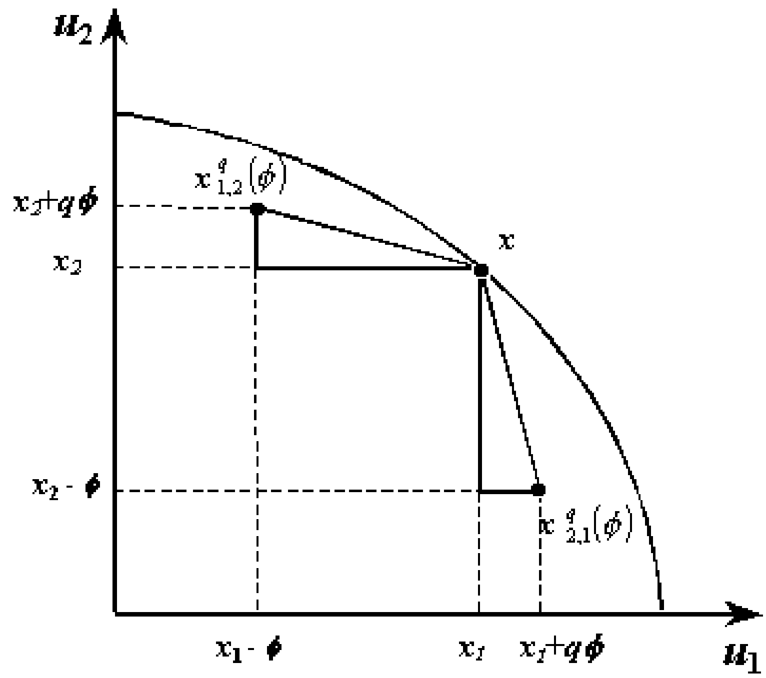

Figure 1.

Let $V_S \subset \mathbf{R}^S$ be a payoff set for $S \subset N$. Given q, $0 \leq q \leq 1$, let $W_S^q \subset \mathbf{R}^S$ be the smallest q-comprehensive set that includes the set V_S. For $V(S) \subset \mathbf{R}^N$ let us define the set $c_q(V(S))$ in the following way:

$$c_q(V(S)) := \left\{ x \in \mathbf{R}^N : x_S \in W_S^q \text{ and } x_a = 0 \text{ for } a \notin S \right\}.$$

Notice that for the relevant components – those assigned to the members of S – the set $c_q(V(S))$ is q-comprehensive, but not for other components. With some abuse of the terminology, we will call this set the *q-comprehensive cover of* $V(S)$. When $q > 0$ we can think of a game as having some degree of "side-paymentness" or as allowing transfers between players, but not necessarily at a one-to-one rate. This is an eminently reasonable assumption for games derived from economic models.

Remark 1. The notion of q-comprehensiveness can be found in Kaneko and Wooders (1996). For the purposes of the current paper, q-comprehensiveness can be relaxed outside the individually rational payoff sets. Note that in the definition of q-comprehensiveness, q places a *lower* bound on the degree of transferable utility for all coalitions S. Note also that, for a q-comprehensive game (N, V), within some coalitions, at some points on the upper boundary of the payoff set $V(S)$, by taking one unit of payoff away from one player we may be able to increase the utility of another player by more than one unit. This cannot hold uniformly however. For if it did, payoff sets would not be bounded above. Thus, property (2.2) of a game rules out q-comprehensive games with $q > 1$. Note also that for $q > 0$, q-comprehensiveness implies strong comprehensiveness.

2.2 Parameterized collections of games.

To introduce the notion of parameterized collections of games we will need the concept of Hausdorff distance. For every two nonempty subsets E and F of a metric space (M, d), define the *Hausdorff distance between E and F* (with respect to the metric d on M), denoted by $dist(E, F)$, as

$$dist(E, F) := \inf \{\theta \in (0, \infty) : \ E \subset B_\theta(F) \text{ and } F \subset B_\theta(E)\},$$

where $B_\theta(E) := \{x \in M : d(x,E) \leq \theta\}$ denotes a θ-neighborhood of E.

Since payoff sets are unbounded below, we will use a modification of the concept of the Hausdorff distance so that the distance between two payoff sets is the distance between the intersection of the sets and a subset of Euclidean space. Let m^* be a fixed positive real number. Let M^* be a subset of Euclidean space $\mathbf{R}^N$ defined by $M^* := \left\{x \in \mathbf{R}^N : x_a \geq -m^* \text{ for any } a \in N\right\}$. For every two nonempty subsets E and F of Euclidean space $\mathbf{R}^N$ let $H_\infty[E,F]$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x-y\|_\infty := \max_i |x_i - y_i|$ and let $H_1[E,F]$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x-y\|_1 := \sum_{i=1}^N |x_i - y_i|$.

The concepts defined below lead to the definition of parameterized collections of games. To motivate the concepts, each is related to analogous concepts in the pregame framework (see, for example, Wooders 1983,1994a or Wooders and Zame 1984). Recall that a pregame is a specification of a set of player types – a finite set or, more generally, a compact metric space of player types – and a worth function ascribing a payoff set to any group of players, where the group is described by the number of players of each type in the group.

$\delta-$substitute partitions. In our approach we approximate games with many players, all of whom may be distinct, by games with finite sets of player types. Observe that for a compact metric space of player types, given any real number $\delta > 0$ there is a

partition (not necessarily unique) of the space of player types into a finite number of subsets, each containing players who are "δ-similar" to each other. Parameterized collections of games do not restrict to a compact metric space of player types, but do employ the idea of a finite number of approximate types.

Let (N, V) be a game and let $\delta \geq 0$ be a non-negative real number. A δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. Formally, given a set $W \subset \mathbf{R}^N$ and a permutation τ of N, let $\sigma_\tau(W)$ denote the set formed from W by permuting the values of the coordinates according to the associated permutation τ. Given a partition $\{N[t] : t = 1, .., T\}$ of N, a permutation τ of N is $type - preserving$ if, for any $i \in N$, $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A *δ-substitute partition* of N is a partition $\{N[t] : t = 1, .., T\}$ of N with the property that, for any type-preserving permutation τ and any coalition S,

$$H_\infty \left[V(S), \sigma_\tau^{-1}(V(\tau(S))) \right] \leq \delta.$$

Note that in general a δ-substitute partition of N is not uniquely determined. Moreover, two games may have the same partitions but have no other relationship to each other (in contrast to games derived from a pregame).

(δ,T)- type games. The notion of a (δ,T)-type game is an extension of the notion of a game with a finite number of types to a game with approximate types. This is

significantly less restrictive than the extension of a finite set of types to a compact metric space.

Let δ be a non-negative real number and let T be a positive integer. A game (N,V) is a (δ,T)-*type game* if there is a T-member δ-substitute partition $\{N[t] : t = 1,..,T\}$ of N. The set $N[t]$ is interpreted as an *approximate type.* Players in the same element of a δ-substitute partition are *δ-substitutes.* When $\delta = 0$, they are *exact substitutes.*

per capita boundedness. Boundedness of average payoffs – in the case of games with side payments, simply finiteness of the supremum of average payoff – has a long history in the study of large games. Here, to keep our framework as simple as possible, we use a very transparent form of the assumption. Let C be a positive real number. A game (N,V) has a *per capita payoff bound* of C if, for all coalitions $S \subset N$,

$$\sum_{a \in S} x_a \leq C\,|S| \text{ for any } x \in V(S).$$

weakly β-effective B-bounded groups. Informally, groups of players containing no more than B members are weakly β-effective if, by restricting coalitions to having fewer than B members, the loss per player is no more than β. This notion formulates the idea of small effective groups in the context of parameterized collections of games. Let (N,V) be a game. Let $\beta \geq 0$ be a given non-negative real number and let B be a given positive integer. For each group $S \subset N$, define a corresponding set

$V(S;B) \subset \mathbf{R}^N$ in the following way:

$$V(S;B) := \bigcup \left[\sum_k V(S^k) : \left\{S^k\right\} \text{ is a partition of } S, \ \left|S^k\right| \leq B \right].$$

The set $V(S;B)$ is the payoff set of the coalition S when groups are restricted to have no more than B members. Note that, by superadditivity, $V(S;B) \subset V(S)$ for any $S \subset N$ and, by construction, $V(S;B) = V(S)$ for $|S| \leq B$. Define the *q-comprehensive cover of* $V(S;B)$, denoted by $c_q(V(S;B))$, analogously to the set $c_q(V(S))$; we might think of $c_q(V(S;B))$ as the payoff set to the coalition S when groups are restricted to have no more than B members and transfers are allowed between groups in the partition. If the game (N,V) has q-comprehensive payoff sets then $c_q(V(S;B)) \subset V(S)$ for any $S \subset N$. The game (N,V) with q-comprehensive payoff sets has *weakly β-effective B-bounded groups* if, for every group $S \subset N$,

$$H_1 \left[V(S), c_q(V(S;B))\right] \leq \beta \left|S\right|.$$

When $\beta = 0$, weakly 0-effective B-bounded groups are called *strictly effective B-bounded groups.*

Remark 2. Our previous papers (Kovalenkov and Wooders 1999a,b) used a more demanding notion of small group effectiveness (*β-effective B-bounded groups*). We discuss the advantages of the present approach in the next section. Note also, as in our other papers and in Wooders (1983), since our proofs proceed by considering equal treatment payoff vectors – those that treat players of the same approximate type symmetrically – the conditions of per capita boundedness and small group effectiveness may be relaxed to hold only for payoff vectors with the equal treatment property. We choose the current forms of the conditions for ease of statement.

Remark 3. The definition of weakly effective β-effective B-bounded groups may appear more restrictive than it actually is. In particular, groups of size B may not be able to exhaust gains to group size. Consider, for example, the collection of TU games where, for any game in the collection, n players can realize the payoff $n - \frac{1}{n}$. Note that there are ever-increasing returns to the size of the total player set. Yet, for each positive integer B and $\beta = \frac{1}{B}$, every game in the collection has weakly β-effective B-bounded groups.

parameterized collections of games $\mathcal{G}_1^q((\delta,T),C,(\beta,B))$. With the above definitions in hand, we can now define parameterized collections of games. Let T and B be positive integers and let C, δ, β, and q be non-negative real numbers. Let $\mathcal{G}_1^q((\delta,T),C,(\beta,B))$ be the collection of all (δ,T)-type games that have q-comprehensive payoff sets, have per capita bound of C, and have weakly β-effective B-bounded groups.

Less formally, given non-negative real numbers C, δ, β, and q, and positive integers T and B, a game (N,V) belongs to the class $\mathcal{G}_1^q((\delta,T),C,(\beta,B))$ if:

(a) the payoff sets satisfy q-comprehensiveness;

(b) there is a partition of the total player set into T sets where each element of the partition consists of players who are δ-substitutes for each other;

(c) maximum per capita gains are bounded by C; and

(d) almost all gains to collective activities (with a per capita maximum possible loss of β) can be realized by partitions of the total player sets into groups containing fewer than B members.

3 The Theorem.

First, we recall some definitions.

The core and epsilon cores. Let (N, V) be a game. A payoff x is *ε-undominated* if for all $S \subset N$ and $y \in V(S)$ it is not the case that $y_S >> x_S + \vec{1}_S \varepsilon$. The payoff x is *feasible* if $x \in V(N)$. The ε-core of a game (N, V) consists of all feasible and ε-undominated allocations. When $\varepsilon = 0$, the ε-core is the *core*.

The equal treatment epsilon core. Given non-negative real numbers ε and δ, we will define the *equal treatment ε-core* of a game (N, V) relative to a partition $\{N[t]\}$ of the player set into δ-substitutes as the set of payoff vectors x in the ε-core with the property that for each t and all i and j in $N[t]$, it holds that $x_i = x_j$.

Let $(N, V) \in \mathcal{G}_1^q((\delta, T), C, (\beta, B))$. The following Theorem provides a lower bound on ε so that for any $\varepsilon' \geq \varepsilon$, the game (N, V) has a nonempty ε'-core. In fact, the Theorem shows nonemptiness of the equal treatment ε-core as well. The lower bound on ε is given by

$$\alpha_N^q((\delta,T),C,(\beta,B)) := \frac{1}{q}(\frac{TC(B-1)}{|N|}+\beta)+\delta.$$

Of course the only interesting cases are those where this bound is small. To avoid trivialities associated with large ε we restrict attention to the case $\alpha_N^q((\delta,T),C,(\beta,B)) \leq m^*$, where m^* is the positive real number fixed in Section 2.2.

Theorem. Let $(N,V) \in \mathcal{G}_1^q((\delta,T),C,(\beta,B))$, where $q>0$. Assume $V(N)$ is convex. Let ε be a positive real number. If $\varepsilon \geq \alpha_N^q((\delta,T),C,(\beta,B))$ then the equal treatment ε-core of (N,V) is nonempty.

The relationships between the lower bound on ε, the parameters describing the game, and the number of players in the total player set are immediate. The idea of the proof of the Theorem is provided in Section 4. Several results used for this proof are presented in appendix.

It is interesting to observe that for inessential games, which always have nonempty cores, the Theorem gives a bound on ε of zero. Let (N,V) be a game where all coalitions are inessential and side payments are available at the rate $q>0$, that is, for any coalition $S \subset N$, $V(S) = c_q(\sum_{i\in S} V(\{i\}))$. Such a game has a nonempty core. Thus, the ε-core is nonempty for any $\varepsilon \geq 0$. To apply our Theorem, let C be a per capita bound for the game (N,V). Then $(N,V) \in \mathcal{G}_1^q((0,|N|),C,(0,1))$. The lower bound given by the expression above, $\frac{1}{q}(\frac{TC(B-1)}{|N|}+\beta)+\delta$ is zero since $\delta=0$, $\beta=0$ and $(B-1)=0$. Even in this extreme case, the bound works well.

Now let us recall the simple example of a parameterized collection of games presented in the introduction. Consider games with transferable utility where, given some positive real number K, within each game, all players are identical, only two-player coalitions are effective, and any two-player coalition can earn a payoff of less than or equal to $2K$. Obviously all these games belong to the class $\mathcal{G}_1^1((0,1),K,(0,2))$. In this case, $\frac{1}{q}(\frac{TC(B-1)}{|N|}+\beta)+\delta=\frac{K}{|N|}$. The core is empty if the number of players is odd. Let $|N|=2n+1$ for some positive integer n. But we can easily determine the least lower bound on ε so that the ε-core is nonempty for any game (N,V) in the collection with $|N|=2n+1$. In particular, suppose we assign each of the first $2n$ players up to $K-\varepsilon$ and the remaining player $2n\varepsilon$. Suppose ε^* solves $K-\varepsilon^*=2n\varepsilon^*$. Then the ε core is nonempty for any game in the collection for any $\varepsilon\geq\varepsilon^*$, but may be empty otherwise. (Take, for example, $K=1$ and $n=1$. Then $\varepsilon^*=\frac{1}{3}$ and the ε-core is empty for any $\varepsilon<\frac{1}{3}$.) Solving for ε^*, we obtain

$$\varepsilon^*=\frac{K}{(2n+1)}=\frac{K}{|N|}.$$

This bound coincides with the bound given by the Theorem. Thus the bound given in the Theorem is the best possible bound for this collection.

Remark 4. Consider again the collection of TU games of Remark 3 where, for any game in the collection, n players can realize the payoff $n-\frac{1}{n}$. It is apparent that given any $\beta>0$ there is a bound B so that this collection is contained in $\mathcal{G}_1^1((0,1),1,(\beta,B))$. Moreover, given $\beta<1$ (and positive, of course) we can choose B equal to the smallest integer greater than $\frac{1}{\beta}$. Thus, we can make $\alpha^1_N((0,T),1,(\beta,B)) = \frac{T(B-1)}{|N|} + \beta$ arbitrarily small by choosing β to be small and $|N|$ to be large. For example, suppose we wish $\alpha^1_N((0,1),1,(\beta,B))$ to be smaller than $\frac{1}{100}$. Setting $\beta = \frac{1}{200}$ and $B = 200$, it holds that $\alpha^1_N((0,1),1,(\beta,B)) < \frac{1}{100}$ for all games in the collection with more than $200 \times 199 = 39,800$ players. This illustrates the remark in the introduction that "For choices of parameter values ensuring that ... relatively small groups of players can realize nearly all gains to collective activities, for games with many players the approximation is close." Example 2 below makes the same point for choices of δ and T.

Now let us consider formally the central case of games with side payments

3.1 Games with side payments: Corollary 1.

A *game with side payments* (also called a *TU game*) is a game (N,V) with 1-comprehensive payoff sets, that is $V(S) = c_1(V(S))$ for any $S \subset N$. This implies that for any $S \subset N$ there exists a real number $v(S) \geq 0$ such that $V_S = \left\{x \in \mathbf{R}^S : \sum_{i\in S} x_i \leq v(S)\right\}$. Since the function $v : 2^N \to \mathbf{R}$, where $v(\emptyset) := 0$, uniquely determine a TU game, a TU game is typically represented by a pair (N,v). All the definitions that we have introduced can be stated for TU games through the function v, called the *characteristic function.* Moreover some of these definitions are essentially simpler and more straightforward than in the general case. For the purposes of the illustration we state below the simpler definitions for TU games:

1). A game (N, v) is *superadditive* if $v(S) \geq \sum_k v(S^k)$ for all groups $S \subset N$ and for all partitions $\left\{S^k\right\}$ of S.

2). Let (N, v) be a game and let $\delta \geq 0$ be a non-negative real number. A *δ-substitute partition* of N is a partition $\{N[t] : t = 1, .., T\}$ of N with the property that, for any type-consistent permutation τ and any coalition S,

$$|v(S) - v(\tau(S))| \leq \delta |S| .$$

3). Let β be a given non-negative real number, and let B be a given integer. A game (N, v) has *weakly β-effective B-bounded groups* if for every group $S \subset N$ there is a partition $\left\{S^k\right\}$ of S into subgroups with $\left|S^k\right| \leq B$ for each k and

$$v(S) - \sum_k v(S^k) \leq \beta |S| .$$

4). Let C be a positive real number. A game (N, v) has a *per capita bound* of C if $\frac{v(S)}{|S|} \leq C$ for all coalitions $S \subset N$.

The case of TU games is central, since first we prove our result for these games and then we extend the result to games without side payments. To make notations simpler in the following sections, we denote *parameterized collections of games with side payments,* $\mathcal{G}_1^1((\delta, T), C, (\beta, B))$, by $\Gamma((\delta, T), C, (\beta, B))$. For the convenience of the reader a corollary of the Theorem corresponding to the case of TU games follows:

Corollary 1. Let $(N, v) \in \Gamma((\delta, T), C, (\beta, B))$ and let ε be a positive real number. If

$$\varepsilon \geq \frac{TC(B-1)}{|N|} + \delta + \beta$$

then the equal treatment ε-core of (N, v) is nonempty.

Now let us state some examples. We begin our examples with the special case of games with exact types and strictly effective B-bounded groups.

Example 1. *Exact types and strictly effective small groups.* Let us consider a game (N, v) with two types of players. Assume that any player alone can get only 0 units or less, that is $v(\{i\}) = 0$ for all $i \in N$. Suppose that any coalition of the two players of types i and j can get up to γ_{ij} units of payoff to divide. Let γ_{11}, $\gamma_{12} = \gamma_{21}$, and γ_{22} be some numbers from the interval $[0, 1]$. An arbitrary coalition can gain only what it can obtain in partitions where no member of the partition contains more than two players.

We leave it to the reader to check that $(N, v) \in \Gamma((0, 2), \frac{1}{2}, (0, 2))$. Thus we have from Corollary 1 that for $\varepsilon \geq \frac{1}{|N|}$ the equal treatment ε-core of (N, v) is nonempty. Notice that this result holds uniformly for all possible numbers γ_{11}, $\gamma_{12} = \gamma_{21}$, and γ_{22}.

The following example illustrates how our result can apply to games derived from pregames with a compact metric space of player types. For brevity, our example is somewhat informal. While the example is worded in terms of firms and workers, as in Crawford and Knoer (1981), for example, it could easily be modified to treat the hospital and intern matching problem as in Roth (1984) or any such assignment problem.

Example 2. *Approximate player types.* Consider a pregame with two sorts of players, firms and workers. The set of possible types of workers is given by the points in the interval $[0,1)$ and the set of possible types of firms is given by the points in the interval $[1,2]$. Formally, let N be any finite player set and let ξ be an *attribute function*, that is, a function from N into $[0,2]$. If $\xi(i) \in [0,1)$ then i is a worker and if $\xi(i) \in [1,2]$ then i is a firm.

Firms can profitably hire up to three workers and the payoff to a firm i and a set of workers $W(i) \subset N$, containing no more than 3 members, is given by $v(\{i\} \bigcup W(i)) = \xi(i) + \sum_{j \in W(i)} \xi(j)$. Workers and firms can earn positive payoff only by cooperating so $v(\{i\}) = 0$ for all $i \in N$. For any coalition $S \subset N$ define $v(S)$ as the maximum payoff the group S could realize by splitting into coalitions containing either workers only, or 1 firm and no more than 3 workers. This completes the specification of the game.

We leave it to the reader to verify that for any positive integer m every game derived from the pregame is a $(\frac{1}{m}, 2m)$-type game and even a member of the class $\Gamma((\frac{1}{m}, 2m), 2, (0,4))$. Then Corollary 1 implies that for any $\varepsilon \geq \frac{12m}{|N|} + \frac{1}{m}$ the equal treatment ε-core of (N, v) is nonempty.

This implies that for any $\varepsilon^0 > 0$ there is a positive integer $N(\varepsilon^0)$ such that for any $|N| \geq N(\varepsilon^0)$ the game (N, v) has a nonempty equal treatment ε^0-core. (For an exact bound take an integer $m^0 \geq \frac{2}{\varepsilon^0}$ and define $N(\varepsilon^0) \geq \frac{24m^0}{\varepsilon^0}$.)

For completeness, we present a simple but formal example with nearly effective groups.

Example 3. *Nearly effective groups.* Call a game (N, v) a k-quota game if any coalition $S \subset N$ of size less than k can realize only 0 units (that is, $v(S) = 0$ if $|S| < k$), any coalition of size k can realize 1 unit (that is, $v(S) = 1$ if $|S| = k$), and an arbitrary coalition can gain only what it can obtain in partitions where no member of the partition contains more than k players. Let Q be a collection, across all k, of all k-quota games with player set N.

We leave it to the reader to verify that, for any positive integer $m > 1$, the class Q is contained in the class $\Gamma((0, 1), 1, (\frac{1}{m}, m - 1))$. Hence Corollary 1 implies that for any $\varepsilon \geq \frac{(m-2)}{|N|} + \frac{1}{m}$ and for any $(N, v) \in Q$ the equal treatment ε-core of (N, v) is nonempty. This implies that for any $\varepsilon^0 > 0$ there is a positive integer $N(\varepsilon^0)$ such that for any $|N| \geq N(\varepsilon^0)$ any game $(N, v) \in Q$ has a nonempty equal treatment ε^0-core. (For an exact bound take an integer $m^0 \geq \frac{2}{\varepsilon^0}$ and define $N(\varepsilon^0) \geq \frac{2(m^0-2)}{\varepsilon^0}$.)

This example also illustrates some differences between parameterized collections of games and games derived from pregames, discussed further in Section 5. A pregame takes as given a topological space of player types and a *single* worth function determining payoff sets for groups of players described by their types. It is immediate that the payoff structure of a pregame is invariant in the sense that only the size and composition of player sets can vary, not the payoff to a given set of players described by their types. Given the player set N, the class Q consists of games generated by varying the payoff structure of the games. Thus, the collection Q cannot be described as a collection of games generated by a pregame.

3.2 Relationships to other results: Corollary 2.

Notice that a feasible payoff is in the ε-core if no coalition of players can improve upon the payoff by at least ε *for each member* of the coalition. This suggests that the distance of coalitions containing fewer than B-members from being effective for the realization of all gains to group formation might most appropriately be defined using the Hausdorff distance with respect to the *sup* norm, and indeed this was the approach that we took in previous papers (Kovalenkov and Wooders 1999a,b). In this section we re-define the notion of small group effectiveness using the sup norm and contrast our current Theorem with our earlier results. Let us first define β-effective B-bounded groups using the Hausdorff distance with respect to the sup norm.

β-effective B-bounded groups. The game (N, V) with q-comprehensive payoff sets has

β-effective B-bounded groups if for every group $S \subset N$

$$H_\infty\left[V(S), c_q(V(S;B))\right] \leq \beta.$$

Notice that β-effective B-bounded groups are always weakly β-effective B-bounded groups, but for TU games these two notions coincide. These notions also coincide in the case when $\beta = 0$.

We now introduce the definition of parametrized collections of games in our prior research.

parameterized collections of games $G^q_\infty((\delta, T), C, (\beta, B))$. Let T and B be positive integers and let C, δ, β, and q be positive real numbers. Let $G^q_\infty((\delta, T), C, (\beta, B))$ be the collection of all (δ, T)-type games that have q-comprehensive payoff sets, have per capita bound of C, and have β-effective B-bounded groups.

Of course $G^q_\infty((\delta, T), C, (\beta, B)) \subset \mathcal{G}^q_1((\delta, T), C, (\beta, B))$, but these two classes coincide for $q = 1$ (games with side payments), that is $G^1_\infty((\delta, T), C, (\beta, B)) = \Gamma((\delta, T), C, (\beta, B))$.

The following statement is a straightforward implication of the Theorem to the class of parameterized games considered in our previous papers.

Corollary 2. Let $(N, V) \in G^q_\infty((\delta, T), C, (\beta, B))$, where $q > 0$. Assume $V(N)$ is convex. Let ε be a positive real number. If $\varepsilon \geq \alpha^q_N((\delta, T), C, (\beta, B))$ then the equal treatment ε-core of (N, V) is nonempty.

The following example illustrates why either convexity or some degree of comprehensiveness is required for our result, even for games with just one exact player type. In brief, this is required so that "left-over" players can be compensated.

Example 4. *Convexity or some positive degree of comprehensiveness.* Let (N, V_0) be a superadditive game where for any two-person coalition $S = \{i, j\},\ j \neq i$,

$$V_0(S) := \{x \in \mathbf{R}^N :\ x_i \leq 1,\ x_j \leq 1, \text{and } x_k = 0 \text{ for } k \neq i, j\}$$

and for each $i \in N$,

$$V_0(\{i\}) := \{x \in \mathbf{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

For an arbitrary coalition S the payoff set $V_0(S)$ is given as the *superadditive cover*, that is,

$$V_0(S) := \bigcup_{\mathcal{P}(S)} \sum_{S' \in \mathcal{P}(S)} V_0(S'),$$

where the union is taken over all partitions $\mathcal{P}(S)$ of S in the sets with one or two elements.

Let m be a positive integer. Let (N^m, V_0^m) be a game where the number of players in the set N^m is $2m+1$ and for any coalition $S \subset N^m$ $V_0^m(S) := V_0(S)$. Thus, each game (N^m, V_0^m) has an odd number of players. It is easy to see that the core of the game is nonempty: any payoff giving 1 to each of $2m$ players is in the core. Since the total number of players is odd, at least one person must be "left out." In a game with side payments this player could upset the nonemptiness of the core. But the games of this example do not satisfy strong comprehensiveness. Thus, a payoff giving 1 to each of $2m$ players cannot be improved upon since the "left-out" player, in a coalition by himself, cannot make both himself and a player in a two-person coalition better off – the player in the two-person coalition cannot be given more than 1. The games, however, can be approximated arbitrarily closely by games with strongly comprehensive payoff sets. (See Wooders 1983, Appendix.)

Let (N^m, V_{sc}^m) be a game with strongly comprehensive payoff sets that approximates the game (N^m, V_0^m). For a sufficiently close approximation, the game (N^m, V_{sc}^m) will have effective small groups and an empty core. This follows from the observations that any payoff must give at least one player less than one and the two worst-off players a total of less than two. The two worst-off players form an improving coalition and hence the core is empty. Moreover, it can be shown with a precise construction of (N^m, V_{sc}^m) that for a small but positive ε the ε-core of (N^m, V_{sc}^m) can be empty even for a great number of players.

Our results rely on convexity and q-comprehensiveness. Since there is only one type of player, in this example either q-comprehensiveness or convexity will suffice. The role of convexity is to average payoffs over similar players. Consider the game (N, V_{conv}^m) where V_{conv}^m is defined as the convex hull of V_{sc}^m. Then the payoff $x = (\frac{2m}{2m+1}, ..., \frac{2m}{2m+1})$ is feasible and in the ε-core of (N^m, V_{conv}^m) for any $\varepsilon \geq \frac{1}{2m+1}$. Now instead of convexity of the total payoff set, suppose that payoff sets are q-comprehensive. In this case for any payoff giving one to each of $2m$ players, it is possible to take some small amount, say ε, away from each of $2m$ players and "transfer" $2m\varepsilon q$ to the leftover player. Thus, for any ε and q satisfying $2m\varepsilon q \geq 1 - \varepsilon$ the ε-core is nonempty.

A crucial feature of Example 4 is the restriction to one player type. Because of this feature and the fact that two-player coalitions are effective, through convexity or q-comprehensiveness we can construct equal treatment payoff vectors in approximate cores. This example suggests that either convexity or q-comprehensiveness is sufficient to get nonemptiness of the epsilon cores for large games. In fact, Theorem 3 in Kovalenkov and Wooders (1999a) supports this intuition in the case of q-comprehensiveness. Theorem 1 in Kovalenkov and Wooders (1999b) shows that convexity is sufficient for nonemptiness but requires "thickness" of the player set (that is, the condition that the proportion of any approximate player type is bounded above zero). Neither of these papers, however, provide exact bounds. Corollary 2 shows that with *both* convexity and q-comprehensiveness, an exact bound can be obtained on ε for nonemptiness of the ε-core. This bound is very simple and is achieved in some examples.

Notice that in some cases our Theorem allows us to obtain a significantly smaller bound than Corollary 2. This arises because of the use of *weakly* β-effective B-bounded groups for the Theorem rather than β-effective B-bounded groups as in Corollary 2. Our next example, continuing Example 4, demonstrates how both the

Theorem and Corollary 2 can be applied to games without side payments and illustrates the advantages of the use of the Theorem.

Example 5. *The advantage of the Theorem in decreasing the bound on* ε. Recall the game (N, V_0) defined in Example 4. Now let us define a game $(N, V_{\frac{1}{3}})$ in the following way. For any $S \subset N$ let $V_{\frac{1}{3}}(S)$ be the $\frac{1}{3}$-comprehensive cover of the convex cover of the payoff set $V_0(S)$; that is,

$$V_{\frac{1}{3}}(S) := c_{\frac{1}{3}}(co(V_0(S))).$$

Obviously the game $(N, V_{\frac{1}{3}})$ has $\frac{1}{3}$-comprehensive convex payoff sets, one player type, and per capita bound of 1. We leave it to the reader to verify that for any positive integer $m \geq 3$ the game $(N, V_{\frac{1}{3}})$ has $\frac{1}{m}$-effective m-bounded groups. Thus the game $(N, V_{\frac{1}{3}})$ is a member of the class $G^{\frac{1}{3}}_{\infty}((0,1), 1, (\frac{1}{m}, m))$. Since $V_{\frac{1}{3}}(N)$ is convex, Corollary 2 states that for any $\varepsilon \geq 3(\frac{(m-1)}{|N|} + \frac{1}{m})$ the equal treatment ε-core of $(N, V_{\frac{1}{3}})$ is nonempty. This implies that for any $\varepsilon^0 > 0$ there is a positive integer $N(\varepsilon^0)$ such that for any $|N| \geq N(\varepsilon^0)$ the game $(N, V_{\frac{1}{3}})$ has a nonempty equal treatment ε^0-core. (To obtain an exact bound take an integer $m^0 \geq \frac{6}{\varepsilon^0}$ and define $N(\varepsilon^0) \geq \frac{6(m^0-1)}{\varepsilon^0}$.)

We leave it to the reader to verify that the game $(N, V_{\frac{1}{3}})$ has weakly $\frac{1}{|N|}$-effective 2-bounded groups. Therefore the game $(N, V_{\frac{1}{3}})$ is a member of the class $\mathcal{G}^{\frac{1}{3}}_{1}((0,1), 1, (\frac{1}{|N|}, 2))$. Recall that $V_{\frac{1}{3}}(N)$ is convex. Then the Theorem states that for any $\varepsilon \geq 3(\frac{1}{|N|} + \frac{1}{|N|}) = \frac{6}{|N|}$ the equal treatment ε-core of $(N, V_{\frac{1}{3}})$

is nonempty which is much better then the bound provided by Corollary 2. (The bound of the Theorem implies that for any $\varepsilon^0 > 0$ and for any $|N| \geq \frac{6}{\varepsilon^0}$ the game $(N, V_{\frac{1}{3}})$ has a nonempty equal treatment ε^0-core.)

4 A sketch of the proof of the Theorem.

We provide here only the main argument. In appendix we present proofs for several results used in this section. Our proof starts from the central case of TU games, then we consider a symmetric case, and finally we deal with the general case with no additional restrictions.

A) The TU case. We start our chain of proofs by first addressing a special case of Corollary 1 (Lemma 1); this case considers games (N, v) from the class $\Gamma((0,T), C, (0,B))$. Given a partition $\{N[t]\}_{t=1}^{T}$ of N into types, for $S \subset N$ define the *profile of* S, denoted by $prof(S)$, by its components

$$prof(S)_t = |S \cap N[t]|$$

for $t = 1, ..., T$. A profile describes a group of players in terms of the numbers of players of each type in the group. For any profile $m \in \mathbf{R}^T$, let $\|m\|$ denote the number of players in a group described by m, that is, $\|m\| = \sum_{t=1}^{T} m_t$. Also, for any profile $m \in \mathbf{R}^T$, define $\overline{v}(m) = v(S)$ for any coalition $S \subset N$ with $prof(S) = m$.

The idea of the proof for the TU case. The intuition for Lemma 1 is the following: The possible emptiness of the core can be viewed as a consequence of the feature of cooperative games that coalitions are not allowed to operate at levels of intensity between zero and one. (Another, but less apt, interpretation, is that coalitions can not operate part-time. The difficulties with this interpretation are discussed in Garratt and Qin 2000.) For games with strictly effective B-bounded groups, we can restrict attention to coalitions containing no more than B members, or, in other words, to coalitions with profiles m satisfying the condition that $\|m\| \leq B$. Let $\left\{m^k\right\}_k$ denote the collection of profiles with $\left\|m^k\right\| \leq B$ for each m^k and let f denote the profile of N. Define $v^b(N)$ as the total payoff to the grand coalition if coalitions were allowed to operate at levels of intensity between zero and one. More formally, define

$$v^b(N) = \max_{\{\omega_k\}} \sum_k \omega_k \overline{v}(m^k)$$

where

$$\sum_k \omega_k m^k = f \text{ and, for each } k,\ \omega_k \geq 0$$

(so that the constraint given by the composition of the total player set is satisfied). Let $\{\omega_k^*\}$ be the positive weights achieving the maximum, that is, $v^b(N) = \sum_k \omega_k^* \overline{v}(m^k)$.

If the weights ω_k^* are *integers* then, from superadditivity, the game (N, v) will have a nonempty core; players can be partitioned into "optimal coalitions" with no left-overs. There exist nonnegative integers r_k such that $\omega_k \geq r_k$ and $h_k := w_k - r_k < 1$ for each k. The smaller the expression

$$\frac{v^b(N) - v(N)}{|N|} \leq \frac{\sum_k (\omega_k^* \overline{v}(m^k) - r_k \overline{v}(m^k))}{|N|} = \frac{\sum_k h_k \overline{v}(m^k)}{|N|}$$

the closer the game is to having a nonempty core. Since the number of profiles in the set $\{m_k\}$ is bounded and since each h_k is less than one, it is clear that for sufficiently large $|N|$ the above expression can be made arbitrarily small. To obtain our bound, we appeal to the Caratheodory theorem, showing that we can restrict the number of elements in the set $\{\omega_k^*\}$ to be no more than T and to the fact that $\sum_k h_k m^k$ must be a vector of integers and, coordinate by coordinate, strictly smaller than a vector of integers $\sum_k m^k$. Therefore, since $\|m_k\| \leq B$ for each k, $\left\|\sum_k h_k m^k\right\| \leq \left\|\sum_k m^k\right\| - T \leq (B-1)T$. Using the per capita bound of C, we obtain the stated lower bound on ε for the case of $\delta = 0$ and $\beta = 0$. These completes the proof of Lemma 1.

The Corollary 1 follows by approximation techniques. It appears, that for positive δ and β we must add both these numbers to the bound. We present the formal proofs of Lemma 1 and of Corollary 1 in *Step 1* of appendix. ■

Remark 5. In effect, Bondareva (1962) and Shapley (1967) showed that a TU game has a nonempty core if and only if $v^b(N) = v(N)$. Wooders (1983) (and earlier SUNY-Stony Brook Department of Economics Working Papers for TU games, especially #184 – published in part in Wooders 1992) developed the approach discussed above for sequences of games with a fixed distribution of player types. The observation that with bounded effective groups and a finite number of types, the emptiness of the core arises from the presence of left overs (a set of players with profile $\sum_k h_k m^k$) already appeared in these papers. This approach has now been extended in a number of papers, most recently to parameterized collections of games in Wooders (1994b) and Kovalenkov and Wooders (1999a,b). The calculation of an exact bound on ε for nonemptiness of ε-cores of large games is new to our reseach on parameterized collections of games.

B) The symmetric case. We continue the proof of the Theorem by first treating games where all players of the same type are exact substitutes of each other. We introduce the following terminology: A set $W \subset \mathbf{R}^N$ is *symmetric across substitute players* if for any player type the set W remains unchanged under any perturbations of the values associated with players of that type.

Formally let us assume that $(N, V) \in \mathcal{G}_1^q((0,T), C, (\beta, B))$. Note that in this case all payoff sets of the game (N, V) are symmetric across substitute players. Let us prove that for any $\varepsilon \geq \alpha_N^q((0,T), C, (\beta, B))$ the equal treatment ε-core of (N, V) is nonempty.

The idea of the proof for the symmetric case. In the proof for the symmetric case we will use the following definitions. Let $A \subset R^m$. A *recession cone* corresponding to A, denoted by $cone(A)$, is defined as follows:

$$cone(A) := \{y \in R^m : \ x + \lambda y \in A \text{ for all } \lambda \geq 0 \text{ and } x \in A\}.$$

The scalar product of $x, y \in R^m$ is denoted by $x \cdot y$. The *negative dual cone* of $P \subset R^m$ is denoted by $dual(P)$ and defined as follows:

$$dual(P) := \{z \in R^m : \ z \cdot y \leq 0 \text{ for any } y \in P\}.$$

A bound on the required size of the parameter ε for our result in the symmetric case is obtained by constructing a family of "λ-weighted transferable utility" games (N, V_λ) corresponding, in a certain way, to the initial game (N, V). Next we consider only those values of λ in a set L^*, defined as the intersection of the equal treatment payoff vectors in the simplex with the negative dual cone of the recession cone of the modified game. (See Figure 2.) For each λ there is corresponding TU game (N, v_λ). We give the formal construction of (N, v_λ) in *Step 2* of appendix.

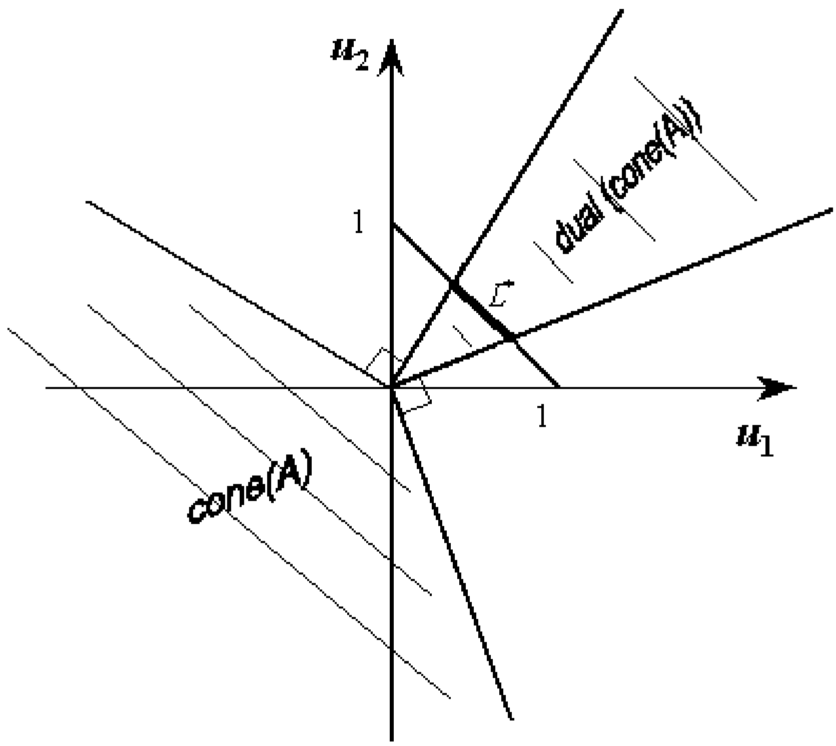

Figure 2.

In *Step 3* of appendix we prove Lemma 2, that, for some parameters C' and ε', any game (N, v_λ) is a member of the parameterized collection of TU games

$\Gamma((0,T),C',(\beta',B))$. This allows us to use Corollary 1 proved in Step 1 of appendix. In Lemma 3 we relate approximate cores of the game (N,v_λ) to approximate cores of the NTU game (N,V_λ). Using the fact that we consider only values of λ in L^*, we obtain an exact bound on ε for the initially given parameters C and β for nonemptiness of the equal treatment ε-core for all games (N,V_λ). This result will give us exactly the bound that we need to deduce for the conclusion of Theorem for the symmetric case.

Now we need only prove that if, given some ε, the equal treatment ε-core of (N,V_λ) is nonempty for all $\lambda \in L^*$, then the equal treatment ε-core will be nonempty for both the modified and initial games as well. With the help of Lemma 4, Lemma 5, and a theorem about excess demand considered in *Step 4*, all in appendix, we complete the proof in the symmetric case. ■

Remark 6. The initial approach in this proof is similar to that introduced in Scarf (1965) and usually used in proofs of the nonemptiness of the exact core for strongly balanced NTU games (for a definition of strong balancedness and for an example of this technique see Hildenbrand and Kirman 1988, Appendix to Chapter 4). But our proof departs from the typical approach in that we construct games (N,V_λ) and (N,v_λ) not for all λ in the simplex as usual, but only for λ belonging to a specific subset L^* of the simplex. The set L^* is the intersection of the equal treatment payoff vectors in the simplex with the dual negative cone to the recession cone of the payoff set for the grand coalition in the modified game. Later we use the structure of the set

L^* and q-comprehensiveness to complete the proof. (The usual technique is applied for $T = N$ and to games having $-\mathbf{R}^N_+$ as the recession cone of the payoff set $V(N)$. The negative dual cone to $-\mathbf{R}^N_+$ is $\mathbf{R}^N_+$. Then $\triangle_+ \subset \mathbf{R}^N_+$, so the relevant intersection is the simplex itself.)

C) The general case. Now let us consider the general case with no additional restrictions. We first modify the game (N, V). For any $S \subset N$ define $V^0(S) := \bigcap \sigma_\tau^{-1}(V(\tau(S)))$, where the intersection is taken over all type-preserving permutations τ of the player set N. Then from the definition of $V^0(S)$ it follows that $V^0(S) \subset V(S)$. (Informally, taking the intersection over all type-preserving permutations makes all players of each approximate type no more productive than the least productive members of that type.) From the definition of δ-substitutes, it follows that $H_\infty[V^0(S), V(S)] \leq \delta$ for any $S \subset N$. Moreover,

$$(N, V^0) \in \mathcal{G}_1^q((0, T), C, (\beta, B)) \text{ and } V^0(N) \text{ is convex.}$$

Therefore, we can apply the result proved in the symmetric case and conclude that the game (N, V^0) has some payoff x in the equal treatment $\frac{1}{q}(\frac{TC(B-1)}{|N|} + \beta)$-core. Now define a payoff vector y by

$$y(\{i\}) := x(\{i\}) - \delta \text{ for each } i \in N.$$

The payoff y will be feasible and $\left(\frac{1}{q}(\frac{TC(B-1)}{|N|} + \beta) + \delta\right)$-undominated in the initial game (N, V). Obviously, y has the equal treatment property. Therefore for $\varepsilon \geq \alpha_N^q((\delta, T), C, (\beta, B))$ the equal treatment ε-core of (N, V) is nonempty. ■

5 Relationships to the literature and conclusions.

Recall that Shapley and Shubik (1966), applying the convexifying effect of large numbers to preferences, showed that large exchange economies with replicated player sets and with quasi-linear utility functions (transferable utility) have nonempty approximate cores. Wooders (1983) showed that per capita boundedness, given $\varepsilon > 0$ all sufficiently large games in a sequence with a fixed distribution of player types have nonempty ε-cores containing payoff vectors with the equal treatment property. A key result is that under a somewhat less restrictive condition than strictly effective B-bounded groups, all payoff vectors in the core of an NTU game have the equal treatment property (Wooders (1983, Theorem 3)). Moreover, under the same condition there is a replication number r_0 with the property that for all positive integers ℓ the $\ell r_0 th$ game has a nonempty equal treatment core. Since then, a number of further results have been obtained for both TU and NTU games, cf., Kaneko and Wooders (1982,1996), Wooders and Zame (1984) and Wooders (1992, 1994a). These results, however, are all obtained in the context of pregames.

A pregame specifies a topological space of player types and a payoff set for every possible coalition in any game induced by the pregame. More precisely, given a compact metric space of player "types" or "attributes" (possibly finite), the payoff function of a pregame assigns a payoff set to every finite list of player types, repetitions allowed. Given any finite player set and an attribute function, assigning a type to each player in the player set, the payoff function of the game is determined by the payoff function of the pregame. Thus, the payoff set to any collection of players having a certain set of attributes is *independent* of the total player set in which it is embedded. In addition, this has significant economic consequences. In particular, widespread externalities are ruled out. Moreover, the pregame structure itself has hidden consequences. For example, within the context of pregame with side payments, there is an equivalence between per capita boundedness, finiteness of the supremum of average payoff, and small group effectiveness, the condition that all or almost all gains to collective activities can be realized by groups bounded in size (Wooders 1994a, Section 5). No such consequences can be hidden within parameterized collections of games since there is no necessary relationship between any of the games in the collection (other than that they are all described by the same parameters).

To study large games generally, without the structure and implicit assumptions imposed by a pregame, Wooders (1994b) introduced the model of parameterized collections of games with side payments and obtained a bound. In the current paper, Corollary 1 significantly improves on the bound. Kovalenkov and Wooders (1999a,b) introduce the concept of parameterized collections of games without side payments and show nonemptiness of approximate cores. With the additional assumption of convexity of the total payoff set, in the current paper Corollary 2 gives an exact bound for the nonemptiness result of Kovalenkov and Wooders (1999a). In addition, the Theorem applies more broadly than Corollary 2 and the bound given by the Theorem is superior to that of Corollary 2. The main result of Kovalenkov and Wooders (1999b) is that all payoff vectors in approximate cores have the equal treatment property. No such result holds for the current model. The nonemptiness result of Kovalenkov and Wooders (1999b) does not require that q-comprehensiveness for $q > 0$. Note, however, that the form of small group effectiveness in that paper is more restrictive than the form in Kovalenkov and Wooders (1999a) and the current paper; this has implications for the class of games covered. Moreover, no explicit bound on the required size of the games is provided in Kovalenkov and Wooders (1999b) and the dependence of the required size on the parameters is not exactly demonstrated. In the current paper, we require both the assumptions of convexity of payoff sets and q-comprehensiveness for $q > 0$, but use a significantly less demanding notion of small group effectiveness than in our prior papers. Using new techniques, we are able to demonstrate an explicit bound on ε for nonemptiness of ε-cores.

A very important literature to which our model and results apply is the assignment or matching literature as in Crawford and Knoer (1981) and Roth (1984), for example. In these problems, there seems to be especially natural parameterizations. For example, in the assignment of interns to hospitals game, the parameter B would be the maximum number of places in a hospital for interns plus one (for the hospital itself), β would be zero, the criteria for selection of interns and rankings of hospitals would determine the number of types T and the closeness of approximate types δ. If side payments are permitted, q would equal one. The per capita bound C could be set. Our results would then straightforwardly apply. It would be interesting to fit parameters to the model and empirically test whether observed outcomes are in approximate cores. For such a test, it appears necessary to have an exact bound on ε to enable testing as well as prediction of the model.

We conclude by noting that the results of this work may have application in economies with local public goods and/or coalition production (see, for example, Conley and Wooders 1995) and other sorts of situations with coalitions. A possible very exciting application is to economies with differential information, as in Allen (1994,1995), Forges and Minelli (1999), or Forges, Heifetz, and Minelli (1999), among others. It may be possible, for example, to derive a bound on the extent of the deviation of cores involving differential information from the full information core.

6 Appendix.

Step 1: Proofs for TU games. Let us first prove the following Lemma, that significantly improves a result achieved in Wooders (1994b).

Lemma 1. Let $(N, v) \in \Gamma((0, T), C, (0, B))$. If $\varepsilon \geq \frac{TC(B-1)}{|N|}$ then the equal treatment ε-core of (N, v) is nonempty.

Proof of Lemma 1. Throughout the reminder of this proof let $\{N[t]\}$ be a partition of N into T subsets each consisting of players who are all substitutes for each other. We will assume for simplicity that there are positive numbers of players of each of the T types. (Otherwise we could reduce the number of types under consideration and obtain a better bound.) Our proof will use the notion of a balanced cover for a game. We first recall the notions of balanced collections of subsets of N and balancing weights. Let Ω denote a collection of subsets of N. The collection Ω is a *balanced collection of subsets* of N if there is a collection of non-negative real numbers $(\widetilde{\omega}_{S'})_{S'\in\Omega}$, called *balancing weights*, such that for each $i \in N$,

$$\sum_{S':i\in S',\, S'\in\Omega} \widetilde{\omega}_{S'} = 1.$$

Let v^b be the function mapping the subsets of N to $\mathbf{R}$ defined by:

$$v^b(S) := v(S) \text{ for all groups } S \neq N$$

and

$$v^b(N) := \max_{\Omega} \textstyle\sum_{S'\in\Omega} \widetilde{\omega}_{S'} v(S'),$$

where the maximum is taken over all balanced collections Ω of N with corresponding balancing weights $(\widetilde{\omega}_{S'})_{S'\in\Omega}$. Then (N, v^b) is a game, called the *balanced cover* of (N, v). Bondareva (1962) and Shapley (1967) have shown that a TU game has a nonempty core if and only if $v^b(N) = v(N)$. Given $\varepsilon \geq 0$, it follows easily from their results that the game (N, v) has a nonempty ε-core if and only if $v^b(N) \leq v(N)+\varepsilon |N|$. The proof proceeds by placing a bound on the difference $v^b(N) - v(N)$.

Let x belongs to the core of (N, v^b); from the preceding paragraph there is such an x. Now consider an equal treatment payoff vector z, defined by its components $z_t := \frac{1}{|N[t]|}\sum_{a\in N[t]} x_a$. It is immediate that z is in the core of the game (N, v^b) (since the core of a TU game is convex, all agents of one type are exact substitutes, and the payoff sets are unaffected by any permutation of substitute players). This vector z will play an important role in the proof later.

Since (N, v) has strictly effective B-bounded groups, there exist a balanced collection of subsets $\left\{S^{\ell}\right\}_l$ of N, where $\left|S^l\right| \leq B$ for each S^l in the collection, all corresponding balancing weights $(\tilde{\omega}_{S^l})$ are strictly positive, and

$$v^b(N) = \sum_l \tilde{\omega}_{S^l} v\left(S^l\right).$$

Let $\left\{m^k\right\}_k$ denote the collection of all profiles of $\left\{S^l\right\}_l$ relative to the partition $\{N[t]\}$ of N, where $\left\|m^k\right\| \leq B$ for each m^k in the collection. Let f denote the profile of N. Define a characteristic function $\bar{v}$ mapping profiles into R^T_+ by

$$\bar{v}(m) := v(M), \text{ for any group } M \text{ with profile } m.$$

Now let us aggregate balancing weights $\tilde{\omega}_{S^l}$ across coalitions with the same profiles. That is, let us consider weights $\omega_k = \sum \tilde{\omega}_{S^l}$, where the summation is taken across all groups S^l that have the same profile m^k. Then from balancedness it holds that $\sum_k \omega_k m^k = f$ and we get

$$v^b(N) = \sum_k \omega_k \overline{v}(m^k).$$

Now let us recall the equal treatment payoff z in the core of the game (N, v^b). It holds that $z \cdot m^k \geq \overline{v}(m^k)$ for any m^k and $z \cdot f = v^b(N)$. Thus we get

$$v^b(N) = \sum_k \omega_k \overline{v}(m^k) \leq \sum_k \omega_k (z \cdot m^k) = z \cdot (\sum_k \omega_k m^k) = z \cdot f = \overline{v}^b(f).$$

But, since $\omega_k > 0$ for any m^k in the collection, this implies that

$$z \cdot m^k = v(m^k) \text{ for each } m^k \text{ in the collection.}$$

Now notice that the condition $\sum_k \omega_k m^k = f$ implies that a vector $f \in R_+^T$ belongs to the convex cone generated by the set of vectors $\{m^k\}$. It follows immediately from the Caratheodory Theorem (Theorem 1.22 in Valentine 1964) that we can select $T^* \leq T$ vectors from the set $\{m^k\}$ so that the vector f will belong to the convex cone of these T^* vectors. Let us renumber the vectors m^k if necessary, so that the selected vectors are $\left\{m^k\right\}_{k=1}^{T^*}$. Since f belongs to the convex cone of the vectors $\left\{m^k\right\}_{k=1}^{T^*}$ there exist nonnegative weights $\{\omega_k^*\}_{k=1}^{T^*}$ such that

$$f = \sum_{k=1}^{T^*} \omega_k^* m^k,\ T^* \leq T.$$

It follows that

$$v^b(N) = z \cdot f = z \cdot (\sum_{k=1}^{T^*} \omega_k^* m^k) = \sum_{k=1}^{T^*} \omega_k^* (z \cdot m^k) = \sum_{k=1}^{T^*} \omega_k^* \bar{v}(m^k)$$

We can write each ω_k^* as an integer plus a fraction, say $\omega_k^* = r_k + h_k$, where $h_k \in [0,1)$. Since the game (N, v) satisfies superadditivity it holds that

$$\sum_k r_k \bar{v}(m^k) \leq v(N).$$

Now

$$\begin{aligned} v^b(N) - v(N) &\leq \sum_{k=1}^{T^*} \omega_k^* \bar{v}(m^k) - \sum_{k=1}^{T^*} r_k \bar{v}(m^k) = \sum_{k=1}^{T^*} (\omega_k^* - r_k) \bar{v}(m^k) \\ &= \sum_{k=1}^{T^*} h_k \bar{v}(m^k) \leq C \sum_{k=1}^{T^*} h_k \left\| m^k \right\| = C \left\| \sum_{k=1}^{T^*} h_k m^k \right\|. \end{aligned}$$

Notice that, since both $\sum_{k=1}^{T^*} \omega_k^* m^k$ and $\sum_{k=1}^{T^*} r_k m^k$ are vectors of integers, $\sum_{k=1}^{T^*} h_k m^k$ is also a vector of integers. Moreover $\sum_{k=1}^{T^*} m^k$ is a vector of nonnegative integers. Now let $h^* = \max_{1 \leq k \leq T^*}(h_k)$. Obviously $h^* < 1$ and thus

$$\sum_{k=1}^{T^*} h_k m^k \leq h^* \sum_{k=1}^{T^*} m^k << \sum_{k=1}^{T^*} m^k$$

which implies that

$$\left\| \sum_{k=1}^{T^*} h_k m^k \right\| \leq \left\| \sum_{k=1}^{T^*} m^k \right\| - T \leq BT^* - T.$$

Thus, since $T^* \leq T$, it holds that

$$v^b(N) - v(N) \leq TC(B-1).$$

Hence, for any $\varepsilon \geq \frac{TC(B-1)}{|N|}$, the ε-core of (N, v) is nonempty. But note that, similarly to the argument before for the core, if some payoff x' belongs to the ε-core then z' defined by its components $z'_t := \frac{1}{|N[t]|} \sum_{a \in N[t]} x'_a$ also belongs to the ε-core. (Like the core, the ε-core of a TU game is convex). Therefore, the equal treatment ε-core of (N, v) is nonempty. ■

Now we will prove Corollary 1.

Proof of Corollary 1: Let $(N, v) \in \Gamma((\delta, T), C, (\beta, B))$ and let ε be a positive real number. We first construct another game with strictly effective groups bounded in size by B. From the definition of strictly effective groups, for any $S \subset N$ there exists a partition $\left\{S^k\right\}$ of S, $\left|S^k\right| \leq B$ for each k, such that $v(S) - \sum_k v(S^k) \leq \beta |S|$. Let us define $w(S) := \max_{\{s^k\}} \sum v(S^k)$ where the maximum is taken over all partitions $\left\{S^k\right\}$ of S with $\left|S^k\right| \leq B$ for each k. Then $(N, w) \in \Gamma((\delta, T), C, (0, B))$ and $\beta |S| \geq v(S) - w(S) \geq 0$ for any $S \subset N$.

Next we construct a related game by identifying all players of the same approximate type. First, for the game (N, w) let $\{N[t]\}$ be a δ-substitute partition of N. Given a group $S \subset N$ let s denote the profile of S. Define

$$w^*(S) := \max \{w(S') : S' \text{ has profile } s\}.$$

Define w^c as the superadditive cover of w^*, i.e. for any $S \subset N$,

$$w^c(S) := \max_{\{S^k\}} \sum_k w^*(S^k),$$

where the maximum is taken over all partitions of S. Then $(N, w^c) \in \Gamma((0, T), C, (0, B))$ and

$$\delta |S| \geq w^c(S) - w(S) \geq 0 \text{ for each } S \subset N.$$

By Lemma 1 the game (N, w^c) has a nonempty equal treatment $\frac{TC(B-1)}{|N|}$-core. Let x belong to the equal treatment $\frac{TC(B-1)}{|N|}$-core of (N, w^c). Hence

$$\sum_{a \in N} x_a \leq w^c(N) \text{ and } \sum_{a \in S} x_a + \frac{TC(B-1)}{|N|} |S| \geq w^c(S).$$

Now define a payoff vector y by

$$y(\{i\}) := x(\{i\}) - \delta$$

for each $i \in N$. Then

$$\sum_{a\in N} y_a = \sum_{a\in N} x_a - \delta |N|$$

$$\leq w^c(N) - \delta |N| \leq w(N) \leq v(N)$$

and for any group S it holds that

$$\sum_{a\in S} y_a + \left(\frac{TC(B-1)}{|N|} + \delta + \beta\right) |S| = \sum_{a\in S} x_a + \frac{TC(B-1)}{|N|} |S| + \beta |S|$$

$$\geq w^c(S) + \beta |S| \geq w(S) + \beta |S| \geq v(S).$$

It follows that y is in the ε-core for any $\varepsilon \geq \frac{TC(B-1)}{|N|} + \delta + \beta$. Since y has equal treatment property by construction, the equal treatment ε-core of (N, v) is nonempty. ■

Step 2: Construction of the TU games. Let us first modify the game (N, V) to avoid boundary problems. Consider the set

$$K := \left\{x \in \mathbf{R}^N : x_a \geq -\varepsilon \text{ for any } a \in N\right\}.$$

Define

$$K^* := K \bigcap V(N)$$

and observe that set K^* is a compact set. Let $Y(V(N), K)$ be the smallest closed cone such that

$$V(N) \subset K^* + Y(V(N), K).$$

Now let us define a modified game (N, V^1) so that

$$\text{(a) } V^1(S) \;:\; = V(S) \text{ for } S \neq N \text{ and}$$

$$\text{(b) } V^1(N) \;:\; = K^* + Y(V(N), K).$$

Notice that $Y(V(N), K)$ is the recession cone of $V^1(N)$; that is,

$$cone(V^1(N)) = Y(V(N), K).$$

Looking ahead, we are going to prove that there exists an equal treatment ε-core payoff x^* for the modified game (N, V^1). Since $V(S) \subset V^1(S)$ for any S, x^* will be ε-undominated in the game (N, V). Thus $x^* \in K \bigcap V^1(N) = K^* \subset V(N)$. So the payoff x^* will be feasible in the game (N, V). It will follow that x^* is an equal treatment ε-core payoff for (N, V).

Define

$$\mathcal{C} := co\left\{x \in \mathbf{R}^N : \exists i, j \in N,\ x_i = -q x_j \geq 0,\ x_k = 0,\ k \neq i, j\right\}$$

and observe that $\mathcal{C}$ is a cone. Since $V^1(N)$ is q-comprehensive and convex, the cone $cone(V^1(N))$ will include $\mathcal{C}$ but will not be more than a half-space. Hence the negative dual cone to the recession cone $dual(cone(V^1(N)))$ will be closed, nonempty and included in the cone dual to $\mathcal{C}$:

$$dual(\mathcal{C}) = \left\{x \in \mathbf{R}^N_{++} : q \leq \frac{x_i}{x_j} \leq \frac{1}{q}\ \forall i, j\right\} \bigcup \{0\}.$$

Now let us consider the simplex in $\mathbf{R}^N_+$:

$$\triangle_+ := \left\{\lambda \in \mathbf{R}^N_+ : \sum_{i=1}^{N} \lambda_i = 1\right\}.$$

Define

$$L := dual(cone(V^1(N))) \bigcap \triangle_+.$$

Given a partition $\{N[t]\}$ of the player set into T types of δ-substitutes, the set of equal treatment allocations is denoted by E^T and defined as follows:

$$E^T := \left\{ x \in \mathbf{R}^N : x_i = x_j \text{ for any } t \text{ and any } i, j \in N[t] \right\}.$$

Now define

$$L^* := L \cap E^T.$$

Observe that L^* is a compact and convex set.

For any $\lambda \in L^*$ there exists a tangent hyperplane to the set $V(N)$ with normal λ such that the whole set $V(N)$ is contained in a closed half-space, and at least one point of the set $V(N)$ lies on the hyperplane. Moreover, since the game is superadditive, for any $\lambda \in L^*$ and any $S \subset N$ there exists a hyperplane in $\mathbf{R}^S$ that has normal parallel to λ_S and that is tangent to V_S. Thus, for a fixed $\lambda \in L^*$ there is a finite real number

$$v_\lambda(S) := \max \left\{ \sum_{a \in S} \lambda_a x_a : x \in V(S) \right\}.$$

The pair (N, v_λ) is a TU game. We construct a "λ-weighted transferable utility" game (N, V_λ) by defining, for each coalition $S \subset N$:

$$V_\lambda(S) := \left\{ x \in \mathbf{R}^N : x_a = 0 \text{ for } a \notin S \text{ and } \sum_{a \in S} \lambda_a x_a \leq v_\lambda(S) \right\}.$$

■

Step 3: Nonemptiness of the epsilon core for the game (N, V_λ).

Consider a fixed $\lambda \in L^*$. Define $\lambda_{\max} := \max\{\lambda_i\}$ and $\lambda_{\min} := \min\{\lambda_i\}$.

Lemma 2. Let $(N, V) \in \mathcal{G}_1^q((0,T), C, (\beta, B))$. Then

$$(N, v_\lambda) \in \Gamma((0,T), C\lambda_{\max}, (\beta\lambda_{\max}, B)).$$

Proof of Lemma 2:

1). We will prove that the $(0,T)$-partition $\{N[t]\}$ of the game (N, V) is a $(0,T)$-partition of the game (N, v_λ). We must check that for any type-consistent permutation τ of N and any coalition S it holds that $v_\lambda(S) = v_\lambda(\tau(S))$. But we have:

$$\begin{aligned} v_\lambda(\tau(S)) &\equiv \max\left\{\sum_{a\in\tau(S)} \lambda_a x_a : x \in V(\tau(S))\right\} \\ &= \max\left\{\sum_{a\in S} \lambda_{\tau(a)} x_a : x \in V(S)\right\} \\ &= \max\left\{\sum_{a\in S} \lambda_a x_a : x \in V(S)\right\} \equiv v_\lambda(S). \end{aligned}$$

The second equality follows from the fact that $V(\tau(S)) = V(S)$, since $\{N[t]\}$ is a $(0,T)$-partition of the game (N, V). The third equality holds since, by construction of L^* and τ, for any a we have $\lambda_a = \lambda_{\tau(a)}$.

2). To show that the number $\lambda_{\max} C$ is a per capita bound for the TU game (N, v_λ), it is necessary to show that $\frac{v_\lambda(S)}{|S|} \leq \lambda_{\max} C$ for each coalition group S. Observe that by the definition of $v_\lambda(S)$, for some $x_a \in V_S$ it holds that

$$\frac{v_\lambda(S)}{|S|} = \frac{\sum_{a \in S} \lambda_a x_a}{|S|} \leq \lambda_{\max} \frac{\sum_{a \in S} x_a}{|S|} \leq \lambda_{\max} C.$$

The last inequality follows from per capita boundedness of the game (N, V).

3). To prove effectiveness of B-bounded $\lambda_{\max}\beta$-effective groups for the TU game (N, v_λ) we need to show that for any $S \subset N$ there exists a partition $\{S_k\}$ of S satisfying $|S_k| \leq B$ for each k and

$$\left| v_\lambda(S) - \sum_k v_\lambda(S_k) \right| \leq \lambda_{\max} \beta \, |S| \, .$$

By superadditivity

$$v_\lambda(S) \geq \sum_k v_\lambda(S_k).$$

By the definition of v_λ there exists a vector x such that $x \in V(S)$ and $v_\lambda(S) = \sum_{a \in S} \lambda_a x_a$. Since (N, V) has weakly β-effective B-bounded groups there exists a vector $y \in c_q(V(S; B))$ such that

$$\sum_{a \in S} |y_a - x_a| \leq \beta \, |S| \, .$$

Then there exists a vector $z \in V(S; B)$ such that $y \in c_q(z)$. Note that since $\lambda \in L^* \subset dual(\mathcal{C})$ and $y \in c_q(z)$ we have

$$\sum_{a \in S_k} \lambda_a y_a \leq \sum_{a \in S_k} \lambda_a z_a.$$

Then since $z \in V(S;B)$ we have that $z_{S_k} \in V_{S_k}$ for some partition $\{S_k\}$ of S (with $|S_k| \leq B$) and we get

$$\sum_{a\in S_k} \lambda_a z_a \leq v_\lambda(S_k).$$

Hence

$$\left|v_\lambda(S) - \sum_k v(S_k)\right| \leq \left|\sum_{a\in S}\lambda_a x_a - \sum_k \sum_{a\in S_k}\lambda_a z_a\right| \leq \left|\sum_{a\in S}\lambda_a x_a - \sum_k \sum_{a\in S_k}\lambda_a y_a\right|$$

$$\leq \sum_{a\in S}\lambda_a |x_a - y_a| \leq \lambda_{\max}\beta |S|.$$

By 1), 2), 3) it holds that $(N, v_\lambda) \in \Gamma((0,T), C\lambda_{\max}, (\beta\lambda_{\max}, B))$. ■

Lemma 3. If the equal treatment ε-core of (N, v_λ) is nonempty, then the equal treatment $\frac{\varepsilon}{\lambda_{\min}}$-core of (N, V_λ) game is nonempty.

Proof of Lemma 3: Consider a payoff y in the equal treatment ε-core of the game (N, v_λ) and define $x_a := \frac{1}{\lambda_a}y_a$. Note that x also has equal treatment property. Then

$$\sum_{a\in N}\lambda_a x_a = \sum_{a\in N} y_a \leq v_\lambda(N);$$

thus x is feasible for the game (N, V_λ). Moreover, for all $S \subset N$,

$$\sum_{a\in S}\lambda_a(x_a + \frac{\varepsilon}{\lambda_{\min}}) = \sum_{a\in S} y_a + \sum_{a\in S}\lambda_a\frac{\varepsilon}{\lambda_{\min}} \geq \sum_{a\in S} y_a + \varepsilon|S| \geq v_\lambda(S)$$

thus x is $\frac{\varepsilon}{\lambda_{\min}}$-undominated in the game (N, V_λ). Therefore, x is in the equal treatment $\frac{\varepsilon}{\lambda_{\min}}$-core of (N, V_λ). ■

We can now finish Step 3: Since $(N,V) \in \mathcal{G}_1^q((0,T),C,(\beta,B))$, by Lemma 2 we have that

$$(N, v_\lambda) \in \Gamma((0,T), C\lambda_{\max}, (\beta\lambda_{\max}, B)).$$

But from Corollary 1 for any game with side payments in $\Gamma((\delta',T),C',(\beta',B))$ and any $\varepsilon^0 \geq \frac{TC'(B-1)}{|N|} + \delta' + \beta'$, the equal treatment ε^0-core is nonempty. Hence, if $\varepsilon^0 \geq \lambda_{\max}(\frac{TC(B-1)}{|N|} + \beta)$, the equal treatment ε^0-core of (N, v_λ) is nonempty. From Lemma 3 this implies that the equal treatment $\frac{\varepsilon^0}{\lambda_{\min}}$-core of (N, V_λ) is nonempty. Thus, since

$$\alpha_N^q((0,T),C,(\beta,B)) = \frac{1}{q}(\frac{TC(B-1)}{|N|} + \beta) \geq \frac{\lambda_{\max}}{\lambda_{\min}}(\frac{TC(B-1)}{|N|} + \beta)$$

$(\frac{\lambda_{\max}}{\lambda_{\min}} \leq \frac{1}{q}$ because $\lambda \in L^* \subset dual(\mathcal{C}))$, we can conclude that if

$$\varepsilon \geq \alpha_N^q((0,T),C,(\beta,B))$$

the equal treatment ε-core of (N, V_λ) is nonempty. This is exactly the bound that we need in the symmetric case. ■

Step 4: Nonemptiness of the epsilon core for the initial game. We need only to prove that if the equal treatment ε-core of (N, V_λ) is nonempty for all $\lambda \in L^*$ then the equal treatment ε-core of (N, V) is nonempty. Define

$$Core_\varepsilon(\lambda) := \left\{ x : \sum_{a \in N} \lambda_a x_a \leq v_\lambda(N), \sum_{a \in S} \lambda_a (x_a + \varepsilon) \geq v_\lambda(S) \right\} \bigcap E^T,$$

the equal treatment ε-core of the (N, V_λ) game. Note that the equal treatment ε-core of (N, V_λ) is nonempty for any $\lambda \in L^*$. For any $\lambda \in L^*$ and any $x \in Core_\varepsilon(\lambda)$,

x cannot be ε'-improved upon in the initial game (N, V) for any $\varepsilon' > \varepsilon$. (If a coalition S could improve, we would have $x_S + \varepsilon'_S \in V_S$ and $\sum_{a \in S} \lambda_a (x_a + \varepsilon'_a) > v_\lambda(S)$, contradicting the definition of $v_\lambda(S)$.) Hence, it remains to show that there exists $\lambda^* \in L^*$ such that some $x^* \in Core_\varepsilon(\lambda)$ is feasible in the initial game.

Lemma 4. The correspondence $\lambda \longmapsto Core_\varepsilon(\lambda)$ from L^* to $\mathbf{R}^N$ is bounded, convex-valued and has a closed graph. Moreover, for any $x \in Core_\varepsilon(\lambda)$ and for any player a it holds that $x_a \geq -\varepsilon$.

Proof of Lemma 4:

1). If f, $g \in Core_\varepsilon(\lambda)$ then $\mu f + (1 - \mu) g$ has equal treatment property and $\mu f + (1 - \mu) g \in Core_\varepsilon(\lambda)$ since:

$$\text{(a)} \sum_{a \in N} \lambda_a (\mu f_a + (1-\mu) g_a) = \mu \sum_{a \in N} \lambda_a f_a + (1-\mu) \sum_{a \in N} \lambda_a g_a \leq \mu v_\lambda(N) + (1-\mu) v_\lambda(N) = v_\lambda(N) \text{ and}$$

$$\text{(b)} \sum_{a \in S} \lambda_a (\mu f_a + (1-\mu) g_a + \varepsilon) = \mu \sum_{a \in S} \lambda_a (f_a + \varepsilon) + (1-\mu) \sum_{a \in S} \lambda_a (g_a + \varepsilon) \geq \mu v_\lambda(S) + (1-\mu) v_\lambda(S) = v_\lambda(S).$$

2). It is straightforward to see that graph is closed since $v_\lambda(S)$ depends continuously on λ.

3). Consider $x \in Core_\varepsilon(\lambda)$. Since x is in the ε-core of (N, V_λ) game, x is ε-individually rational, that is, $x_a \geq -\varepsilon$.

4). Consider $x \in Core_\varepsilon(\lambda)$. By construction,

$$\sum_{a \in N} \lambda_a x_a \leq v_\lambda(N) \leq \frac{1}{q} C \left| N \right|.$$

Since $\lambda \in L^* \subset L \subset \triangle_+$, there exists i such that $\lambda_i \geq \frac{1}{|N|}$. Then $\lambda \in L$ implies $\lambda_a \geq q\lambda_i \geq \frac{q}{|N|}$. Therefore, using 3) above we have that

$$\frac{q}{|N|} x_a \leq \lambda_a x_a \leq \frac{1}{q} C \left| N \right| - (1 - \frac{q}{|N|})(-\varepsilon).$$

This proves that

$$x_a \leq \frac{1}{q^2} C \left| N \right|^2 + (\frac{|N|}{q} - 1)\varepsilon. \blacksquare$$

Now let us define

$$\Psi(\lambda) := Core_\varepsilon(\lambda) - \left\{ x \in K^* : \sum_{a \in N} \lambda_a x_a = \max_{z \in V(N)} \sum_{a \in N} \lambda_a z_a \right\} \bigcap E^T.$$

For $\lambda \in L^*$ both the first term and the second term of this sum are nonempty, bounded, convex-valued correspondences with closed graphs; this follows from Lemma 3 and the observations that (a) $V(N)$ is convex and symmetric across substitute players and (b) K^* is compact. Hence the sum $\Psi(\lambda)$ is also bounded, closed and convex-valued for $\lambda \in L^*$. By construction $\sum_{a \in N} z_a \lambda_a \leq 0$ for any $z \in \Psi(\lambda)$.

Now we can use the following theorem of excess demand, which is in fact a version of Kakutani's theorem. (For a proof see Hildenbrand and Kirman 1988, Lemma AIV.1)

Theorem (Debreu, Gale, Nikaido): Let $\triangle^*$ be a closed and convex subset of $\triangle_+$. If the correspondence Ψ from $\triangle^*$ is bounded, convex-valued, has closed graph and it holds that for all $p \in \triangle^*$, $p \cdot z \leq 0$ for all $z \in \Psi(p)$, then there exists $p^* \in \triangle^*$ and $z^* \in \Psi(p^*)$ such that $p \cdot z^* \leq 0$ for all $p \in \triangle^*$.

It follows, from the Debreu-Gale-Nikaido Theorem, that there exists $\lambda^* \in L^*$ and $z^* \in \Psi(\lambda^*)$ such that $\lambda \cdot z^* \leq 0$ for all $\lambda \in L^*$. Since $z^* \in \Psi(\lambda^*)$, z^* can be represented as $z^* = x^* - y^*$ with $x^* \in Core_\varepsilon(\lambda)$, $y^* \in K^* \bigcap E^T$. Therefore $z^* \in E^T$. As we argued at the beginning of this Step, x^* is ε-undominated in the initial game (N, V). In addition, x^* has the equal treatment property.

We now deduce that x^* is feasible for the game (N, V). Observe that $x^* = y^* + z^*$, where $y^* \in K^* \bigcap E^T$ and $\lambda \cdot z^* \leq 0$ for all $\lambda \in L^*$. Hence $z^* \in dual(L^*) \bigcap E^T$.

Lemma 5. Let X be a convex and symmetric across substitute players subset of $\mathbf{R}^N$. Let $X^* := X \bigcap E^T$. Then $dual(X^*) \bigcap E^T \subset dual(X)$.

Proof of Lemma 5: For any $x \in X$, let us construct $\bar{x} \in \mathbf{R}^N$ as follows: for each $1 \leq t \leq T$, for any $a \in N[t]$ define

$$\bar{x}_a := \frac{1}{|N[t]|} \sum_{i \in N[t]} x_i.$$

Since X is convex and symmetric across substitute players, $\bar{x} \in X$. Obviously, $\bar{x} \in E^T$. Therefore $\bar{x} \in X \bigcap E^T = X^*$.

Now consider any $y \in dual(X^*) \bigcap E^T$. For any $x \in X$ we have

$$\begin{aligned} y \cdot x &= \sum_{i \in N} y_i x_i = \sum_{1 \leq t \leq T} y_t \Big(\sum_{i \in N[t]} x_i \Big) \\ &= \sum_{1 \leq t \leq T} |N[t]| \, y_t \bar{x}_t = \sum_{i \in N} y_i \bar{x}_i \leq 0, \end{aligned}$$

where the last inequality follows from the fact $y \in dual(X^*)$ and $\bar{x} \in X^*$. Hence, by the definition of the dual negative cone, $dual(X^*) \bigcap E^T \subset dual(X)$. ■

Since $V(N)$ is convex and symmetric across substitute players, it follows from construction of $cone(V^1(N))$ that $L = dual(cone(V^1(N))) \bigcap \triangle_+$ is convex and symmetric across substitute players. Therefore, by Lemma 5,

$$z^* \in dual(L^*) \bigcap E^T \subset dual(L) = cone(V^1(N)).$$

Moreover

$$x^* \in K^* + cone(V^1(N)) \subset V^1(N),$$

that is, x^* is feasible in the modified game. We also have $x^* \in Core_\varepsilon(\lambda)$. It follows from Lemma 3 that $x^*_a \geq -\varepsilon$. It now follows from the definition of K and K^* that $x^* \in K^* \subset V(N)$, that is, x^* is feasible in the initial game (N, V). We have now proven that x^* is in the equal treatment ε-core of the initial game; therefore the equal treatment ε-core is nonempty. ■

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