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# APPROXIMATE CORES OF LARGE GAMES

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The core of a game, which is an abstraction of the core or set of cooperative equilibrium states of an economy, is a fundamental notion of social equilibrium. However, except for games derived from special kinds of economic situations or satisfying restrictive (balancedness) conditions, the core is usually empty. In contrast, this paper shows that, with mild and economically natural assumptions, large games always have non-empty approximate cores. The game-theoretic framework is sufficiently general to cover a wide variety of economic situations.

## 1. INTRODUCTION

THE CORE IS A FUNDAMENTAL CONCEPT of social and economic equilibrium. However, except in idealized situations or with special assumptions (such as balancedness), the core may well be empty. In the study of private goods exchange economies, this problem has been addressed by the introduction of concepts of approximate cores and the derivation of conditions ensuring approximate cores are non-empty; the structure of private goods exchange economies, is itself, however, special. Relatively little work of this sort has been done on other kinds of economies.[2]

In this paper, we establish the non-emptiness of approximate cores for a wide diversity of large economies with weak and natural economic assumptions (no balancedness is assumed). The framework we adopt is that of games in characteristic function form. This framework is sufficiently general to accommodate, not only private goods economies, but also economies with pure and local public goods, with coalition production, with hedonic coalitions, and with heterogeneous and/or indivisible goods. In this framework, our results establish the non-emptiness of approximate cores for large games (i.e., games with many players) and show that the "approximation" can be made arbitrarily good as the games grow large. Moreover, there always exist payoffs with the property that agents who are "scarce" and hence, "in demand," can command all the excess profits from formation of coalitions. (It is gratifying that this intuitive economic idea has a valid formal statement in such a general framework.)

To model large games, we introduce the notion of a pregame with attributes. This formalizes a situation in which the value of a coalition depends only on the attributes of its members, and changes only slightly if some members are replaced by others with similar attributes. (If the space of attributes is finite, this reduces to a situation in which each game has a finite number of types of players, and

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[2] We discuss the literature in more detail in Section 2.

players having the same type are identical.) Our formalization is thus analogous to ones used for large private goods exchange economies, where an agent is characterized by his endowment and preferences (which would here simply represent a point in attribute space). We stress that our framework is, in most respects, more general than that of private goods exchange economies, and our techniques are quite different than those usually used to study such economies. In particular, the convexifying effect of large numbers plays no role in our work. Indeed, Bewley [**5**] has shown that the usual techniques used in private goods exchange economies have limited applicability in other economic situations. Another example illustrating this point for economies with local public goods is in Shubik and Wooders [**3**, Appendix]. In that example agents are "divisible," preferences and production possibilities are convex, yet the core may well be empty; this occurs because of congestion effects—a "local public bad."

In this paper, we restrict our attention to games with sidepayments. Although the framework of games without sidepayments would be more general, many economic situations are already covered by the framework of games with sidepayments and this framework is more easily understood and applied. We plan to treat the more general framework in a later paper, but that treatment will require rather different (and more complicated) assumptions and techniques and, of necessity, yield somewhat different results. In particular, the exact analogs of our results for games with sidepayments are simply not valid for games without sidepayments.

Detailed applications of our results are too lengthy for this paper. We intend to give a number of diverse examples, including ones with local public goods and with production of differentiated commodities, in subsequent work. Here, we briefly note a few additional applications; many others are possible. In fact, the results of this paper have already been applied to a spatial model by Cremer, de Kerchove, and Thisse [**9**]. Currently, they are being applied by Berliant and the authors to a model of an economy with land, developed by Berliant [**3**] and Berliant and Dunz [**4**]. For models in the literature on assignment-type problems (cf. Kaneko [**14**] and Crawford and Knoer [**8**]), our results can be used to obtain existence of approximate equilibria without some of the restrictions required for existence of (exact) equilibria. Finally, we remark that our results can be applied to economies with "club structures"—ones where agents may belong to different clubs for the purposes of consumption and/or production of different goods (public or private).

The remainder of this paper is organized as follows. In the next section, we discuss the relationship of our work to the literature. The third (short) section reviews some well-known game theoretic concepts, including the $\varepsilon$-core, and introduces the individually rational $\varepsilon$-core. A motivating example is provided in Section 4. Section 5 contains the formal framework of our model and a statement of results. We then provide a set of examples illustrating the necessity of our various hypotheses. Although this section could be skipped by the uninterested reader, we feel it provides insights into the mechanisms of the proofs. The detailed proofs themselves are presented in the last section.

## 2. RELATIONSHIPS TO THE LITERATURE

The concept underlying the core was introduced by Edgeworth [11] to describe the set of states of an economy upon which no group of agents can improve. Edgeworth called this set the "contract curve" although it is now frequently called the "core of the economy." The term "core" was introduced in the game-theoretic framework by Gillies [12]. These two notions were related by Shubik [27] who showed that, for private goods economies with transferable utility, states in the core of an economy correspond to payoffs in the core of the game generated by the economy.

The notion of balancedness was introduced independently by Bondareva [6, 7] and Shapley [22], who showed that a game (with sidepayments) is balanced if and only if it has a non-empty core. A number of classes of games are known to be balanced and thus have non-empty cores, including market games (Shapley and Shubik [24]), assignment games (Shapley and Shubik [26]), and convex games (Shapley [23]). Kaneko and Wooders [15] introduced the notion of partitioning games, generalizing assignment games, and determined which such games have non-empty cores. Numerous particular games are known to have non-empty cores; however, numerous examples are known which illustrate that, even for games derived from natural economic models, the core may well be empty (cf. Shapley and Shubik [25], Wooders [35], Shubik [28], and Kaneko and Wooders [15]).

The $\varepsilon$-core was introduced by Shapley and Shubik [25]. Using an extension of Scarf's [21] balancedness condition (for games without sidepayments), Weber [32] has shown that certain games with a continuum of players have non-empty $\varepsilon$-cores (for arbitrarily small $\varepsilon$). The study of approximate cores of large games, without balancedness assumptions, was initiated by Wooders [33], where the concept of a replica game was introduced and conditions determined under which large replica games have non-empty $\varepsilon$-cores. Stronger forms of Wooders' result are obtained by Kaneko and Wooders [15] for the class of partitioning games.

The literature on private goods exchange economies is vast and we make no attempt to survey it here. The portion which is most relevant to the present work deals with existence and convergence questions for cores, approximate cores, competitive equilibria and approximate competitive equilibria for large economies; see for example Debreu and Scarf [10], Shapley and Shubik [25], Kannai [16, 17], Hildenbrand, Schmeidler, and Zamir [13], Mas-Colell [20], Anderson [1], Khan and Yamazaki [19], Khan and Rashid [18], and Anderson, Khan, and Rashid [2].

To explain the relationship of the present work to this literature, let us point out that Shapley and Shubik [25] and Debreu and Scarf [10] allow only a finite number of types of agents, and their economies "become large" by replication. Wooders [33] adopts a game-theoretic framework but maintains the restrictions to a finite number of types and to replication. Our results do not require the restriction to a finite number of types or to replication. Rather, our games "become large" in a manner similar to that employed by, for example, Hildenbrand, Schmeidler, and Zamir [13] or Anderson [1]. As we have already mentioned, our game-theoretic framework is much more general than the framework of private goods exchange economies, although in this paper we restrict ourselves to games with sidepayments. (We hope to treat the more general case of games without sidepayments in later work.)

In the non-private-goods-exchange economy literature, some results have been obtained showing non-emptiness of approximate cores of large economies, without balancedness assumptions. We reference Wooders [34], in which it is shown that a class of large economies with local public goods have non-empty approximate cores and note that, as in this paper, the results are obtained without use of the convexifying effect of large numbers. Also, using the results in Wooders [33], in Shubik and Wooders [29] it is shown that a more general class of economies with local public goods has non-empty approximate cores; in Shubik and Wooders [30, 31], similar results are obtained for coalition production economies. These papers all involve "types" and replication.

## 3. GAMES

A *game with sidepayments* or simply a *game* is an ordered pair $(N, v)$ where $N$ is a finite set, called the set of *players*, and $v$, called the *characteristic function*, maps subsets of $N$ into the non-negative reals $\mathbb{R}^+$ with $v(\phi) = 0$. Given a subset $S$ of $N$, *the value of* $S$ is $v(S)$. Two players $i$ and $j$ are *substitutes* if for all subsets $S$ of $N$ where $i \notin S$ and $j \notin S$, we have $v(S \cup \{i\}) = v(S \cup \{j\})$. The game is *superadditive* if for all disjoint subsets $S$ and $S'$ of $N$, we have $v(S) + v(S') \leqslant v(S \cup S')$. A *payoff* for the game is a vector $x = (x^1, \ldots, x^i, \ldots, x^N)$ in $\mathbb{R}^N$ where $x^i$ is interpreted as the payoff of the $i$th player. A payoff $x$ is *feasible* if $x(N) \leqslant v(N)$ where, for any subset $S$ of $N$, we define $x(S) = \sum_{i \in S} x^i$. Given a real number $\varepsilon \geqslant 0$, a payoff $x$ belongs to the $\varepsilon$*-core* of $(N, v)$ if (a) $x(N) = v(N)$, (b) $x(S) \geqslant v(S) - \varepsilon|S|$ for all subsets $S$ of $N$, where $|S|$ denotes the cardinal number of the set $S$. The payoff $x$ belongs to the *individually rational* $\varepsilon$*-core* if it belongs to the $\varepsilon$-core and, in addition, (c) $x^i \geqslant v(\{i\})$ for all $i$ in $N$. When $\varepsilon$ equals zero, the $\varepsilon$-core is simply the *core*.

Less formally, the $\varepsilon$-core for small but positive $\varepsilon$ is characterized by the requirement that the members of every coalition $S$ receive at least nearly the value of that coalition—$x(S) \geqslant v(S) - \varepsilon|S|$. As pointed out by Shapley and Shubik [25], a payoff $x$ in the $\varepsilon$-core can be interpreted as a stable payoff given that players are satisficing or given some organizational cost prerequisite to cooperative action and proportional to the parameter $\varepsilon$. We consider approximate cores with and without the requirement of individual rationality since in some situations this requirement may be natural whereas in others it may be unnecessarily restrictive.

## 4. MOTIVATION—A PRODUCTION ECONOMY

The intuitive ideas we wish to capture in considering games with types or attributes run along the following lines. Consider a two-sided economy with only firms and workers. We imagine that the firms are identical and also that the workers are identical. If we model this economy as a game, it will have as many players as the total of firms and workers but these players are evidently of only two "types," and the value of any coalition depends, not on the precise make-up, but rather only on the number of players of each type in a coalition. More generally, we may imagine that the firms (respectively, workers) are not identical, but merely very similar, and the value of a coalition depends, again not on the precise make-up, but rather only on (some of) the attributes of its members, and that this value changes only a little with small changes in the attributes of the members of the coalition.

It may help if we discuss in detail a very simple two-sided economy and actually describe its $\varepsilon$-core.

Consider an economy with a number $F$ of identical firms and a number $W$ of identical workers. Neither firms nor workers can make a profit independently. A coalition of one firm and $k$ workers can make a profit $p(k)$ given by

$$p(k)=\begin{cases} k^2 & \text{for } 1 \leqslant k \leqslant 10, \\ 100 & \text{for } k \geqslant 10. \end{cases}$$

(This represents a situation where marginal profit is linearly increasing and positive for $1 \leqslant k \leqslant 10$ and zero for $k > 10$; see Figures 1 and 2.)

We ask: For what values of $F$, $W$, and $\varepsilon$ is the individually rational $\varepsilon$-core non-empty? Let us take $\varepsilon = 1$ and analyze the situation in detail.

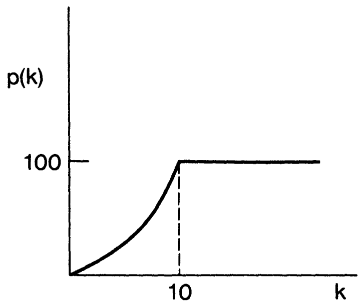


FIGURE 1—Profits.

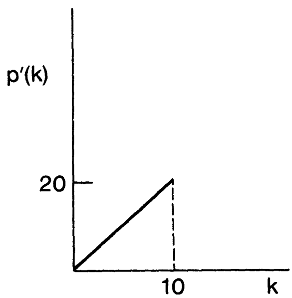


FIGURE 2—Marginal profits.

Let us suppose first that $W \leqslant 10F$. In that case, we can write $W = 10q + r$ where $q$ and $r$ are non-negative integers with $0 \leqslant r < 10$ and $q \leqslant F$. The most profitable hiring profile then assigns 10 workers to each of $q$ firms, $r$ workers to one of the firms, and no workers to the remaining firms (if any). The total profit is then $100q + r^2$; if we distribute the entire profit equally among all workers, they each obtain $(100q + r^2)/(10q + r)$, while the firms obtain nothing. This payoff is certainly individually rational. For it to be in the individually rational $\varepsilon$-core with $\varepsilon = 1$, we need to know that no coalition could improve upon it by more than one unit per member. Obviously, the most profitable coalitions consist of one firm and ten workers (earning a total of 100 units) so we need to have

$$10\left(\frac{100q + r^2}{10q + r}\right) \geqslant 100 - 11,$$

i.e.,

$$1000q + 10r^2 \geqslant 89(10q + r),$$

$$1000q + 10r^2 \geqslant 890q + 89r,$$

or

$$G(r, q) := 10r^2 - 89r + 110q \geqslant 0.$$

If we minimize $G(r, q)$ as a function of $r$ (holding $q$ fixed) we see that the minimum is attained when $r = 89/20$ and the minimum value is then

$$10\left(\frac{89}{20}\right)^2 - \frac{(89)^2}{20} + 110q.$$

In other words, $G(r, q)$ is positive whenever $q \geqslant 3$. We conclude that the individually rational $\varepsilon$-core (for $\varepsilon = 1$) is non-empty whenever there are at least 30 workers and $W \leqslant 10F$.

On the other hand, suppose that $W > 10F$. Then the total profit (i.e., $100F$) can be allocated equally among the firms (who obtain 100 each); this allocation is actually in the core, which is automatically individually rational.

We conclude that the individually rational $\varepsilon$-core is non-empty (for $\varepsilon = 1$) whenever

$$W \leqslant 10F \quad \text{and} \quad W \geqslant 30$$

or

$$W > 10F.$$

In other words, the individually rational $\varepsilon$-core is non-empty (for $\varepsilon = 1$) whenever the total number of workers is at least 30.

We remark that when individual rationality is not required, the profit allocations we have constructed in the individually rational $\varepsilon$-core could all be altered slightly while remaining in the $\varepsilon$-core.

This example has a number of features we wish to stress, since they are typical of the general situation. Most importantly, notice that non-emptiness of the $\varepsilon$-core depends only on the large number of firms and workers, while the distribution of profits for an allocation in the $\varepsilon$-core depends strongly on the relative abundance of firms and of workers. The particular shape of the profit function is of no importance (except to facilitate computation); its salient feature is boundedness, which means that the marginal contributions of agents are bounded. We chose a profit function for which the marginal profit function was increasing in an interval to emphasize that the core may be empty even if the $\varepsilon$-core is not.

## 5. GAMES WITH ATTRIBUTES

We need first of all to formalize the notion of a game with attributes. The intuition we wish to capture is that the value of a coalition depends on the attributes of its members and coalitions containing similar players have similar values.

Let $\mathscr{A}$ be a compact metric space (*the space of attributes*) with distance function $d$; it is convenient to assume that $d(a, b) \leq 1$ for each $a, b$ in $\mathscr{A}$. By the *support* of a function $f: \mathscr{A} \to Z^+$ we mean the set $\operatorname{supp}(f)$ of all points $a$ in $\mathscr{A}$ where $f(a) \neq 0$. Let $\mathscr{F}(\mathscr{A})$ denote the set of all functions from $\mathscr{A}$ to $Z^+$ with finite support. Notice that $\mathscr{F}(\mathscr{A})$ is an additive semigroup, with addition of functions given by:

$$(f+g)(a) = f(a) + g(a).$$

The zero element of the semigroup $\mathscr{F}(\mathscr{A})$ is the zero function 0. For $k$ a positive integer, we will write $kf$ for the sum of $f$ with itself $k$ times. We set $|f| = \sum_{a \in \mathscr{A}} f(a)$; notice that this sum is finite since $f$ has finite support, and can be interpreted as the number of elements in the support of $f$ if we count each element as many times as its "multiplicity" $f(a)$.

We define a distance function $\bar{d}$ on $\mathscr{F}(\mathscr{A})$ in the following way. If $f, g$ are in $\mathscr{F}(\mathscr{A})$ and $|f| \neq |g|$, then $\bar{d}(f, g) = 1$. If $|f| = |g|$, then we list the points in the supports of $f$ and $g$, with each point $a$ occurring as many times as its multiplicity $f(a)$ or $g(a)$ (depending on which list we are considering):

$$\operatorname{supp}(f) = \{a_1, \ldots, a_k\},$$

$$\operatorname{supp}(g) = \{b_1, \ldots, b_k\}.$$

Since $|f| = |g|$, these two lists are the same length; we set

$$\bar{d}(f, g) = \min \max d(a_i, b_{\pi(i)})$$

where the maximum extends over the indices $i = 1, 2, \ldots, k$ and the minimum extends over all permutations $\pi$ of the index set $\{1, 2, \ldots, k\}$. With this distance function, $\mathscr{F}(\mathscr{A})$ becomes a metric space.

We will say that a function $\Omega: \mathscr{F}(\mathscr{A}) \to \mathbb{R}^+$ is *uniformly continuous per capita* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $f, g \in \mathscr{F}(\mathscr{A})$ with $|f| = |g|$ and $\bar{d}(f, g) < \delta$ then $|\Omega(f) - \Omega(g)| < \varepsilon |g| = \varepsilon |f|$. (Note that we can ignore the case $|f| \neq |g|$ since any two such functions are at distance 1 from each other.) The function $\Omega$ is *superadditive* if $\Omega(f+g) \geq \Omega(f) + \Omega(g)$ for all $f, g$ in $\mathscr{A}$ and $\Omega(\underset{\sim}{0}) = 0$.

By a *pregame with attributes* we mean a pair $(\mathscr{A}, \Omega)$ where $\mathscr{A}$ is a compact metric space and $\Omega : \mathscr{F}(\mathscr{A}) \to \mathbb{R}^+$ is uniformly continuous per capita and superadditive. By an *individual marginal bound* for the pregame $(\mathscr{A}, \Omega)$ we mean a real number $M$ such that

$$\Omega(f+\chi_a) \leqslant \Omega(f) + M$$

for each $f$ in $\mathscr{F}(\mathscr{A})$ and each $a$ in $\mathscr{A}$, where $\chi_a$ is the function which is 1 at $a$ and 0 elsewhere on $\mathscr{A}$.

A few words of explanation seem in order. We think of the points of $\mathscr{A}$ as representing attributes which may be possessed by a player in a game. Two points of $\mathscr{A}$ are close together if they represent similar attributes. A function $f$ in $\mathscr{F}(\mathscr{A})$ represents a coalition of players, of whom $f(a)$ possess the attribute $a$. Two functions $f, g$ in $\mathscr{F}(\mathscr{A})$ are close together if they represent coalitions having the same number of players and having the same number of players with similar attributes. The value of $\Omega$ at a function $f$ is just the value of the coalition represented by $f$ (so that coalitional values depend only on the attributes of the members). Our continuity requirement is that a small change in the attributes of members of a coalition produces a change in the coalitional value which is small (in per capita terms). The existence of an individual marginal bound simply means that the marginal contribution of any player to any coalition is bounded. Note that an individual marginal bound implies what Wooders [**33**] and Kaneko and Wooders [**15**] call a per capita bound, i.e., $v(S)/|S|$ is bounded. The reverse is not generally true, although it is true in the replication framework used by Kaneko and Wooders.

To derive a game from a pregame with attributes, we assume given a finite non-empty set $N$ and a function $\alpha : N \to \mathscr{A}$ (an *attribute function*). For $a$ in $\mathscr{A}$, the cardinality $|\alpha^{-1}(a)|$ of the set $\alpha^{-1}(a)$ is thus the number of players in $N$ who possess the attribute $a$. For each subset $S$ of $N$, we write $f_S$ for the function in $\mathscr{F}(\mathscr{A})$ whose value at the attribute $a$ is $|\alpha^{-1}(a)|$. The *derived game* $(N, v_\alpha)$ is then defined by setting

$$v_\alpha(S) = \Omega(f_S)$$

for each subset of $S$ of $N$. Superadditivity of the game $(N, v_\alpha)$ follows easily from superadditivity of the function $\Omega$.

Our first result can now be stated in the following way:

THEOREM 1: *Let $(\mathscr{A}, \Omega)$ be a pregame with attributes which has an individual marginal bound. Let $\varepsilon > 0$ be a positive number. Then there is an integer $n_0$ such that if $N$ is any finite set with at least $n_0$ elements and $\alpha : N \to \mathscr{A}$ is any attribute function, then the derived game $(N, v_\alpha)$ has a non-empty $\varepsilon$-core.*

We remark that the payoff vectors we construct in the $\varepsilon$-core will have an equal-treatment property: players whose attributes are similar will obtain the same payoff.

Note that in Theorem 1 we do not assert non-emptiness of the individually rational $\varepsilon$-core; this is not so, as can be seen by simple examples (e.g., Example 2 of Section 6). In order to guarantee non-emptiness of the individually rational $\varepsilon$-core, stronger assumptions need to be made. Examination of the proofs in Section 7 should convince the reader that a sufficient requirement is that "enough" players can effect a non-trivial profit in some coalition. One way to ensure this is to require that "enough" players are similar to each other.

THEOREM 2: *Let $(\mathscr{A}, \Omega)$ be a pregame with attributes which has an individual marginal bound. Let $\varepsilon > 0$ be a positive number. Then there is an integer $n_0$ and a positive number $\delta$ such that: if $N$ is any finite set and $\alpha : N \to \mathscr{A}$ is any attribute function with the property that for each player $i$ in $N$ there are distinct players $j_1, \ldots, j_{n_0}$ in $N$ such that $d(\alpha(i), \alpha(j_k)) < \delta$ for $k = 1, 2, \ldots, n_0$, then the derived game $(N, v_\alpha)$ has a non-empty individually rational $\varepsilon$-core.*

Once again, the vectors we construct will have an equal-treatment property.

A special case of particular interest occurs when the space $\mathscr{A}$ of attributes is finite (hence discrete). In this case, the space of functions $\mathscr{F}(\mathscr{A})$ is also discrete, so every function mapping $\mathscr{F}(\mathscr{A})$ into $\mathbb{R}^+$ is automatically uniformly continuous per capita. In this situation it is natural to think of attributes as *types*; players of the same type are exact substitutes for one another (at least insofar as coalitional values are concerned). The example presented in Section 4 is a simple instance of this special case.

The types case, in addition to its conceptual simplicity and economic interest, is of particular use to us because it can be used to approximate the general attributes case. For this reason, it is convenient to adopt a slightly different notation. If the space of attributes is finite, we will refer to *types* rather than attributes, and write $T$ for the set of types and $\Lambda : \mathscr{F}(T) \to \mathbb{R}^+$ for the function that gives coalitional values. We refer to the pair $(T, \Lambda)$ as a *pregame with types.* We call a function $\tau$ from a finite non-empty set $N$ to $T$ a *type function.* Superadditivity, individual marginal bounds and derived games are of course defined just as before. It is convenient to formulate precisely the types cases of our main results, since the general cases will be derived from these special cases. For the statement of the following theorems we formally define the equal-treatment property. A payoff $x$ for a derived game $(N, v_\tau)$ has the *equal-treatment property* if $x_i = x_j$ whenever $i$ and $j$ are in $N$ and $\tau(i) = \tau(j)$.

THEOREM 1 (Types): *Let $(T, \Lambda)$ be a pregame with types which has an individual marginal bound. Let $\varepsilon > 0$ be a given positive number. Then there is an integer $n_0$ such that if $N$ is any finite set with at least $n_0$ elements and $\tau : N \to T$ is any type function then the derived game has a non-empty $\varepsilon$-core containing a payoff with the equal-treatment property.*

THEOREM 2 (Types): *Let $(T, \Lambda)$ be a pregame with types which has an individual marginal bound. Let $\varepsilon > 0$ be a given positive number. Then there is an integer $n_0$*

*such that if $N$ is any finite set and $\tau: N \to T$ is any type function for which the cardinality of $\tau^{-1}(s)$ is either 0 or at least $n_0$ for each $s$ in $T$ (so these are either no players of a given type or at least $n_0$), then the derived game $(N, v_\tau)$ has a non-empty individually rational $\varepsilon$-core containing a payoff with the equal treatment property.*

## 6. COUNTER-EXAMPLES

In this section, we present four simple counter-examples. They show that: the requirement of an individual marginal bound cannot be relaxed to a per capita bound (in Theorem 1); that the requirement that there be enough players of each type cannot be relaxed to enough players in total (in Theorem 2); that the integer $n_0$ in the theorems depends on the pregame; and that uniform continuity per capita cannot be replaced by continuity. We present these counterexamples not only because we are concerned with the sharpness of our results, but also because we feel they provide additional insight into the underlying structure.

EXAMPLE 1: A pregame with types which has a per capita bound (but no individual marginal bound) for which arbitrarily large derived games have empty $\varepsilon$-cores.

Consider a set $T = \{1, 2, 3, 4\}$ of 4 elements and a function $\lambda : \mathscr{F}(T) \to \mathbb{R}^+$ defined as follows:

$$\begin{aligned} \lambda(f) &= n^2 && \text{if } f = n\chi_1 + n\chi_2 + n^2\chi_4, \\ & && \text{or } f = n\chi_1 + n\chi_3 + n^2\chi_4, \\ & && \text{or } f = n\chi_2 + n\chi_3 + n^2\chi_4, \\ \lambda(f) &= 0 && \text{otherwise.} \end{aligned}$$

This function $\lambda$ is obviously not superadditive, but we can define its superadditive cover function $\Lambda : \mathscr{F}(T) \to \mathbb{R}^+$ by setting:

$$\Lambda(f) = \sup\,[\lambda(f_1) + \lambda(f_2) + \cdots + \lambda(f_k)]$$

where the supremum extends over all finite sets of functions $f_1, \ldots, f_k$ in $\mathscr{F}(T)$ such that $f = f_1 + \cdots + f_k$. It is easily checked that $(T, \Lambda)$ is a pregame with types which has a per capita bound (to wit $\Lambda(f) \leq |f|$) but no individual marginal bound. We can easily produce many large derived games with empty $\varepsilon$-cores (for small $\varepsilon$). For example, if $N$ is a set of $3n + 2n^2$ elements, where there are $n$ elements of type 1, $n$ elements of type 2, $n$ elements of type 3, and $2n^2$ elements of type 4, then the derived game $(N, v)$ has an empty $\varepsilon$-core for each $\varepsilon < \frac{1}{12}$. (To the reader familiar with Wooders [33] we point out that in the framework used in that paper, individual marginal bounds and per capita bounds coincide.)

EXAMPLE 2: A pregame with types which has an individual marginal bound and for which large derived games have empty individually rational $\varepsilon$-cores.

This is nearly trivial. Consider a set $T=\{1,2\}$ of two types with payoff function $\lambda:\mathscr{F}(T)\to\mathbb{R}^+$ given by:

$$\lambda(f)=1 \qquad \text{if } f(1)=2,$$
$$\lambda(f)=0 \qquad \text{otherwise}$$

(so that players of type 2 are dummies).

As in the previous example, the function $\lambda$ is not superadditive so we pass to its superadditive cover function $\Lambda:\mathscr{F}(T)\to\mathbb{R}^+$ to obtain a pregame with types $(T,\Lambda)$ which has an individual marginal bound. If we consider a set of $N$ of $3+n$ players: 3 of type 1 and $n$ of type 2, we easily see that the derived game $(N,v)$ has an empty individually rational $\varepsilon$-core for any $\varepsilon<\frac{1}{2}$ (no matter what $n$ is). (Of course, it has a non-empty $\varepsilon$-core if $(1/2n)<\varepsilon$; we simply allocate $\frac{1}{2}$ to each player of type 1 and $-1/2n$ to each player of type 2.)

Example 3: A sequence of pregames $(T,\Lambda_k)$ where each game in the sequence has the same individual marginal bound and where $n_0^k$ (the integer in Theorem 1), goes to infinity as $k$ goes to infinity.

Let $T=\{1,2,3\}$. Given a positive integer $k$, consider the function $\lambda_k:\mathscr{F}(T)\to\mathbb{R}^+$ defined as follows:

$$\begin{aligned}\lambda_k(f)&=n-k &&\text{if } f=n\chi_1+n\chi_2,\\ & &&\text{or } f=n\chi_1+n\chi_3,\\ & &&\text{or } f=n\chi_2+n\chi_3,\\ \lambda_k(f)&=0 &&\text{otherwise.}\end{aligned}$$

Define the superadditive cover of $\lambda_k$, $\Lambda_k$, as in Example 1.

Now consider what number of players we need for a non-empty $\frac{1}{8}$-core; if we have $4k$ players of each type in the player set, the most the coalition of the whole can earn is $3k$ (or $\frac{1}{4}$ per capita) while $4k$ players of each of two types can also earn $3k$ ($\frac{3}{8}$ per capita). Thus we cannot be assured of a non-empty $\frac{1}{8}$-core unless the number of the players is greater than $4k\cdot 3=12k$. Note that an individual marginal bound is 1 (this is independent of $k$) and $n_0^k$ must be greater than or equal to $12k$.

Taking $k=1,2,\ldots$ gives a sequence of pregames with 3 types and the same individual marginal bounds such that the smallest integer $n_0^k$ that works for the $k$th pregame blows up.

We note that, for replications of a given game, Kaneko and Wooders [**15**] show that there is an $r_0$ such that for all replication numbers greater than $r_0$, the $\varepsilon$-core of the replicated game is non-empty independently of the characteristic function of the original game. The example above shows that no such uniformity result is possible, without further restrictions, outside the replication framework.

Example 4: A pair $(\mathscr{A},\Omega)$ for which the function $\Omega$ is continuous, but not uniformly continuous per-capita, whose derived games have empty $\varepsilon$-cores.

For $\mathscr{A}$ take $(\{1/n: n \in Z^+\} \cup 0\}) \times \{1, 2, 3\}$. For $\Omega$ take the superadditive cover of the function which, for each $k \in Z^+$, agrees with the function $\Lambda_k$ of Example 3 on $\mathscr{F}(\{1/k\} \times \{1, 2, 3\})$ and is zero elsewhere. This $\Omega$ is continuous, but not uniformly continuous per-capita, and, as in Example 3, arbitrarily large derived games may have empty $\frac{1}{8}$-cores.

## 7. PROOFS OF THE THEOREMS

Before beginning the proofs of the theorems, we recall some facts about balancedness and the balanced cover of a game.

Let $(N, v)$ be a superadditive game. A collection $\{B_i\}$ of subsets of $N$ is said to be *balanced* if there exist non-negative real numbers $\beta_i$, called *balancing weights*, such that for each $j$ in $N$,

$$\sum_i \beta_i \sigma_j(B_i) = 1$$

where $\delta_j(B_i) = 1$ if $j \in B_i$ and $\delta_j(B_i) = 0$ if $j \notin B_i$. The collection $\{B_i\}$ is *minimal balanced* if it is balanced and contains no proper balanced subcollection. If $\{B_i\}$ is minimal balanced, then the balancing weights are unique, strictly positive, and rational. Since there are only a finite number of minimal balanced collections of subsets of $N$, there is a least positive integer $D$, which we will call the *depth* of $N$, such that $D\beta$ is an integer for every balancing weight $\beta$ of every minimal balanced collection of subsets of $N$.

The *balanced cover* $(N, \tilde{v})$ of $(N, v)$ is the game with the same set of players, and with the characteristic function given by:

$$\tilde{v}(S) = v(S) \qquad \text{for } S \subsetneqq N,$$
$$\tilde{v}(N) = \max \sum \beta_i v(B_i),$$

where the maximum extends over all minimal balanced collections $\{B_i\}$ of subsets of $N$, and the coefficients $\beta_i$ are the associated balancing weights. The game $(N, v)$ is *balanced* if $\tilde{v} = v$. Note that the balanced cover $(N, \tilde{v})$ is always balanced; by a fundamental result of Bondareva [**6, 7**] and, independently, Shapley [**22**], the game $(N, \tilde{v})$ has a non-empty core.

Since the proofs are rather long and complicated, it seems useful to give an overview of the strategy before embarking on the technical details.

As we mentioned previously, the results for general pregames with attributes are obtained from the special case of pregames with types by an approximation argument. The heart of the argument is thus in pregames with types. This is in fact one of the reasons for introducing pregames with types.

The proof of Theorem 1 (types) proceeds by induction on the number of types. If Theorem 1 were false (for a given prègame with types), we would have a sequence $\{(N_n, v_n)\}$ of derived games with empty $\varepsilon$-cores, where the number of players $|N_n|$ tends to infinity. We look at the fraction of players of each type; passing to a subsequence if necessary we assume these fractions approach limits as $n$ tends to infinity. If none of these limits is zero, we show that the games $(N_n, v_n)$ are approximately balanced (for large $n$); this leads to a payoff vector in the $\varepsilon$-core. If one or more of these limits is zero, we construct a reduced game, use the inductive hypothesis to construct a payoff vector in the $(\varepsilon/2)$-core of the reduced game, and adjust this payoff vector to produce a payoff vector in the $\varepsilon$-core of the game $(N_n, v_n)$, for large $n$.

In Theorem 2 (types) much greater care must be exercised to assure that we obtain an individually rational payoff vector, but the principles are the same.

We now turn to the detailed proofs.

PROOF OF THEOREM 1 (Types): The proof is by induction on the cardinality $|T|$ of $T$; i.e., the number of types. If $|T|=0$, the result is vacuous, so there is nothing to prove. Let us therefore assume the Theorem to be valid for every positive $\varepsilon$ and for every pregame, with fewer than $t$ types, which has an individual marginal bound. Fix a pregame $(T, \Lambda)$ with $|T|=t$ and an individual marginal bound of $M$, and fix a positive $\varepsilon$. Write $T=\{1,2,\dots,t\}$.

Suppose that the theorem were false for this pregame $(T, \Lambda)$ and this $\varepsilon$. Then there would be a sequence $\{(N_n, \tau_n)\}$ of finite sets and type functions such that $|N_n| \to \infty$ and the derived games $(N_n, v_{\tau_n})$ all have empty $\varepsilon$-cores. We are going to derive a contradiction. The argument will fall into two cases, depending on the relative abundance of players of each type.

For each integer $n$ and each $j$ in $T$, write

$$p_j(n)=\frac{|\tau_n^{-1}(j)|}{|N_n|},$$

so $p_j(n)$ is the fraction of players in the $n$th derived game who are of type $j$. Note that $0 \leqslant p_j(n) \leqslant 1$; by passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty} p_j(n)=p_j$ exists for each $j$ in $T$. Note that $\sum_j p_j(n)=1$ for each $n$, so that $\sum_j p_j=1$. Henceforth, we will write $v_n$ for $v_{\tau_n}$.

We now separate the argument into two cases:

CASE 1: All the numbers $p_j$ are strictly positive.

Let $(N_n, \tilde{v}_n)$ be the balanced cover of the $n$th derived game. Since $(N_n, \tilde{v}_n)$ is balanced, it has a non-empty core. We are going to use the core of $(N_n, \tilde{v}_n)$ to construct a point in the $\varepsilon$-core of $(N_n, v_n)$.

Our first task is to estimate $\tilde{v}_n(N_n)$. By definition, there is a minimal balanced collection $\{B_i\}$ of subsets of $N_n$ with balancing weights $\beta_i$ such that

$$\tilde{v}_n(N_n)=\sum_i \beta_i v_n(B_i).$$

Since $(N, v_n)$ is derived from the pregame $(T, \Lambda)$ which has an individual marginal bound of $M$, we see that

$$\begin{aligned}\tilde{v}_n(N_n) &= \sum_i \beta_i v_n(B_i)\\ &\leq \sum_i \beta_i M|B_i|\\ &= M\sum_i \beta_i|B_i|\\ &= M\sum_i \beta_i \sum_{m\in N_n} \delta_m(B_i)\\ &= M\sum_{m\in N_n}\sum_i \beta_i\delta_m(B_i).\end{aligned}$$

Since the coefficients $\beta_i$ are balancing weights, the inner sum in the last expression is 1 for each $m$ in $N_n$. We thus obtain the estimate we need:

$$\tilde{v}_n(N_n) \leq M|N_n|,$$

and hence

$$\frac{\tilde{v}_n(N_n)}{|N_n|} \leq M.$$

Now, by passing to a subsequence if necessary, we may assume that

$$\lim_{n\to\infty} \frac{\tilde{v}_n(N_n)}{|N_n|} = L$$

exists; evidently $L \leq M$. We make the following claim.

CLAIM:

$$\lim_{n\to\infty} \frac{v_n(N_n)}{|N_n|} = L = \lim_{n\to\infty} \frac{\tilde{v}_n(N_n)}{|N_n|}.$$

Leaving the proof of this claim aside for the moment, we complete the argument in Case 1.

Set $p = \min(p_1, \ldots, p_t) > 0$. In view of the claim, we may choose an integer $n_0$ such that for $n \geq n_0$,

$$\left|\frac{\tilde{v}_n(N_n)}{|N_n|} - \frac{v_n(N_n)}{|N_n|}\right| < p\varepsilon$$

and

$$p_j(n) > p/2 \qquad \text{for each } j.$$

Since $(N_n, \tilde{v}_\cdot)$ is balanced, it has a non-empty core (for each $n$). Fix an integer $n \geq n_0$ and a payoff vector $x$ in the core of $(N_n, \tilde{v}_n)$. We are going to perturb $x$ to obtain an equal-treatment payoff in the $\varepsilon$-core of $(N_n, v_n)$.

Let $\Gamma$ be the set of permutations $\pi$ of $N_n$ for which $\tau_n(\pi(i)) = \tau_n(i)$ for each $i$ in $N_n$ (i.e., $\pi$ leaves types fixed). For each $\pi$ in $\Gamma$, define the vector $x^\pi$ by $x_i^\pi = x_{\pi(i)}$ for each $i$ in $N_n$. Since $(N_n, v_n)$ is derived from the pregame $(T, \Lambda)$, the value of any coalition depends only on the types of its members, and the same is true for $(N_n, \tilde{v}_n)$. In particular, $x^\pi$ belongs to the core of $(N, \tilde{v}_n)$ for each $\pi$ in $\Gamma$. Since the core of a game is convex, the vector

$$\bar{x} = \frac{1}{|\Gamma|} \sum_{\pi \in \Gamma} x^\pi$$

also belongs to the core of $(N_n, \tilde{v}_n)$, and it is evidently an equal-treatment payoff. Define[3] the vectors $y, \bar{y}$ by

$$y_i = \max(\bar{x}_i - \varepsilon, v_n(i)) \qquad \text{for } i \text{ in } N_n;$$

$$\bar{y}_i = y_i \frac{v_n(N_n)}{y(N_n)} \quad \text{for } i \text{ in } N_n.$$

Evidently, $\bar{y}$ is an equal-treatment payoff; we show that it is in the (individually rational) $\varepsilon$-core of $(N_n, v_n)$.

We need first of all to estimate $y(N_n)$. Let $A = \{i \in N_n \colon \bar{x}_i - \varepsilon \geq v_n(i)\}$, $B = N_n - A$. Then, by construction of $y$ and since $\bar{x}$ belongs to the core,

$$\begin{aligned} y(N_n) &= \sum_{N_n} y_i = \sum_A y_i + \sum_B y_i \\ &= \sum_A (\bar{x}_i - \varepsilon) + \sum_B v_n(i) \\ &\leq \bar{x}(A) - \varepsilon|A| + v_n(B) \\ &\leq \bar{x}(N_n) - \varepsilon|A| \\ &= \tilde{v}_n(N_n) - \varepsilon|A|. \end{aligned}$$

If $A = \varnothing$, then $y_i = v_n(i)$ for each $i$ so we certainly have $y(N_n) = \sum v_n(i) \leq v_n(N_n)$ (because $v_n$ is superadditive). On the other hand, if $A \neq \varnothing$ then the facts that $\bar{x}$ is an equal-treatment payoff and $p_j(n) \geq p/2$ for each $j$ allow us to conclude that $|A| \geq \frac{1}{2}p|N_n|$, whence

$$\tilde{v}_n(N_n) - \varepsilon|A| \leq \tilde{v}_n(N_n) - \frac{\varepsilon p |N_n|}{2} \leq v_n(N_n).$$

In either case, we then obtain $y(N_n) \leq v_n(N_n)$.

It now follows that $\bar{y}_i \geq y_i$ for each $i \in N$, and in particular that $\bar{y}_i \geq v_n(i)$ for each $i$. Moreover, since $\bar{x}$ is in the core of $(N_n, \tilde{v}_n)$, if $S$ is any proper subset of $N_n$, then

$$\bar{y}(S) \geq y(S) \geq \tilde{v}_n(S) - \varepsilon|S| = v_n(S) - \varepsilon|S|.$$

[3] For ease in notation, we use $v_n(i)$ for $v_n(\{i\})$ and later, given a payoff $y$, we will use $y(i)$ for $y(\{i\})$.

Finally, since $\bar{y}(N_n)=v_n(N_n)$, we conclude that $\bar{y}$ is an equal-treatment payoff in the (individually rational) $\varepsilon$-core of $(N_n, v_n)$, as desired.

(In the context of Theorem 1 we did not need to show that $\bar{y}$ is individually rational, but this will greatly simplify the argument in Theorem 2.)

It remains to establish the Claim. We are going to show that for all large integers $m$ and all sufficiently large (in terms of $m$) integers $n$, the game $(N_n, v_n)$ contains a large multiple of the balanced cover $(N_m, \tilde{v}_m)$ of the game $(N_m, v_m)$. This will enable us to obtain the inequality we require between $v_n(N_n)$ and $\tilde{v}_m(N_m)$.

To this end, fix a large positive integer $R$ and an integer $m$ so large that for $n \geqslant m$ we have for each $j$ in $T$:

$$1-\frac{1}{R}<\frac{p_j(n)}{p_j}<1+\frac{1}{R}.$$

(This is possible since $p_j(n)\to p_j \neq 0$.) Let $D$ be the depth of $N_m$ and choose any integer $n \geqslant m$ so large that

$$|N_n|>|N_m|DR^2.$$

Now, there is a unique positive integer $q$, which is necessarily at least as large as $R$, such that

$$q|N_m|DR \leqslant |N_n|<(q+1)|N_m|DR.$$

Since $n \geqslant m$, we obtain from the estimate on $p_j(n)/p_j$.

$$1-\frac{2}{R}<\frac{p_j(n)}{p_j(m)}<1+\frac{2}{R} \qquad \text{for each } j.$$

Hence, for each $j$ we have a lower bound:

$$\begin{aligned} |\tau_n^{-1}(j)| = p_j(n)|N_n| &> \left(1-\frac{2}{R}\right) p_j(m)|N_n| \\ &\geqslant \left(1-\frac{2}{R}\right) p_j(m)q|N_m|DR \\ &= (R-2)p_j(m)q|N_m|D \\ &= (R-2)qD|\tau_m^{-1}(j)|. \end{aligned}$$

In other words, for every player of type $j$ in $N_m$, there are at least $(R-2)qD$ distinct players of type $j$ in $N_n$.

Let us now look at $\tilde{v}_m(N_m)$. By definition, there is a minimal balanced collection $B_1, \ldots, B_K$ of subsets of $N_m$, with balancing weights $\beta_1, \ldots, \beta_K$ such that

$$\tilde{v}_m(N_m)=\sum \beta_k v_m(B_k).$$

Since $D$ is the depth of $N_m$, the numbers $\beta_k D$ are all integers, and $\sum_k \beta_k \delta_j(B_k)D = D$ for each player $j$ in $N_m$. Since there are so many players of every type in $N_n$, we can choose functions $\psi_d^l: N_m \to N_n$ (for $1 \leqslant d \leqslant D$, $1 \leqslant l \leqslant (R-2)q$) such that:

(a) for each $d$ and $l$, and for each player $i$ in $N_m$, the player $\psi^l_d(i)$ has the same type as $i$; (b) $\psi^l_d(i) \neq \psi^{l'}_{d'}(i')$ unless $i = i'$, $d = d'$, and $l = l'$. We now define coalitions $C^l_{k,s}$ in $N_n$, where $1 \leqslant l \leqslant (R-2)q$, $1 \leqslant k \leqslant K$, and $1 \leqslant s \leqslant \beta_k D$, by first writing $\sigma(k) = \sum_{u=1}^{k-1} \beta_u D$ and setting $C^l_{k,s} = \psi^l_{\sigma(k)+s}(B_k)$. Notice that the coalitions $C^l_{k,s}$ are disjoint from each other. Moreover, since the functions $\psi^l_d$ preserve types and both $(N_m, v_m)$ and $(N_n, v_n)$ are derived from the same pregame $(T, \Lambda)$, $v_n(C^l_{k,s}) = v_m(B_k)$.

Finally, we can now estimate $v_n(N_n)$:

$$\begin{aligned} v_n(N_n) &\geqslant \sum_{l,k,s} v_n(C^l_{k,s}) \\ &= \sum_{l,k} \beta_k D v_m(B_k) \\ &= D \sum_l \sum_k \beta_k v_m(B_k) \\ &= D \sum_l \tilde{v}_m(N_m) \\ &= D(R-2)q\tilde{v}_m(N_m). \end{aligned}$$

Hence

$$\begin{aligned} \frac{v_n(N_n)}{|N_n|} &\geqslant \frac{D(R-2)q\tilde{v}_m(N_m)}{|N_n|} \\ &\geqslant \frac{D(R-2)q\tilde{v}_m(N_m)}{(q+1)|N_m|DR} \\ &= \left(\frac{q}{q+1}\right)\left(1-\frac{2}{R}\right)\frac{\tilde{v}_m(N_m)}{|N_m|}. \end{aligned}$$

Since $R$ was arbitrary (but as large as we like) and $q \geqslant R$, we see that

$$\liminf_n \frac{v_n(N_n)}{|N_n|} \geqslant \lim_m \frac{\tilde{v}_m(N_m)}{|N_m|} = L.$$

On the other hand, since $\tilde{v}_n(N_n) \geqslant v_n(N_n)$, we also have

$$\limsup_n \frac{v_n(N_n)}{|N_n|} \leqslant \lim_m \frac{\tilde{v}_m(N_m)}{|N_m|} = L.$$

Combining the lim sup and lim inf estimates establishes the Claim and completes the proof of Case 1.

CASE 2: Some of the limiting ratios $p_1, p_2, \ldots, p_t$ are zero.

Set $T_1 = \{j \in T: p_j \neq 0\}$; $T_1$ is a proper subset of $T$ and is non-empty (since $\sum p_j = 1$). Every function $f: T_1 \to Z^+$ may be extended to a function $\hat{f}: T \to Z^+$ by setting $\hat{f}(j) = f(j)$ if $j \in T_1$ and $\hat{f}(j) = 0$ if $j \notin T_1$. If we then define $\Lambda_1: \mathscr{F}(T_1) \to Z^+$ by $\Lambda_1(f) = \Lambda(\hat{f})$, it is clear that $(T_1, \Lambda_1)$ is a pregame with fewer than $t$ types, so our inductive hypothesis can be applied: there is an integer $n_1$ such that if $N$ is any set with at least $n_1$ elements and $\tau : N \to T_1$ is any type function then the derived game $(N, v_\tau)$ has an equal treatment vector in its $(\varepsilon/2)$-core.

We now return to our sequence $(N_n, v_n)$. Using the definition of $T_1$, we see that we can choose an integer $n_0$ so that for $n \geq n_0$,

$$|\tau_n^{-1}(T_1)| > n_1,$$

and

$$\frac{|\tau_n^{-1}(T - T_1)|}{|\tau_n^{-1}(T_1)|} < \frac{\varepsilon}{2M},$$

where $M$ is the individual marginal bound for $(T, A)$. If we write $N'_n = \tau_n^{-1}(T_1)$ and $\tau'_n = \tau_n | N_n^1|$, then our inductive hypothesis enables us to choose an equal-treatment payoff $x$ in the $(\varepsilon/2)$-core of the game $(N'_n, v_{\tau'_n})$ derived from $(T_1, A_1)$. Set

$$\sigma = \frac{M|\tau_n^{-1}(T - T_1)| + v_n(N'_n) - v_n(N_n)}{|\tau_n^{-1}(T_1)|} < \frac{\varepsilon}{2}$$

and define the vector $y$ by

$$y_i = M \qquad \text{for } i \text{ in } \tau_n^{-1}(T - T_1);$$

$$y_i = x_i - \sigma \qquad \text{for } i \text{ in } \tau_n^{-1}(T_1).$$

It is easily checked that $y$ is an equal-treatment payoff in the $\varepsilon$-core of the derived game $(N_n, v_n)$. (Note that, unlike the situation in Case 1, $y$ need not be individually rational.) This completes the proof of Theorem 1 (types).

PROOF OF THEOREM 2 (Types): We proceed exactly as before, using induction on the number of types. If the result is false for $t$ types, we find a sequence $\{(N_n, \tau_n)\}$ of sets and type functions such that $|\tau_n^{-1}(j)|$ is either zero or at least $n$ (for each $j$ in $T$) but the derived games $(N_n, v_n)$ do not have equal-treatment vectors in their individually rational $\varepsilon$-cores. By passing to a subsequence if necessary, we may assume that the same types occur in all the games, and by discarding types which do not occur we may assume that all types do occur. That is, there is no loss of generality in assuming that $|\tau_n^{-1}(j)| \to \infty$ for each $j$ in $T$. We form the ratios $p_j(n)$ and their limits $p_j$ and consider the same two cases. In the first case (where none of the limiting ratios are zero), the previous argument goes through verbatim since we were careful to construct an individually rational payoff vector.

In the second case, however, a problem arises: the perturbation used to produce the payoff vector $y$ from the payoff vector $x$ need not preserve individual rationality. In order to deal with this difficulty, we will need to be more careful about the relative scarcity of players.

So, we pick up the proof of Theorem 1 (types) at the beginning of Case 2: some of the limiting ratios are zero. There is no loss of generality (since we may renumber the set $T$ of types and pass to a subsequence of the games $(N_n, v_n)$ if necessary) in assuming that for each $n$,

$$|\tau_n^{-1}(1)| \leqslant |\tau_n^{-1}(2)| \leqslant \cdots \leqslant |\tau_n^{-1}(t)|.$$

Form the ratios $q_j(n) = |\tau_n^{-1}(1)|/|\tau_n^{-1}(j)| = p_n(1)/p_n(j)$, so that $1 = q_1(n) \geqslant q_2(n) \geqslant \cdots \geqslant q_t(n)$. As before, we may assume that $q_j = \lim_n q_j(n)$ exists for each $j$. Since some of the limiting ratios $p_j$ are zero, there is an integer $K$, $1 \leqslant K < t$ such that $1 = q_1 \geqslant q_2 \geqslant \cdots \geqslant q_K > q_{K+1} = \cdots = q_t = 0$. Write $T^0 = \{1, 2, \ldots, K\}$, $T^1 = \{K+1, \ldots, t\}$; players whose type is in $T^0$ are scarce, others are relatively abundant. For a function $f$ in $\mathscr{F}(T^1)$, let $\hat{f}$ again denote its trivial extension to $T$ and define $\Lambda^1(f) = \Lambda(\hat{f})$; as before we see that $(T^1, \Lambda^1)$ is a pregame with types and has an individual marginal bound. We now distinguish two subcases; since we are, in effect, already discussing Case 2, we call the first of these Case 2A.

CASE 2A: There is a function $F$ in $\mathscr{F}(T^1)$ such that $\Lambda^1(F) > \sum_j F(j)\Lambda^1(\chi_j)$.

Validity of this inequality simply means that the pregame $(T^1, \Lambda^1)$ is not trivial; or put another way, that some coalition of abundant players can obtain a payoff which is strictly greater than the sum of their individual payoffs. Write

$$\mu = \Lambda^1(F) - \sum_j F(j)\Lambda^1(\chi_j).$$

Consider the game $(N_n, v_n)$ for large $n$ (to be determined). If we set $N_n^1 = \tau_n^{-1}(T^1)$, and let $v_n^1$ be the restriction of $v_n$ to $N_n^1$, then the game $(N_n^1, v_n^1)$ is derived from the pregame $(T^1, \Lambda^1)$. We can then apply the induction hypothesis (for large $n$) to select a payoff vector $x$ in the individually rational $(\varepsilon/2)$-core of $(N_n^1, v_n^1)$. We assert that, for sufficiently large $n$, we can find a type $j$ in $T^1$ and a positive number $\sigma$ such that the payoff vector $y$ given by:

$$\begin{aligned} y_i &= M && \text{for } i \text{ in } \tau_n^{-1}(T^0);\\ y_i &= x_i && \text{for } i \text{ in } \tau_n^{-1}(T^1 - \{j\});\\ y_i &= x_i - \sigma && \text{for } i \text{ in } \tau_n^{-1}(j) \end{aligned}$$

is in the individually rational $\varepsilon$-core of $(N_n, v_n)$.

Analogously to Case 2 of Theorem 1, set

$$\sigma = \frac{v_n(N_n^1) - v_n(N_n) + M|\tau_n^{-1}(T^0)|}{|\tau_n^{-1}(j)|}$$

so that $y(N_n) = v_n(N_n)$. Since $T^0$ consists of scarce types while $j$ is an abundant type, and $v_n(N_n^1) - v_n(N_n)$ is negative, $\sigma$ will be less than $\varepsilon/2$ if $n$ is large. This shows that $y$ is in the $\varepsilon$-core of $(N_n, v_n)$, for any choice of $j$, and leaves us to deal with individual rationality.

Since $x$ is an individually rational equal-treatment payoff, the difference $x(i) - v_n(i)$ is nonnegative and constant on each of the sets $\tau_n^{-1}(k)$, for $k$ in $T^1$. Let $j$ be a type in $T^1$ for which this difference is as large as possible; we show that for this choice of $j$, the payoff vector $y$ is individually rational; that is,

$$\sigma \leq x(i) - v_n(i) \qquad \text{for } i \text{ in } \tau_n^{-1}(j).$$

Combining with the definition of $\sigma$ yields

$$|\tau_n^{-1}(j)|(x(i) - v_n(i)) \geq v_n(N_n^1) - v_n(N_n) + M|\tau_n^{-1}(T^0)|.$$

The left-hand side of this inequality is just $x(\tau_n^{-1}(j)) - \sum_{\tau_n^{-1}(j)} v_n(i)$, which, by choice of $\sigma$ and the fact that there are only $t$ types, is at least $t^{-1}[x(N_n^1) - \sum_{N_n^1} v_n(i)]$. Since $v_n(N_n^1) - v_n(N_n)$ is negative, it will suffice to verify that, for large $n$,

$$(*) \qquad x(N_n^1) - \sum_{N_n^1} v_n(i) \geq tM|\tau_n^{-1}(T^0)|.$$

To do this, we make use of the function $F$. For each $n$, there is a coalition $S \subset N_n^1$ such that the function $f_S$ is an integer multiple of $F$; say $f_S = RF$ where we choose $S$ to make $R$ as large as possible. For large values of $n$, the set $N_n^1$ contains many players of every type in $T^1$ (abundance!) so $R$ is large, too. In fact, since the ratios $|\tau_n^{-1}(T^0)|/|\tau_n^{-1}(k)|$ can be made as small as we like (for large values of $n$ and for every $k$ in $T^1$), we see that the ratio $|\tau_n^{-1}(T^0)|/R$ can also be made as small as we like. On the other hand

$$x(N_n^1) - \sum_{N_n^1} v_n(i) \geq R\mu$$

so that the inequality $(*)$ is satisfied if

$$R\mu \geq tM|\tau_n^{-1}(T^0)|$$

or equivalently if

$$\frac{\mu}{tM} \geq \frac{|\tau_n^{-1}(T^0)|}{R}.$$

As we have just seen, this last inequality is satisfied for large values of $n$, so that inequality $(*)$ is also satisfied for large values of $n$. Verification of this inequality takes care of individual rationality of $y$ and completes the proof of Case 2A.

CASE 2B: For every function $F$ in $F(T^1)$ we have $\Lambda^1(F) = \sum F(j)\Lambda^1(\chi_j)$.

This simply means that the pregame $(T^1, \Lambda^1)$ is trivial; or put another way, no coalition of abundant players can obtain a payoff which is strictly greater than the sum of their individual payoffs.

We define a function $\Lambda^0 : \mathscr{F}(T^0) \to \mathbb{R}^+$ by

$$\Lambda^0(f) = \sup \Lambda(\bar{f}) - \sum_{j \in T^1} \bar{f}(j)\Lambda^1(\chi_j)$$

where the supremum extends over all functions $\bar{f}$ in $\mathscr{F}(T)$ which agree with $f$ on $T^0$. Using the individual marginal bound for $\Lambda$ and the triviality of $(T^1, \Lambda^1)$, it is easily checked that $\Lambda^0(f)$ is indeed finite, that $\Lambda^0$ is super-additive, and that

$(T^0, \Lambda^0)$ is a pre-game with types which has an individual marginal bound. Let us write $N_n^0 = \tau_n^{-1}(T^0)$, and $(N_n^0, w_n)$ for the game derived from $(T^0, \Lambda^0)$.

Observe that the limiting ratios of types for the sequence of games $\{(N_n^0, w_n)\}$ are all different from zero, so that we can apply the arguments of Case 1 of Theorem 1 (types). We conclude immediately that, for large $n$, the games $(N_n^0, w_n)$ have non-empty individually rational $(\varepsilon/2)$-cores. Moreover, passing to a subsequence if necessary, the ratios $\psi_n = w_n(N_n^0)/|N_n^0|$ approach a limit $W \leq M$. We want to see that the ratios

$$\rho_n = \frac{v_n(N_n) - v_n(N_n^1)}{|N_n^0|}$$

also approach the same limit $W$. To see this, notice first that $\psi_n \geq \rho_n$ for each $n$ (because $(T^1, \Lambda^1)$ is trivial and $\Lambda$ is superadditive). Now fix a large integer $m$ and a positive number $\mu$. By definition, there is a function $\bar{f}$ in $\mathscr{F}(T)$ such that the restriction of $\bar{f}$ to $T^0$ is just $f_{N_m^0}$ (on $T^0$) and

$$\begin{aligned} w_m(N_m^0) &= \Lambda^0(f_{N_m^0}) \\ &\leq \Lambda(\bar{f}) - \sum_{j \in T^1} \bar{f}(j)\chi_j + \mu. \end{aligned}$$

Arguing as in the claim of Case 1, Theorem 1 (types), we see that for large $n$, the game $(N_n, v_n)$ contains a large multiple, say $R$, of $(N_n^0, w_m)$, because the scarce players have approximately the same distribution in $N_n^0$ as in $N_m^0$, while in $N_n$ there will be a very large number of abundant players. We can thus write $f_{N_n} = R\bar{f} + g$ for some function $g$ in $\mathscr{F}(T)$. Arguing as before we see that $w_n(N_n^0)$ is approximately $Rw_m(N_m^0)$, that $|N_n^0|$ is approximately $R|N_m^0|$, and that

$$v_n(N_n) - v_n(N_n^1) \geq R[w_m(N_m^0) - \mu].$$

After doing a little arithmetic, we see that $\rho_n \geq \psi_m - \mu$ so that $\lim \rho_n = W$ as desired.

Now choose an equal-treatment payoff vector $x$ in the individually rational $(\varepsilon/2)$-core of $(N_n^0, w_n)$, and define the vector $y$ by

$$\begin{aligned} y_i &= x_i \quad \text{for } i \text{ in } N_n^0, \\ y_i &= v_n(i) \quad \text{for } i \text{ in } N_n^1. \end{aligned}$$

In view of the previous discussion, $y(N_n) \leq v_n(N_n) + (\varepsilon/2t)|N_n|$ if $n$ is sufficiently large. Proceeding as before, we can find a type $k$ in $T^0$ and a positive number $\rho$ such that the vector $\bar{y}$ defined by

$$\begin{aligned} \bar{y}_i &= y_i - \sigma \quad \text{if } i \in \tau_n^{-1}(k), \\ \bar{y}_i &= y_i \quad \text{if } i \notin \tau_n^{-1}(k) \end{aligned}$$

is an equal treatment payoff vector in the individually rational $\varepsilon$-core of $(N_n, v_n)$. This completes the proof of Case 2B and with it the proof of Theorem 2 (types).

Having done the most difficult work, we now present the approximation arguments which allow us to obtain our general results for pregames with attributes from the special case of pregames with types.

PROOF OF THEOREM 1: Given $\varepsilon > 0$, we construct a pregame with types $(T, \Lambda)$ which "approximates" $(\mathscr{A}, \Omega)$ to within $\varepsilon/2$, and derive the desired result by applying Theorem 1 (types) to $(T, \Lambda)$. To this end, choose a $\delta$, $0<\delta<1$ such that if $f$, $g$ are in $\mathscr{F}(\mathscr{A})$ and $\bar{d}(f, g)<\delta$ then $|\Omega(f)-\Omega(g)|<(\varepsilon/2)|f|=(\varepsilon/2)|g|$. Use the compactness of $\mathscr{A}$ to choose a finite subset $T$ of $\mathscr{A}$ such that every point of $\mathscr{A}$ is within $\delta$ of some point of $T$. Each function $f: T\to Z^+$ has an extension $\hat{f}: \mathscr{A}\to Z^+$ defined by

$$\hat{f}(a)=f(a) \quad \text{for } a \text{ in } T;$$

$$\hat{f}(a)=0 \quad \text{for } a \text{ not in } T.$$

Now define $\Lambda: \mathscr{F}(T)\to \mathbb{R}^+$ by

$$\Lambda(f)=\Omega(\hat{f}).$$

It is evident that $(T, \Lambda)$ is a pregame with types and has an individual marginal bound (since $(\mathscr{A}, \Omega)$ does). By Theorem 1 (types) there is an integer $n_0$ such that for any set $N$ having at least $n_0$ elements and any type function $\tau: N\to T$, the derived game $(N, v_\tau)$ has a non-empty $(\varepsilon/2)$-core.

Now let $N$ be any set with at least $n_0$ elements and let $\alpha: N\to\mathscr{A}$ be any attribute function. Because of the way $T$ is located in $\mathscr{A}$, we can find a function $\tau: N\to T$ such that $d(\tau(i), \alpha(i))<\delta$ for each $i$ in $N$. Since $\tau$ is a type function, we have a derived game $(N, v_\tau)$ and we know there is a vector $x$ in the $\varepsilon/2$-core of $(N, v_\tau)$. Set

$$\mu=\frac{v_\alpha(N)-v_\tau(N)}{|N|}$$

and define the vector $y$ by

$$y_i=x_i-\mu \quad \text{for each } i \text{ in } N.$$

Our choice of $\delta$ guarantees that $|\mu|<\varepsilon/2$. Moreover, for each subset $S$ of $N$, $|v_\alpha(S)-v_\tau(S)|<\varepsilon/2$. Putting these facts together with the fact that $x$ is in the $\varepsilon/2$-core of $(N, v_\tau)$, it is easy to check that $y$ is in the $\varepsilon$-core of $(N, v_\tau)$, as desired. We note that if $x$ was chosen to be an equal-treatment payoff, then $y$ will have the property that players with similar attributes obtain the same payoff.

PROOF OF THEOREM 2: Modulo one small difficulty, this result follows from Theorem 2 (types) in exactly the same way as Theorem 1 follows from Theorem 1 (types). The small difficulty is that in adjusting the vector $x$ in the $\varepsilon/2$-core of the types game $(N, v_\tau)$ to produce a vector $y$ in the $\varepsilon$-core of the attributes game $(N, v_\tau)$, we must be careful to retain individual rationality. This can be accomplished in exactly the same way as in the proof of Theorem 2 (types) itself; we leave the details to the readers.

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