Soc Choice Welfare (1999) 16: 337–346

Social Choice and Welfare

Arbitrage and global cones: Another counterexample*

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Received: 18 August 1997/Accepted: 30 January 1998

Abstract. Chichilnisky (1997) claims that another variant of her condition limiting arbitrage is necessary and sufficient for existence of equilibrium and nonemptiness of the core in an economy with short sales allowing half lines in indifference surfaces. Her proof, however, is based on a proposition purporting to relate her notion of "global cone" (see Chichilnisky (1997) for references) to the Page-Wooders "increasing cone." In this paper, we present a counterexample showing that parts (i) and (ii) of Chichilnisky's proposition are false. Thus, Chichilnisky's claimed result is without proof.

Introduction

In an exchange economy with unbounded short sales, Chichilnisky (1997) states that another variant of her condition limiting arbitrage is necessary and sufficient for existence of equilibrium and nonemptiness of the core.[1] An aspect of her model is that, in a limited way, half lines in indifference surfaces are allowed.[2] Chichilnisky's claimed result rests on Proposition 5 of the paper,

* The authors are grateful to Kenneth Arrow, Peter Hammond, and an anonymous referee for helpful comments.
** F. H. Page gratefully acknowledges financial support from the RGC program at the University of Alabama.
*** M. H. Wooders is indebted to the Social Sciences and Humanities Research Council of Canada for support.

[1] Monteiro, Page, and Wooders (1997) discuss previous versions of her condition and, in particular, the global cones used in her previous conditions (see Sect. 1 below).

[2] For example, her model does not allow an agent to have half lines in indifference surfaces in some directions and no half lines in others.

which purports to establish the relationship between global cones, introduced in her prior papers, and Page-Wooders increasing cones.[3] In this paper, we present a counterexample showing that parts (i) and (ii) of Chichilnisky's proposition are false. In a related paper, Monteiro, Page, and Wooders (1997) present a counterexample showing that Chichilnisky's previous notions of limited arbitrage (no arbitrage conditions based on global cones) are either not well defined or fail to be sufficient for existence of equilibrium or nonemptiness of the core. Thus, her claimed existence and nonemptiness results remain without proof.

When agents' indifference surfaces contain no half lines, Chichilnisky's (1997) condition of limited arbitrage is in fact equivalent to the conditions limiting arbitrage typically used in the literature.[4] Moreover, in various exchange economy models, some more general than Chichilnisky's (for example, Werner (1987)), a number of authors have shown that whenever agents' indifference surfaces contain no half lines, no unbounded arbitrage is necessary and sufficient for existence of equilibrium.[5] In addition, Page and Wooders (1993, 1994, 1996a,b) have shown that no unbounded arbitrage is necessary and sufficient for compactness of the set of utility possibilities and nonemptiness of the core.[6] Thus, what remains to be shown is whether the connections between unbounded arbitrage, equilibrium, and the core continue to hold outside of the no half lines case, that is, in the case where half lines in indifference surfaces are allowed. This appears to have been one of the main objecties of Chichilnisky's papers on arbitrage. The counterexample below, together with the counterexample in Monteiro, Page, and Wooders (1995), show that this objective has not yet been achieved.

1. The model

Chichilnisky (1997a) considers an exchange economy $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ where each agent j has consumption set R^L, endowment $\omega_j \in R^L$, and preferences

[3] Increasing cones were introduced in Page (1982, 1989) to obtain necessary and sufficient conditions for the existence of equilibrium in an asset market model with short sales and named in Page and Wooders (1993). Page and Wooders (1994, 1996a,b) extend the notion of increasing cone to accommodate thick indifference curves.

[4] See Hart (1974), Milne (1980), Kreps (1981), Hammond (1983), Grandmont (1977, 1982), Page (1982, 1984, 1987, 1989, 1996), Werner (1987), Nielsen (1989), and Page and Wooders (1993, 1994, 1996a,b).

[5] See Grandmont (1977, 1982), Milne (1980), Hammond (1983), Page (1982, 1989, 1996), Werner (1987), and Page and Wooders (1993, 1994, 1996a,b).

[6] The equivalence of nonemptiness of the core and existence of an equilibrium is immediate from Debreu and Scarf (1963). In unbounded economies, Page and Wooders (1996a,b) make this connection directly for the core and the partnered core introduced in Reny, Winter, and Wooders (1994) and Reny and Wooders (1996). Page and Wooders (1996a) show that no unbounded arbitrage is even necessary and sufficient for the existence of a Pareto optimal point. Aliprantis et al. (1997) extend the equivalence of nonemptiness of the core and existence of equilibrium to infinite dimensional Riesz spaces.

over R^L specified via a utility function $u_j(\cdot) : R^L \rightarrow R$. Chichilnisky makes the following special assumptions:

[G-1] for each agent j, $u_j(\cdot)$ is continuous, quasi-concave, and monotonically increasing with $u_j(0) = 0$ and $\sup\{u_j(x) : x \in R^L\} = \infty$;[7]

[G-2] for each agent j, either (i) (*no half lines*): all indifference surfaces contain no half lines, or (ii) (*closed gradient*): the set of gradient directions along any indifference surface is closed;

[G-3] (*uniform nonsatiation*): for each agent j, preferences are uniformly nonsatiated, that is, $u_j(\cdot)$ has a bounded rate of increase. In particular, for preferences represented by a twice continuously differentiable utility function $u_j(\cdot)$, there exists positive numbers K and ε such that $K > \|\nabla u_j(x)\| > \varepsilon$ for all $x \in R^L$.

2. Global cones and increasing cones

According to Chichilnisky (1997), the *global cone* corresponding to the jth agent's utility function $u_j(\cdot)$ at consumption vector x' is given by,

$$A_j(x') = \{y \in R^L : \forall\, x \in R^L\ \exists\, \lambda_x > 0 \text{ such that } u_j(x' + \lambda_x y) > u_j(x)\}. \tag{1}$$

Thus, a consumption vector y is an element of the global cone $A_j(x')$ if and only if given *any* consumption vector x, there exists a scalar $\lambda_x > 0$ such that the jth agent strictly prefers $x' + \lambda_x y$ to x.

The *increasing cone*, introduced in Page (1982) (also see Page (1989, 1996) and Page and Wooders (1993)), is given by,

$$I_j(x') = \{y \in R^L : u_j(x' + \lambda y) > u_j(x' + \mu y) \text{ if } \lambda > \mu \geq 0\}. \tag{2}$$

An example in Monteiro et al. (1997) shows that the cones defined by (1) and (2) are not equivalent. In Page and Wooders (1994, 1996a,b) the definition of the increasing cone is extended to accommodate thick indifference curves:

$$\hat{I}_j(x') = \{y \in R^L : \forall \mu \geq 0,\ \exists\, \lambda > \mu \text{ such that } u_j(x' + \lambda y) > u_j(x' + \mu y)\}. \tag{3}$$

Chichilnisky (1997) modifies her arbitrage condition by using the increasing cone $\hat{I}_j(x')$, but alternatively stated in her paper as:

$$G_j(x') = \{y \in R^L : \neg\exists \max\nolimits_{\lambda \geq 0} u_j(x' + \lambda y)\}. \tag{4}$$

The equivalence of the Page-Wooders increasing cone defined by (3) and the increasing cone defined by (4) is immediate.[8] Note that under condition [G-3], the cones defined by (2), (3) and (4) are *all* equivalent, and not equivalent to the cone defined by (1) – see Monteiro, Page, and Wooders (1997).

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[7] According to Chichilnisky, $u_j(0) = 0$ and $\sup\{u_j(x) : x \in R^L\} = \infty$ can be assumed without loss of generality (see Kannai and Yusim 1992).

[8] We are indebted to Kenneth Arrow for private correspondence bringing to our attention this change in Chichilnisky (1995) from her prior condition based on her global cone, $A_j(\omega_j)$, to a condition based on the Page-Wooders increasing cone, $I_j(\omega_j)$.

3. Chichilnisky's proposition

Chichilnisky's (1997) Proposition 5, counterexampled below, can now be stated formally as follows:

Proposition 5 (Chichilnisky 1997): *Let $u_j(\cdot) : R^L \to R$ be a utility function satisfying [G-1], [G-2], and [G-3] (i.e., uniform nonsatiation). Then the following statements are true:*

(i) *The interior (denoted int) of the cone $G_j(\omega_j)$ is nonempty and equal to the interior of global cone $A_j(\omega_j)$. Moreover, the interior of $G_j(\omega_j)$ contains those directions in which utility increases to infinity. Thus,*

$$\begin{aligned} \operatorname{int} G_j(\omega_j) &= \operatorname{int} A_j(\omega_j) \\ &= \{y \in G_j(\omega_j) : \lim_{\lambda\to\infty} u_j(\omega_j + \lambda y) = \infty\} \neq \emptyset. \end{aligned} \tag{5}$$

(ii) *The boundary of the cone $G_j(\omega_j)$, denoted by $\partial G_j(\omega_j)$, contains*
   (a) *those directions along which utility increases toward a bounded value that is never reached, and*
   (b) *those directions along which the utility eventually achieves a constant value.*

(iii) *The interiors, boundaries, and closures of the global cones $A_j(x')$ are uniform across all vectors $x' \in R^L$. The cones $G_j(x')$ are also uniform across all vectors $x' \in R^L$.*

(iv) *For general nonsatiated preferences (i.e., preferences that are not* uniformly *nonsatiated), the cones $G_j(x')$ may not be uniform across all vectors $x' \in R^L$.*

Before stating Proposition 5, Chichilnisky writes:

"*The following proposition establishes the structure of the global cones, and is used in proving the connection between limited arbitrage, equilibrium and the core*:"

The counterexample presented below shows that parts (i) and (ii) of Chichilnisky's Proposition 5 are false. No proof of Proposition 5 nor of the claimed connection between limited arbitrage, equilibrium and the core is provided in Chichilnisky (1997). For the proof of Proposition 5, Chichilnisky instead refers the reader to forthcoming papers and Chichilnisky (1995) which states and purports to prove the same proposition. Note that Monteiro, Page, and Wooders (1997) also provides a counterexample to footnote 16 of Chichilnisky (1997) which claims the equivalence of $G_j(\omega_j)$ and $A_j(\omega_j)$. Note also that we do not present a counterexample to the current version of Chichilnisky's claim that limited arbitrage is necessary and sufficient for existence. However, given the importance of Proposition 5 to the validity of this claim, our counterexample shows that the claim is still without proof.

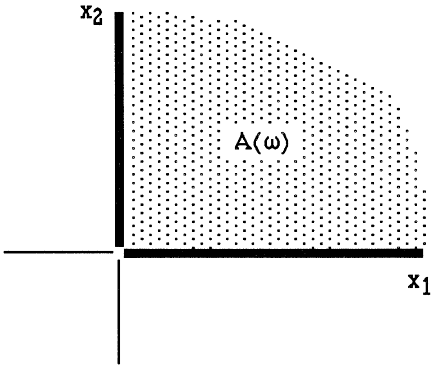

Fig. 1

4. The counterexample

Overview

We give an example of a utility function $u(\cdot) : R^2 \to R$ which satisfies assumptions [G-1], [G-2] (i) (*no half lines*), and [G-3] (*uniform nonsatiation*). The utility function in our example has global cone, $A(\omega)$, depicted in Fig. 1. Here, global cone $A(\omega)$ is given by the *nonnegative orthant* minus the origin.

We show for the utility function in our example that for any $x \in R^L$

$$u(x + \lambda y) \text{ is increasing in } \lambda \geq 0 \quad \text{for } y \in A(\omega),$$

$$\lim_{\lambda \to \infty} u(x + \lambda y) = \infty \quad \text{for all } y \in A(\omega),$$

and

$$\lim_{\lambda \to \infty} u(x + \lambda y) = -\infty \quad \text{for all } y \notin A(\omega).$$

Thus, in our example the global cone $A(x)$ and the increasing cone $G(x)$ are equal for all $x \in R^L$, and,

$$\text{int}\, A_j(\omega_j) \neq \{y \in G_j(\omega_j) : \lim_{\lambda \to \infty} u_j(\omega_j + \lambda y) = \infty\},$$

negating part (i) of Chichilnisky's Proposition 5. Also, in our example

$$\lim_{\lambda \to \infty} u_j(\omega_j + \lambda y) = \infty \quad \text{for all nonzero } y \in \partial G(x),$$

negating part (ii) of Chichilnisky's Proposition 5.

Details

Consider the utility function $u(\cdot) : R^2 \to R$ given by

$$u(x_1, x_2) = \left(\frac{1}{2}\right) \cdot \int_0^{x_1 - (x_2/3)} \left(2 - \frac{2}{\pi} \text{ArcTan}(c)\right) dc + \left(\frac{1}{2}\right) \cdot \int_0^{x_2 - (x_1/3)} \left(2 - \frac{2}{\pi} \text{ArcTan}(c)\right) dc. \tag{6}$$

Lemma 1 (Basic Properties of Utility Function (6)): *Utility function (6) is strictly concave, strictly increasing, and continuously differentiable infinitely many times (i.e.* C^∞*), with* $u(0) = 0$.

Lemma 2 (Evaluation of Limits): *For utility function (6), the following limit statements are true:*

(a) $\lim_{\lambda\to\infty} u(x + \lambda y) = \infty$ *for any x in* R^2 *and any nonzero vector y contained in the* nonnegative *orthant (i.e., any vector* $y = (y_1, y_2) \neq 0$ *with* $y_1 \geq 0$ *and* $y_2 \geq 0$*).*
(b) $\lim_{\lambda\to\infty} u(x + \lambda y) = -\infty$ *for any x in* R^2 *and any y not contained in the nonnegative orthant.*

From Lemma 2 we can conclude that the global cone corresponding to the utility function given in (6) is invariant with respect to x and is given by

$$A(x) = \{y \in R^2 \mid y = (y_1, y_2) \neq 0 \text{ with } y_1 \geq 0 \text{ and } y_2 \geq 0\}, \tag{7}$$

as in Fig. 1.

It only remains to show that utility function (6) satisfies [G-3], uniform nonsatiation.

Lemma 3 (Utility Function (6) Satisfies Uniform Nonsatiation): *For utility function (6), uniform nonsatiation is satisfied (i.e., [G-3] holds). In particular, for all* $x \in R^L$, $\frac{2}{3} < \left|\frac{\partial u(x_1, x_2)}{\partial x_1}\right| + \left|\frac{\partial u(x_1, x_2)}{\partial x_2}\right| < 2.$

Appendix

Proof of Lemma 1: The strict concavity of $u(\cdot)$ is an immediate consequence of the strict concavity of the function $v(\cdot)$ given by

$$v(\bar{c}) = \int_0^{\bar{c}} \left(2 - \frac{2}{\pi}\text{ArcTan}(c)\right) dc,$$

and the fact that the mapping $(x_1, x_2) \to \left(x_1 - \frac{x_2}{3}, x_2 - \frac{x_1}{3}\right)$ specifying the limits of integration in (3) is linear injective. Moreover because $v(\cdot)$ is continuously differentiable infinitely many times, the utility function $u(\cdot)$ is C^∞. The fact that $u(\cdot)$ is strictly increasing follows by inspection of the partial derivatives

$$\frac{\partial u(x_1, x_2)}{\partial x_1} = \frac{2}{3} - \frac{1}{\pi}\text{ArcTan}\left(x_1 - \frac{x_2}{3}\right) + \frac{1}{3\pi}\text{ArcTan}\left(x_2 - \frac{x_1}{3}\right), \quad \text{and}$$

$$\frac{\partial u(x_1, x_2)}{\partial x_2} = \frac{2}{3} + \frac{1}{3\pi}\text{ArcTan}\left(x_1 - \frac{x_2}{3}\right) - \frac{1}{\pi}\text{ArcTan}\left(x_2 - \frac{x_1}{3}\right). \qquad \text{Q.E.D.}$$

Proof of Lemma 2: First note that the antiderivative of ArcTan(c) is

$$c \cdot \text{Arctan}(c) - \frac{\log(1+c^2)}{2} + \text{a constant.}$$

Therefore,

$$u(x) = \frac{2(x_1+x_2)}{3} - \frac{1}{\pi}\left\{ c \cdot \text{ArcTan}(c)\Big|_0^{x_2-(x_1/3)} - \frac{\log(1+c^2)}{2}\Big|_0^{x_2-(x_1/3)} \right\}$$

$$-\frac{1}{\pi}\left\{ c \cdot \text{ArcTan}(c)\Big|_0^{x_1-(x_2/3)} - \frac{\log(1+c^2)}{2}\Big|_0^{x_1-(x_2/3)} \right\}.$$

Let $y \in R^2$ and $\lambda > 0$ and consider $\frac{u(x+\lambda y)}{\lambda}$. We have

$$\frac{u(x+\lambda y)}{\lambda} = \frac{2(x_1+x_2)}{3\lambda} + \frac{2(y_1+y_2)}{3}$$

$$-\frac{1}{\pi}\left\{ \left(\frac{3x_2-x_1}{3\lambda} + y_2 - \frac{y_1}{3}\right) \text{ArcTan}\left(\frac{3x_2-x_1}{3} + \lambda\left(y_2 - \frac{y_1}{3}\right)\right)\right\}$$

$$-\frac{1}{\pi}\left\{ \left(\frac{3x_1-x_2}{3\lambda} + y_1 - \frac{y_2}{3}\right) \text{ArcTan}\left(\frac{3x_1-x_2}{3} + \lambda\left(y_1 - \frac{y_2}{3}\right)\right)\right\}$$

$$+\frac{1}{2\pi\lambda}\left\{ \log\left[1 + \left(x_2 - \frac{1}{3}x_1 + \lambda\left(y_2 - \frac{y_1}{3}\right)\right)^2\right]\right.$$

$$\left. + \log\left[1 + \left(x_1 - \frac{1}{3}x_2 + \lambda\left(y_1 - \frac{y_2}{3}\right)\right)^2\right]\right\}.$$

As $\lambda \to \infty$, the limit of the terms above involving log is zero. Moreover,

$$\lim_{\lambda\to\infty} \frac{z}{\pi} \cdot \text{ArcTan}(c + \lambda z) = \frac{|z|}{2}.$$

Thus,

$$\lim_{\lambda\to\infty} \frac{u(x+\lambda y)}{\lambda} = \frac{2(y_1+y_2)}{3} - \frac{1}{2} \cdot \left| y_1 - \frac{y_2}{3} \right| - \frac{1}{2} \cdot \left| y_2 - \frac{y_1}{3} \right|. \quad (8)$$

Limit statement (b) follows immediately from (8). Also it follows from (8) that

$\lim_{\lambda\to\infty} u(x+\lambda y) = \infty$ for all x in R^2 and all vectors $y = (y_1, y_2)$ with $y_1 > 0$ and $y_2 > 0$.

Now let us evaluate the limit $\lim_{\lambda\to\infty} u(x+\lambda y)$ for the more delicate case $y = (1,0)$. The case $y = (0,1)$ is totally analogous. For the case $y = (1,0)$

we have

$$u(x+\lambda y) = u(x_1+\lambda, x_2) = \frac{2(x_1+\lambda+x_2)}{3}$$
$$-\frac{1}{\pi}\left\{ c\cdot \mathrm{ArcTan}(c)\Big|_0^{x_2-((x_1+\lambda)/3)} - \frac{\log(1+c^2)}{2}\Big|_0^{x_2-((x_1+\lambda)/3)} \right\}$$
$$-\frac{1}{\pi}\left\{ c\cdot \mathrm{ArcTan}(c)\Big|_0^{x_1+\lambda-(x_2/3)} - \frac{\log(1+c^2)}{2}\Big|_0^{x_1+\lambda-(x_2/3)} \right\}.$$

Thus,

$$u(x_1+\lambda, x_2) = \frac{2(x_1+x_2)}{3} + \frac{2\lambda}{3} - \frac{1}{\pi}\left\{\left(\frac{3x_2-x_1}{3} - \frac{\lambda}{3}\right)\mathrm{ArcTan}\left(\frac{3x_2-x_1}{3} - \frac{\lambda}{3}\right)\right\}$$
$$-\frac{1}{\pi}\left\{\left(\frac{3x_1-x_2}{3}+\lambda\right)\mathrm{ArcTan}\left(\frac{3x_1-x_2}{3}+\lambda\right)\right\}$$
$$+\frac{1}{2\pi}\left\{\log\left[1+\left(x_2-\frac{1}{3}x_1-\frac{\lambda}{3}\right)\right)^2\right]$$
$$+\log\left[1+\left(x_1-\frac{1}{3}x_2+\lambda\right)^2\right]\right\}.$$

We need to calculate

$$\lim_{\lambda\to\infty}\left[\frac{2\lambda}{3} - \frac{1}{\pi}\left\{-\frac{\lambda}{3}\mathrm{ArcTan}\left(\frac{3x_2-x_1}{3}-\frac{\lambda}{3}\right) + \lambda\ \mathrm{ArcTan}\left(\frac{3x_1-x_2}{3}+\lambda\right)\right\}\right.$$
$$\left.+\frac{1}{2\pi}\left\{\log\left[1+\left(x_2-\frac{1}{3}x_1-\frac{\lambda}{3}\right)^2\right] + \log\left[1+\left(x_1-\frac{1}{3}x_2+\lambda\right)^2\right]\right\}\right].$$

First note that the limit of the terms *not* containing log is *finite*. To see this we apply L'Hopital rule:

$$\lim_{\lambda\to\infty}\frac{\left[\frac{2}{3}+\frac{\mathrm{ArcTan}\,(x_2-(x_1/3)-(\lambda/3))}{3\pi} - \frac{\mathrm{ArcTan}\,(x_1+\lambda-(x_2/3))}{\pi}\right]}{1/\lambda}$$

by L'Hôpital's rule

$$= \lim_{\lambda\to\infty}\frac{-\frac{1}{9\pi}\cdot\frac{1}{1+(x_2-(x_1/3)-(\lambda/3))^2} - \frac{1}{\pi}\cdot\frac{1}{1+(x_1+\lambda-(x_2/3))^2}\Big]}{-1/\lambda^2} = \frac{2}{\pi}.$$

Thus, we have

$$\lim_{\lambda\to\infty}\left[\frac{2\lambda}{3}-\frac{1}{\pi}\left\{-\frac{\lambda}{3}\mathrm{ArcTan}\left(\frac{3x_2-x_1}{3}-\frac{\lambda}{3}\right)+\lambda\mathrm{ArcTan}\left(\frac{3x_1-x_2}{3}+\lambda\right)\right\}\right.$$
$$\left.+\frac{1}{2\pi}\left\{\log\left[1+\left(x_2-\frac{1}{3}x_1-\frac{\lambda}{3}\right)^2\right]+\log\left[1+(x_1-\frac{1}{3}x_2+\lambda)^2\right]\right\}\right]=\infty,$$

and thus we can conclude that $\lim_{\lambda\to\infty}u(x_1+\lambda,x_2)=\infty$ for all $(x_1,x_2)\in R^2$. Q.E.D.

Proof of Lemma 3: First recall that

$$\frac{\partial u(x_1,x_2)}{\partial x_1}=\frac{2}{3}-\frac{1}{\pi}\,\mathrm{ArcTan}\left(x_1-\frac{x_2}{3}\right)+\frac{1}{3\pi}\,\mathrm{ArcTan}\left(x_2-\frac{x_1}{3}\right),\quad \text{and}$$
$$\frac{\partial u(x_1,x_2)}{\partial x_2}=\frac{2}{3}+\frac{1}{3\pi}\,\mathrm{ArcTan}\left(x_1-\frac{x_2}{3}\right)-\frac{1}{\pi}\,\mathrm{ArcTan}\left(x_2-\frac{x_1}{3}\right).$$

To see that utility function (6) satisfies uniform nonsatiation [G-3] it suffices to consider the sum norm. Thus, for all $x\in R^L$ we have

$$\left|\frac{\partial u(x_1,x_2)}{\partial x_1}\right|+\left|\frac{\partial u(x_1,x_2)}{\partial x_2}\right|$$
$$=\left|\frac{2}{3}-\frac{1}{\pi}\,\mathrm{ArcTan}\left(x_1-\frac{x_2}{3}\right)+\frac{1}{3\pi}\,\mathrm{ArcTan}\left(x_2-\frac{x_1}{3}\right)\right|$$
$$+\left|\frac{2}{3}+\frac{1}{3\pi}\,\mathrm{ArcTan}\left(x_1-\frac{x_2}{3}\right)-\frac{1}{\pi}\,\mathrm{ArcTan}\left(x_2-\frac{x_1}{3}\right)\right|$$
$$=\frac{4}{3}-\frac{2}{3\pi}\,\mathrm{ArcTan}\left(x_1-\frac{x_2}{3}\right)-\frac{2}{3\pi}\,\mathrm{ArcTan}\left(x_2-\frac{x_1}{3}\right).$$

Since for $c\in R$, $-\frac{\pi}{2}<\mathrm{ArcTan}(c)<\frac{\pi}{2}$, we have

$$\frac{2}{3}<\left|\frac{\partial u(x_1,x_2)}{\partial x_1}\right|+\left|\frac{\partial u(x_1,x_2)}{\partial x_2}\right|<2.$$ Q.E.D

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