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Arbitrage, equilibrium, and gains from trade: A counterexample

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Abstract

We present a counterexample to a theorem due to Chichilnisky (*Bulletin of the American Mathematical Society*, 1993, 29, 189–207; *American Economic Review*, 1994, 84, 427–434). Chichilnisky's theorem states that her condition of limited arbitrage is necessary and sufficient for the existence of an equilibrium in an economy with unbounded short sales. Our counterexample shows that the condition defined by Chichilnisky is not sufficient for existence of equilibrium. We also discuss difficulties in Chichilnisky (*Economic Theory*, 1995, 5, 79–107).

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1. Introduction

Werner (1987) introduces a model of an economy with unbounded short sales and shows that a condition limiting arbitrage opportunities is sufficient for the

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existence of an equilibrium and remarks that if indifference surfaces contain no half lines then the condition is also necessary. [1] Under a different set of assumptions on the economic model, Chichilnisky introduces a new condition, called *limited arbitrage*, and asserts that within the context of her model, limited arbitrage is necessary and sufficient for the existence of an equilibrium. [2] Chichilnisky also claims that her condition is necessary and sufficient for boundedness of gains from trade. This paper provides a counterexample to Chichilnisky's claims that limited arbitrage is necessary and sufficient for the existence of an equilibrium and shows that Chichilnisky's notion of gains from trade has no relevance for the existence of an equilibrium.

Chichilnisky defines arbitrage as an opportunity for an agent to increase his utility costlessly beyond the level associated with any given vector in his consumption set. Chichilnisky's condition of limited arbitrage rules out such arbitrage. However, ruling out only such arbitrage is not sufficient to ensure the existence of an equilibrium. Consider, for example, an agent who can costlessly make net trades in direction y on an arbitrarily large scale, and suppose that the utility levels of these trades increase with their magnitudes. [3] It is possible that in direction y the utility function is bounded above by some finite utility level smaller than the supremum of utility over the entire consumption set. Under almost any definition of arbitrage, such a trading opportunity would be viewed as unbounded arbitrage, in the sense that costless utility increases in the y direction cannot be exhausted by trade bounded in size. This type of arbitrage can be fatal for the existence of an equilibrium. In Chichilnisky's approach, since trade in direction y cannot increase utility beyond any given level, trade in this direction is not considered to be arbitrage and is not ruled out by limited arbitrage.

From the perspective of the existence of an equilibrium, the critical feature of arbitrage is not the magnitude of utility increases that can be achieved costlessly, but rather the magnitude of net trade required to exhaust increases in utility costlessly. Earlier papers on the existence of an equilibrium introduce conditions limiting arbitrage that endogenously place a bound on the magnitudes of net trades that are both utility increasing and costless, for example Page (1987) and Werner (1987). [4] Page's condition of no unbounded arbitrage is a condition on arbitrage cones of individual agents. The arbitrage cone of an agent describes the set of net

[1] A proof of Werner's necessity result is provided in Page and Wooders (1993).

[2] This claim appears as Theorem 1 in Chichilnisky (1993) and as Proposition 2 in Chichilnisky (1994a).

[3] More formally, $u(\omega + \lambda y)$ is strictly increasing as λ goes to infinity.

[4] Werner (1987) limits arbitrage by placing a condition on the positive duals of the arbitrage cones. In some cases the conditions in Werner (1987) and Page (1987) are equivalent. The condition in Page (1987) is directly on arbitrage opportunities and applies to a broader class of models. See Nielsen (1989) for such an application.

trade vectors that are utility *non-decreasing* on any scale. [5] No unbounded arbitrage, as defined by Page (1987), requires that if the sum, over the set of agents in an economy, of net trade vectors in arbitrage cones of individual agents is zero, then each net trade vector must be zero. [6] In other words, there is no set of feasible (i.e. mutually compatible) and non-trivial net trades that is utility non-decreasing on any scale for each agent. [7] Werner (1987) uses the dual form of no unbounded arbitrage.

In contrast to conditions limiting arbitrage in the earlier literature, Chichilnisky's condition is only on those net trade vectors in directions in which the utilities of agents approach their suprema. Chichilnisky defines the global cone of an agent as the set of directions in which utility approaches the supremum as trades in those directions become infinite. Limited arbitrage is satisfied if the intersection of positive duals of global cones of all agents is non-empty. For example, if the supremum of the utility function of each agent is infinite, then Chichilnisky's condition bounds arbitrage in all directions where utilities become arbitrarily large as the magnitude of trade increases arbitrarily. Chichilnisky (1994a) states that gains from trade are bounded if and only if the sum of agents' incremental increases in utility from feasible trade is bounded. In our counterexample, gains from trade are bounded under Chichilnisky's definition, yet gains from trade cannot be exhausted by trades bounded in size. Thus, Chichilnisky's claim that boundedness of gains from trade is necessary and sufficient for limited arbitrage to hold has no relevance for the existence of an equilibrium.

One of the main objectives of Chichilnisky (1993, 1994a) appears to have been to obtain necessary and sufficient conditions for the existence of an equilibrium in terms of conditions limiting arbitrage for economic models in which agents' indifference curves are allowed to contain half lines. Almost all other results on arbitrage-based necessary and sufficient conditions for the existence of an equilibrium assume that agents' indifference curves contain no half lines. [8] In view of the discussion of no unbounded arbitrage above, it is easy to see why such a condition may play an important role. The existence of an equilibrium does not rule out arbitrarily large utility non-decreasing trade. If there are half lines in indifference curves, then an equilibrium may exist but some agents may be able to engage in

[5] That is, the vector of trades y is in the arbitrage cone of the jth agent if $u_j(\omega_j + \lambda y)$ is non-decreasing in λ for $\lambda \geq 0$.

[6] Debreu (1962) applies a similar condition to the consumption sets of agents to bound feasible and individually rational net trades.

[7] Page (1989, 1996) and Page and Schlesinger (1993) elaborate on the distinction between bounded and unbounded (utility) arbitrage. As Page (1987) notes, no unbounded arbitrage is a similarity assumption on preferences; if two agents have diametrically opposed preferences, then no unbounded arbitrage cannot be satisfied.

[8] For example, see Grandmont (1977, 1982), Hammond (1983), Werner (1987), Page (1982, 1989, 1996), Page and Schlesinger (1993), and Page and Wooders (1993).

arbitrarily large trade while realizing only their equilibrium utility levels. Thus, in general with half lines in indifference curves, the existence of an equilibrium does not imply no unbounded arbitrage in the sense of Page (1987) and Werner (1987). Certainly for some economic models where half lines in indifference curves are allowed it is possible to state necessary and sufficient conditions for existence in terms of conditions limiting arbitrage. See, for example, Milne (1980), who deduces a necessary and sufficient condition for existence from Hart's (1974) condition limiting arbitrage. Milne accomplishes this in an asset market model by using the structure of the distribution of asset returns. Page and Wooders (1995, 1996a) show that Page's (1987) condition of no unbounded arbitrage is necessary and sufficient for the existence of an equilibrium in an exchange economy with unbounded short sales provided at least all but one agent has no half lines in indifference curves. Chichilnisky (1993, 1994a) requires conditions on the set of gradients to indifference curves unbounded from below as well as on the norms of gradients to indifference curves. As it turns out Chichilnisky's conditions on the economic model do not suffice.

In the penultimate section of the paper, we discuss some of Chichilnisky's more recent research on arbitrage. Chichilnisky (1995a) asserts that a different condition limiting arbitrage is necessary and sufficient for the existence of an equilibrium. We note that our counterexample grew out of remarks made by Page in two referee's reports on the *original* versions of Chichilnisky (1995a) submitted to *Economic Theory*. Our counterexamples apply to these submissions. [9] Modifications to the definition of the global cone which appeared in the published version (i.e., in Chichilnisky (1995a)) were made at the page proof stage. The problems discussed here concern the published version of the paper and begin with a lack of well-definedness of the global cone. Chichilnisky (1994b) introduces yet another condition limiting arbitrage, but provides no proof of the claimed results. [10] Chichilnisky (1995b) adopts the 'increasing cone' of Page (1982, 1989) and Page and Wooders (1993) and thus a still different condition limiting arbitrage. A crucial proposition on which the proofs in Chichilnisky (1995b) rest, howver, is counterexampled in Monteiro, et al. (1995).

We conclude the paper with some brief comments on the literature. We note here, however, that since the existence of an equilibrium implies non-emptiness of the core, the necessity of no unbounded arbitrage for non-emptiness of the core, shown in Page and Wooders (1993), is a stronger result than Werner's (1987)

[9] According to Chichilnisky's correspondence with C.D. Aliprantis (Managing Editor of *Economic Theory*) obtained during the legal process of discovery, she and Nicholas Yannelis (a Co-Editor of *Economic Theory*) agreed, prior to its submission, that Page would be asked to referee her paper. Thus, unbeknownst to Page, his identity as the referee was known to Chichilnisky from the outset of the review process. During the course of five months, Page reviewed the submission two times.

[10] We refer the reader to Hurwicz (1996) for further discussion of the several versions of 'limited arbitrage' used in Chichilnisky's papers and of our counterexamples.

result on the necessity of equilibrium. In Page and Wooders (1995, 1996a) it is shown that in strictly reconcilable economies (i.e., economies allowing at most one agent to have half lines in indifference surfaces) merely the existence of a Pareto optimal allocation implies no unbounded arbitrage. This is the strongest of the necessity results.

2. The basic model, notation, and terminology

2.1. The basic model

Let $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ denote an unbounded exchange economy. Each agent j has a consumption set R^L, an endowment $\omega_j \in R^L$, and preferences over R^L specified via a utility function $u_j(\cdot): R^L \to R$. We shall assume throughout that

A1. For each agent, $u_j(\cdot)$ is continuous and quasiconcave.

For an unbounded exchange economy $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ the *set of feasible and individually rational allocations* is given by

$$F(\omega) = \left\{ (x_1, \dots, x_n) \in R^L \times \cdots \times R^L : \sum_{j=1}^n x_j = \sum_{j=1}^n \omega_j, \right.$$

$$\left. \text{and } \forall j,\ u_j(x_j) \geq u_j(\omega_j) \right\}. \tag{1}$$

Given prices $p \in R^L$, the cost of a consumption vector $x = (x_1, \dots, x_L) \in R^L$ is $\langle p, x \rangle = \sum_{\ell=1}^L p_\ell x_\ell$. The *budget set* for the jth agent is given by

$$B(p, \omega_j) = \left\{ x \in R^L : \langle p, x \rangle \leq \langle p, \omega_j \rangle \right\}.$$

Without loss of generality, we can assume that commodity prices are contained in the unit ball $B \equiv \{p \in R^m : \| p \| \leq 1\}$.

An *equilibrium* for the economy $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ is an $(n+1)$-tuple of vectors $(\bar{x}_1, \dots, \bar{x}_n, \bar{p})$ such that

(a) $(\bar{x}_1, \dots, \bar{x}_n) \in F(\omega)$ (the allocation is feasible and individually rational);
(b) $\bar{p} \in B \setminus \{0\}$ (prices are in the unit ball and not all prices are zero);
(c) and for each j,
  (i) $\langle \bar{p}, \bar{x}_j \rangle = \langle \bar{p}, \omega_j \rangle$ (budget constraints are satisfied), and
  (ii) $u_j(\bar{x}_j) = \max\{u_j(x) : x \in B(\bar{p}, \omega_j)\}$ (there are no affordable preferred net trades).

In Chichilnisky (1994a) *gains from trade* are defined by

$$\sup\left\{ \sum_{j=1}^n \left(u_j(x_j) - u_j(\omega_j) \right) : (x_1, \dots, x_n) \in F(\omega) \right\}.$$

Chichilnisky (1993, 1994a) considers an unbounded exchange economy $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ satisfying A1 and imposes additional assumptions. We summarize these additional assumptions in Section 3.

2.2. Recession directions and positive dual cones

Given any subset S of R^L, we define

$$R(S) = \{y \in R^L : \forall x \in S \text{ and } \forall \lambda \geq 0,\ x + \lambda y \in S\} \tag{2}$$

and

$$D(S) = \{p \in R^L : \langle p, y \rangle > 0,\ \forall y \in S \setminus \{0\}\}. \tag{3}$$

R(S) is the set of recession directions of S. If S is a closed, convex set, then $R(S)$ is a closed, convex cone containing the origin. Moreover, if S is a closed, convex set, then $x' + \lambda y \in S$ *for some* $x' \in S$ and all $\lambda \geqslant 0$ implies that $x + \lambda y \in S$ *for all* $x \in S$ and all $\lambda \geq 0$. The set $D(S)$ is the positive dual cone corresponding to S.

2.3. Recession cones and increasing cones

Let $U_j(x') = \{x \in R^L \mid u_j(x) \geq u_j(x')\}$. The set $U_j(x')$ is the 'preferred to' set at the consumption vector x'. The recession cone corresponding to the jth agent's utility function $u_j(\cdot)$ at the consumption vector x' is given by

$$R(U_j(x')) = \{y \in R^L : \forall x \in U_j(x'),\ u_j(x + \lambda y) \geq u_j(x') \forall \lambda \geq 0\}. \tag{4}$$

Thus, the recession cone $R(U_j(x'))$ is simply the set of recession directions corresponding to the closed, convex 'preferred to' set $U_j(x')$. If the utility function is concave, then

$$R(U_j(x)) = R(U_j(\bar{x})) \quad \text{for all } x \text{ and } \bar{x} \text{ in } R^L \tag{5}$$

(see theorem 8.7 in Rockafellar, 1970). [11]

A cone closely related to the recession cone is the increasing cone. [12] The increasing cone corresponding to the jth agent's utility function $u_j(\cdot)$ at the consumption vector x' is given by [13]

$$I_j(x') = \{y \in R^L : \forall \lambda \geq 0\ \exists \lambda' > \lambda \text{ such that } u_j(x' + \lambda' y) > u_j(x' + \lambda y)\}. \tag{6}$$

[11] Page and Wooders (1993) named condition (5) *uniformity*.

[12] The increasing cone was introduced in Page (1982) in an asset market model to obtain necessary and sufficient conditions for existence. It has since been used in several papers including Page, (1989, 1996), Page and Schlesinger (1993), Page and Wooders (1993, 1995, 1996a,b).

[13] If the utility function $u_j(\cdot)$ is concave, then $I_j(x') = \{y \in R^L : u_j(x' + \lambda y)$ is increasing in $\lambda\}$. Expression (6) extends the increasing cone of Page (1982) and Page and Wooders (1993) to allow thick indifference curves (see Page and Wooders, 1995, 1996a,b).

In many economic models recession cones (minus the origin) and increasing cones are equivalent. We call the condition of equality of recession cones and increasing cones *extreme desirability*. To obtain a clear picture of when the equality might hold, we introduce the *no-half-lines condition* and formally define the *condition of extreme desirability*.

Definition 1.

(a) *No half lines* (Werner, 1987). We say that agent j's utility function $u_j(\cdot)$ satisfies the no-half-lines condition if there do not exist vectors x and y in R^L with $y \neq 0$ such that $u_j(x) = u_j(x + \lambda y)$ for all $\lambda \geq 0$. [14]

(b) *Extreme desirability* (Page and Wooders, 1993). We say that agent j's utility function $u_j(\cdot)$ satisfies the extreme desirability condition if $R(U_j(x)) \setminus \{0\} = I_j(x)$, $\forall x \in R^L$. [15]

The proof of the following proposition is straightforward and is thus omitted.

Theorem 1 (Equivalence of no half lines and extreme desirability). *Let* $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ *be an unbounded exchange economy such that* $u_j(\cdot)$ *is continuous and quasiconcave. Then the following are equivalent*:

(a) *Agent j's utility function satisfies the no-half-lines condition.*

(b) *Agent j's utility function satisfies the extreme desirability condition.*

2.4. Global cones

Chichilnisky (1993, 1994a) defines the global cone. The global cone corresponding to the jth agent's utility function $u_j(\cdot)$ at the consumption vector x' is given by

$$A_j(x') = \left\{ y \in R^L : \forall x \in R^L \ \exists \lambda_x > 0 \text{ such that } u_j(x' + \lambda_x y) > u_j(x) \right\}. \tag{7}$$

Thus, if the consumption vector y is an element of the global cone $A_j(x')$, then given *any* consumption vector x, there exists a scalar $\lambda_x > 0$ such that the jth

[14] Within the context of a general equilibrium model similar to the model presented here, Werner (1987) shows that the dual form of no unbounded arbitrage based on recession cones is sufficient for the existence of an equilibrium and states that if agents' utility functions satisfy no half lines, then no unbounded arbitrage is also necessary.

[15] Within the context of a general equilibrium model of asset trading, Hammond (1983) and independently, Page (1982) show that if agents' utility functions satisfy extreme desirability, then a no-unbounded-arbitrage condition based on recession cones is necessary and sufficient for the existence of an equilibrium in an asset market.

agent strictly prefers $x' + \lambda_x y$ to x. Note that Chichilnisky's global cone and the Page and Wooders increasing cone are not equivalent.

3. Chichilnisky's model

3.1. Assumptions

Let $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ be an unbounded exchange economy satisfying A1. The additional assumptions required by Chichilnisky (1993, 1994a) are the following:

C1. For all agents, utility functions are concave, monotonically increasing, and smooth (i.e. twice continuously differentiable), with $u_j(0) = 0$.

C2. For all agents, the set of gradient directions along any indifference curve unbounded from below is closed (*the closed gradient condition*). [16]

C3. There exists a positive number ϵ such that for each agent, $\| \partial u_j(x)/\partial x \| > \epsilon$ for all $x \in R^L$ (*the norm of the gradient condition*).

C4. There exists a positive number K such that for each agent, $\| \partial^2 u_j(x)/\partial x^2 \| < K$ for all $x \in R^L$.

3.2. Arbitrage cones and limited arbitrage

Chichilnisky (1993, 1994a) considers only unbounded arbitrage opportunities contained in agents' global cones (as defined in (7)). Her condition of *limited arbitrage* is given by

$$\bigcap_j D(A_j(\omega_j)) \neq \varnothing. \tag{8}$$

This condition simply means that the set of prices assigning a *positive* value to any vector of net trades y contained in any global cone $A_j(\omega_j)$ is non-empty.

Unfortunately, Chichilnisky's condition of limited arbitrage is inadequate. Note that each global cone contains utility-increasing net trades. Even under the strong assumptions of Chichilnisky's model, global cones can fail to contain *all* utility-

[16] Let $L_j(x') = \{x \in R^L \mid u_j(x) = u_j(x')\}$ denote the indifference curve containing x'. For any $x' \in R^L$, the indifference curve $L_j(x')$ is bounded from below if there exists a vector $z \in R^L$ such that $z \leq x$ for all $x \in L_j(x')$. Let $\text{grad}_j(x') = \{\nabla u_j(x)/ \|\nabla u_j(x)\| : x \in L_j(x')\}$. Thus, $\text{grad}_j(x')$ is the set of gradient directions along the indifference curve containing x'. Chichilnisky's closed gradient condition C2 can be stated formally as follows: if $L_j(x')$ is not bounded from below, then $\text{grad} j(x')$ is closed.

increasing net trades (more on this when we discuss the intuition behind our counterexample). In contrast, recession cones include all possible unbounded arbitrage opportunities. This is precisely why the existing literature, [17] almost without exception, uses recession cones. Recession cones were introduced into the study of economies with unbounded consumption sets by Debreu (1962) and have since been used in very general economic models (see, for example, Duffie, 1986). In papers by Grandmont (1982), Hammond (1983), Page (1982, 1989, 1996), Werner (1987), Page and Wooders (1993, 1995, 1996a,b), and Page and Schlesinger (1993), it has been demonstrated that conditions limiting arbitrage stated in terms of recession cones are necessary and sufficient for the existence of an equilibrium precisely in those cases where recession cones and increasing cones coincide (i.e. the extreme desirability case). [18]

3.3. Chichilnisky's theorems

Chichilnisky's Theorems can now be stated formally as follows.

Theorem 1 (Chichilnisky, 1993, theorem 1; Chichilnisky, 1994a, proposition 2). *For an exchange economy* $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ *satisfying C1–C4, the following statements are equivalent*:

(a) $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ *has an equilibrium.*

(b) $\cap_j D(A_j(\omega_j)) \neq \emptyset$.

Theorem 2 (Chichilnisky, 1994a, proposition 1). *For an exchange economy* $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ *satisfying C1–C4, the following statements are equivalent*:

(a) $\sup\{\Sigma_{j=1}^n (u_j(x_j) - u_j(\omega_j)) : (x_1, \ldots, x_n) \in F(\omega)\}$ *is finite.*

(b) $\cap_j D(A_j(\omega_j)) \neq \emptyset$.

Our counterexample shows that Theorem 1 is false. Moreover, our counterexample shows that Chichilnisky's notion of gains from trade is unrelated to the existence of an equilibrium.

4. The counterexample

4.1. The intuition behind the counterexample

Let us consider an unbounded exchange economy populated by two agents in which two goods are traded. Suppose agent 1 has global cone $A_1(\omega_1)$, shown in

[17] This includes papers by Hart (1974), Grandmont (1977, 1982), Milne (1980), Hammond (1983), Page (1982, 1987, 1989, 1996), Werner (1987), Nielsen (1989), Page and Schlesinger (1993), Page and Wooders (1993), and others.

[18] In Page (1989, 1996) and Page and Schlesinger (1993), the definition of unbounded arbitrage is given in terms of increasing cones.

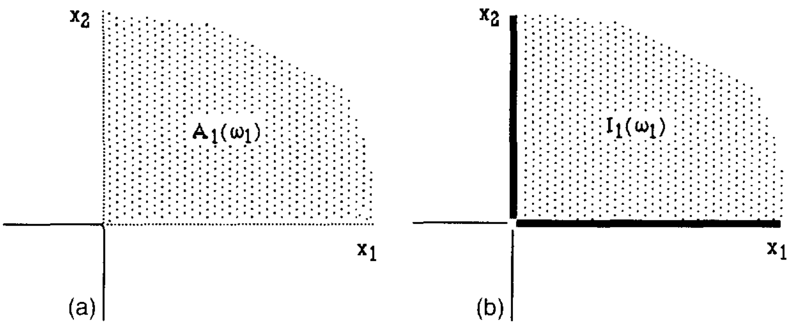

Fig. 1.

Fig. 1(a), and an increasing cone $I_1(\omega_1)$, shown in Fig. 1(b). Here, the global cone $A_1(\omega_1)$ is given by the *positive orthant*, while the increasing cone $I_1(\omega_1)$ is given by the *non-negative orthant* minus the origin. Thus, the net trade vector $y^1 = (0, 1)$ is contained in the increasing cone, but not in the global cone. Note that the price vector $p = (1, 0)$ is contained in the positive dual cone $D(A_1(\omega_1))$ corresponding to the global cone. Note also that $\langle p, y^1 \rangle = 0$. Thus, agent 1 can *costlessly* trade in the y^1 direction, increasing his utility as the scale of net trades y^1 increases. [19]

The presence of unbounded arbitrage creates a problem if and only if agent 1 has a compatible trading partner. Now consider agent 2. Agent 2 has global cone $A_2(\omega_2)$, shown in Fig. 2(a), and a recession cone $R(U_2(\omega_2))$, shown in Fig. 2(b). Here, the the global cone $A_2(\omega_2)$ is given by an open half space, while the recession cone $R(U_2(\omega_2))$ is given by the closure of this half space. Thus the net trade vector $y^2 = (0, -1)$ is contained in agent 2's recession cone but not in his global cone. Note that the price vector $p = (1, 0)$ is contained in the positive dual cone $D(A_2(\omega_2))$ corresponding to the global cone. Note also that $\langle p, y^2 \rangle = 0$. Thus, agent 2 can *costlessly* trade in the y^2 direction, *maintaining* his utility as the scale of net trades y^2 increases. [20]

Conclusion: agent 1 and agent 2 can make feasible and Pareto improving net trades on an arbitrarily large scale. In particular, because $y^1 + y^2 = (0, 0)$, net trades $y^1 = (0, 1)$ and $y^2 = (0, -1)$ are feasible on any scale, and because $y^1 \in I_1(\omega_1)$ and $y^2 \in R(U_2(\omega_2))$, net trades y^1 and y^2 are Pareto improving on any scale.

In this example, no equilibrium exists, the core of the economy is empty, individually rational allocations are unbounded, and gains from trade cannot be

[19] Thus, $u_1(\omega_1 + \lambda y^1)$ is an increasing function of λ for $\lambda \geq 0$.

[20] Thus, the function $u_2(\omega_2 + \lambda y^2)$ is a non-decreasing function of λ for $\lambda \geq 0$.

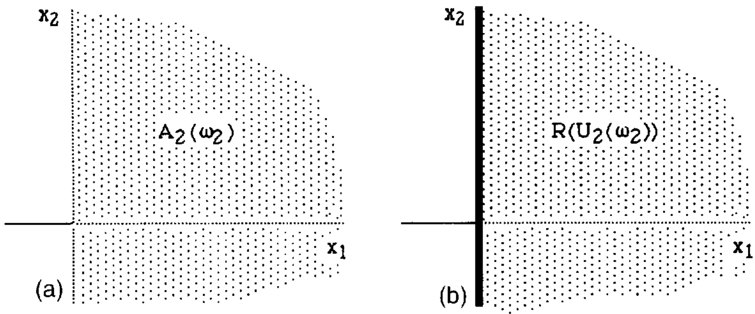

Fig. 2.

exhausted by trades bounded in size. We observe, however, that the price vector $p=(1, 0)$ is contained in $D(A_1(\omega_1)) \cap D(A_2(\omega_2))$. Thus *limited arbitrage is satisfied.*

4.2. The details

Can we construct an economic model that generates Figs. 1 and 2 and at the same time satisfies Chichilnisky's assumptions? The answer is yes.

Let us consider an unbounded exchange economy $(R^2, \omega_j, u_j(\cdot))_{j=1}^2$ populated by two agents in which two commodities are traded. Suppose agents' endowments are given by $\omega_1 = \omega_2 = (0, 0)$. Now let agent 1's utility function be given by

$$u_1(x_1, x_2) = x_1 + x_2 + 2 - \sqrt{(x_1 - x_2)^2 + 4} \tag{9}$$

and agent 2's utility function by

$$u_2(x_1, x_2) = x_1. \tag{10}$$

Theorem 2. The unbounded exchange economy $(R^2, \omega_j, u_j(\cdot))_{j=1}^2$ *with* $\omega_1 = \omega_2 = (0, 0)$ *and utility functions over* $x = (x_1, x_2)$ *given by*

$$u_1(x_1, x_2) = x_1 + x_2 + 2 - \sqrt{(x_1 - x_2)^2 + 4}$$

and

$$u_2(x_1, x_2) = x_1$$

satisfies Chichilnisky's assumptions C1–C4.

Proof. See the appendix.

Theorem 3.

(a) *Agent 1's global cone at endowment* ω_1 *is given by*

$$A_1(\omega_1) = \{y \in R^2 : y = (y_1, y_2), \text{ with } y_1 > 0 \text{ and } y_2 > 0\},$$

while agent 1's increasing cone at endowment ω_1 *is given by*

$$I_1(\omega_1) = \{y \in R^2 : y = (y_1, y_2) \neq 0, \text{ with } y_1 \geq 0 \text{ and } y_2 \geq 0\}.$$

(b) *Agent 2's global cone at endowment* ω_2 *is given by*

$$A_2(\omega_2) = \{y \in R^2 : y = (y_1, y_2) \text{ with } y_1 > 0\},$$

while agent 2's recession cone at endowment ω_2 *is given by*

$$R(U_2(\omega_2)) = \{y \in R^2 : y = (y_1, y_2) \text{ with } y_1 \geq 0\}.$$

Proof. See the appendix.

5. A summary of the counterexample

The example presented in Subsection 4.2 satisfies Chichilnisky's assumptions C1–C4. In addition, the price vector $p = (1, 0)$ is contained in $\cap_j D(A_j(\omega_j))$. Thus, our example *satisfies limited arbitrage*. But it follows from Theorem 3 and the discussion in Subsection 4.1 that the example has no equilibrium.

Note also that in our example,

$$\sup\left\{\sum_{j=1}^{n}(u_j(x_j) - u_j(\omega_j)) : (x_1, \dots, x_n) \in F(\omega)\right\}$$

is finite, but the supremum is *not* attained. In particular, letting $y^1 = (0, 1) \in I_1(\omega_1)$ and $y^2 = (0, -1) \in R(U_2(\omega_2))$, we have

$$(\omega_1 + \lambda y^1) + (\omega_2 + \lambda y^2) \in F(\omega) \quad \text{for all } \lambda \geq 0.$$

In addition,

$$(u_1(\omega_1 + \lambda y^1) - u_1(\omega_1)) + (u_2(\omega_2 + \lambda y^2) - u_2(\omega_2))$$

is strictly increasing (but bounded from above) for all $\lambda \geq 0$. Thus in the sense of Chichilnisky, gains from trade are bounded. But gains from trade cannot be exhausted by trades bounded in size.

6. Discussion of Chichilnisky (1995a)

In Chichilnisky (1995a) a key assumption made in her (1993, 1994a) model $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ was changed, and the definition of the global cone is altered. As a consequence, the condition of limited arbitrage she relies on must also change, as must her theorem. The relevant changes can be summarized as follows:

6.1. Changes in assumptions [21]

Assumption C2 is changed to the following (see Chichilnisky, 1995a, p. 84, Assumption 2):

*C2**. For each agent, *either* (a) the set of gradient directions along any indifference curve unbounded from below is closed (i.e. the closed gradient condition holds), *or* (b) the no-half-lines condition holds (see Definition 1(a)).

6.2. Changes in the definition of the global cone

In Chichilnisky (1995a, p. 85, case 1) the author defines 'global cone' as follows: For agent j the global cone at the consumption vector x' is now given by

$$G_j(x') = \begin{cases} A_j(x'), & \text{if agent } j\text{'s utility function } u_j(\cdot) \text{ satisfies C2}^*\text{(a)}, \\ cl\, A_j(x'), & \text{if agent } j\text{'s utility function } u_j(\cdot) \text{ satisfies C2}^*\text{(b)}. \end{cases} \tag{11}$$

Here $A_j(x')$ denotes the cone defined in expression (7) above and 'cl' denotes closure. As we discuss later in the paper, $G_j(x')$ is not well-defined.

6.3. Changes in the condition of limited arbitrage

As a consequence of changes in assumption C2 and changes in the definition of the global cone, in Chichilnisky (1995a) limited arbitrage is given by

$$\bigcap_j D\big(G_j(\omega_j)\big) \neq \varnothing. \tag{12}$$

6.4. Changes in the main theorem

The main theorem in Chichilnisky (1995a) can be stated as follows:

Theorem 1 (Chichilnisky 1995a, theorem 1, p. 94). *For an exchange economy* $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ *satisfying C1, C2*, C3, and C4, the following statements are equivalent*:

(a) $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ *has an equilibrium.*

(b) $\cap_j D(G_j(\omega_j)) \neq \varnothing$.

[21] Assumptions C1, C3, and C4 remain the same in Chichilnisky (1995a) as they were in Chichilnisky (1993, 1994a).

6.5. Impact of changes in Chichilnisky (1995a)

Below we argue that global cones in Chichilnisky (1995a) are not well defined. Thus (1995a) theorem 1 is not comprehensible. Consider agent 1 in our counterexample. Agent 1's utility function $u_1(\cdot)$ (see (9)) satisfies the no-half-lines condition and has indifference curves *bounded from below*. Thus agent 1's utility function satisfies *both* C2*(a) *and* C2*(b). Is the global cone for agent 1 given by $A_1(\omega_1)$? Or is it given by cl $A_1(\omega_1)$? Note that for agent 1, $A_1(\omega_1) \neq$ cl $A_1(\omega_1)$. Limited arbitrage is not well defined, and thus in this case, Chichilnisky's (1995a) theorem 1 is without meaning.

One way to rid Chichilnisky's (1995a) model of the global cone ambiguity problem is to treat the closed gradient case C2*(a) and the no-half-lines case C2*(b) separately. This approach is consistent with Chichilnisky's prior papers. It is also natural in view of Chichilnisky's reference in footnote 16 to Werner's (1987) treatment of case (b) [22]. This approach is simple to implement. All that is required is that C2* be changed to the following:

*C2***. (a) *For all agents* the closed gradient condition holds, *or* (b) *for all agents* the no-half-lines condition holds.

While this approach solves the ambiguity problem, under C2** Chichilnisky's (1995a) theorem 1 is false by our counterexample.

A natural question to ask is whether or not there is some modification of Chichilnisky's model that would make (1995a) theorem 1 meaningful *and* amenable to proof. One possible modification may be to restate C2*(a) to require that the set of gradient directions along *any* indifference curve be closed. In particular, let us consider the following modification of C2*:

*C2****. (a) For all agents the set of gradient directions along *any* indifference curve is closed, *or* (b) for all agents the no-half-lines condition holds. [23]

With such a modification of C2* the global cone is well defined and it may be that Chichilnisky's (1995a) theorem 1 is provable.

[22] Werner (1987) shows that the dual form of Page's (1987) condition of no unbounded arbitrage is necessary and sufficient for existence assuming *all* agents' utility functions satisfy no half lines. Case (b) (i.e. the no-half-lines case specified via C2*(b)) and a few short remarks concerning this case were apparently added to Chichilnisky (1995a) at the page proof stage. This also suggests that the purported proofs were intended to treat the closed gradient case (case (a)) and the no-half-lines case (case (b)) separately.

[23] According to Walt Heller, Ted Groves, Mo Hirsch, Wayne Shafer, and Duncan Foley (private communication) this is the intended meaning of Chichilnisky's assumption C2*.

In Chichilnisky (1995a) the author claims that theorem 1 (1995a) is true for the case in which all agents' utility functions satisfy no half lines – C2*** (b) above – and refers the reader to Chichilnisky (1994b) or Werner (1987) for a proof (see footnote 16 on p. 84 and p. 106 in Chichilnisky 1995a). Werner (1987) shows that the dual form of no unbounded arbitrage is necessary and sufficient for the existence of an equilibrium in an economic model with no half lines that strictly subsumes Chichilnisky's (1995a) model for C2*** (b). Chichilnisky (1994b) does not provide a proof that limited arbitrage is sufficient for existence in the no-half-lines case. [24] Moreover, the proof of sufficiency for the no-half-lines case does not automatically follow from Werner (1987). As noted by Chichilnisky (see p. 85 and 86 in Chichilnisky 1995a) – and as has been demonstrated by our counterexample – global cones differ from recession cones as used by Werner (1987). [25] Thus in general Werner's condition of no unbounded arbitrage differs from Chichilnisky's condition of limited arbitrage.

6.6. Other facts concerning Chichilnisky (1995a)

Proposition 1 in Chichilnisky (1995a) states that

For an exchange economy $(R^L, \omega_j, u_j(\cdot))_{j=1}^n$ *satisfying C1, C2*, C3, and C4, global cones* $A_j(\omega_j)$ *are open convex sets.*

Since global cones and $A_j(\omega_j)$ cones are not necessarily equal, the meaning of proposition 1 is ambiguous. In particular, does Chichilnisky mean to say that $A_j(\omega_j)$ cones are open and convex? Or does she mean to say that global cones (defined in terms of $A_j(\omega_j)$ cones on p. 85 of Chichilnisky (1995a)) are open and convex? If she means global cones, then proposition 1 is automatically false because in case C2*(b) the global cone is *defined* to be closed (via the closure operation). If she means the $A_j(\omega_j)$ cones, then proposition 1 is false by counterexample (see Monteiro et al., 1995). [26]

[24] Chichilnisky's (1994b) paper was substituted for another paper and inadvertently published in the December (1994) issue of *Economics Letters*. However, because Chichilnisky's paper was published without proper review, North-Holland recalled the December issue of *Economics Letters* and re-issued it with Chichilnisky's paper expunged (see the March 1995 issue of *Economics Letters*).

[25] Page and Wooders (1995) prove that under some conditions the closures of global cones coincide with recession cones as conjectured in Chichilnisky (1995a).

[26] Perhaps because of problems with $A_j(\omega_j)$ cones, in recent papers (e.g. Chichilnisky, 1995b), Chichilnisky uses the increasing cone of Page (1982, 1989), Page and Schlesinger (1993), and Page and Wooders (1993) to model arbitrage. Chichilnisky (1995b) contains a statement that $A_j(\omega_j)$ cones are the interiors of increasing cones (see proposition 2 in Chichilnisky, 1995b). Monteiro et al. (1995), however, provide a counterexample to this statement. This interiority property is crucial for Chichilnisky's purported proof of existence. In revisions of Chichilnisky (1995b), the author adopts the generalized increasing cone introduced in Page and Wooders (1995). (See also expression (6) above and Page and Wooders (1996a,b)).

Chichilnisky (1995a) also includes a definition of limited arbitrage for economies where all consumption sets are equal to the non-negative orthant R_+^L. For this case, Chichilnisky's concepts are well defined, but theorem 1 of Chichilnisky (1995a) is false. The difficulty already appears when one considers figs. 4B and 7 in Chichilnisky (1995a). Contrary to Chichilnisky's assertions, an economy such as that depicted in fig. 4B – where one agent likes only one good and is endowed with only the other good – may have a competitive equilibrium. This is quite intuitive: the agent who has only one good and likes only the other will be willing to sell all his endowment for any non-negative price. Assuming the other agent enjoys both goods, at some sufficiently low price he will buy all the first agent's endowment. The difficulty with fig. 7 and the accompanying description is that the economy has no equilibrium, but Chichilnisky's condition of limited arbitrage is satisfied. A counterexample to Chichilnisky's (1995a) theorem 1 for the R_+^L case, based on fig. 7, appears in Monteiro et al. (1996).

For the case where consumption sets are equal to R_+^L, Chichilnisky (1995b) redefines the global cone as the increasing cone of Page (1982, 1989)), Page and Schlesinger (1993), and Page and Wooders (1993). However, this does not avoid, the counterexample of Monteiro et al. (1996).

7. Conclusions

While imposing additional conditions on the economic model, Chichilnisky (1995a,b) attempts to go beyond the necessity result of Werner (1987) by allowing half lines in indifference surfaces. (In fact, in all versions of her (1995a) paper available to us prior to June 1994, Chichilnisky treated only economies satisfying the closed gradient condition – see assumption C2.) It appears that even with limited arbitrage defined in terms of the increasing cone of Page and Wooders (1993) or its generalizations in Page and Wooders (1995, 1996a,b), Chichilnisky's condition of limited arbitrage has not been shown to be sufficient for the existence of an equilibrium. [27] Since Chichilnisky's global cone is, in general, contained in the Page-Wooders increasing cone, it does hold, however, that Chichilnisky's condition is necessary for existence of equilibrium. From Page and Wooders (1996a) it follows that Chichilnisky's condition is even necessary for the existence of a Pareto optimal allocation.

The current state of the literature is that the broadest economic model (without price-dependent preferences) under which no unbounded arbitrage is shown to be

[27] Even when defined in terms of the increasing cone, Chichilnisky's condition of limited arbitrage is not, in general, equivalent to the Page and Wooders condition of no unbounded arbitrage. However, in those cases where limited arbitrage and no unbounded arbitrage are equivalent, sufficiency follows from Page and Wooders (1993, 1995, 1996a,b).

sufficient for the existence of an equilibrium is given in Nielsen (1989). See also Page and Wooders (1995). Nielsen doesn't consider necessity. To the best of our knowledge Page and Wooders (1995, 1996a,b) provide the broadest economic model for which conditions limiting arbitrage are necessary as well as sufficient for the existence of an equilibrium, non-emptiness of the core, and existence of a Pareto optimal allocation. For the necessity results of Page and Wooders (1995, 1996a) one agent is permitted to have half lines in his indifference surfaces. It is an open question whether the Page and Wooders no unbounded condition can be modified to obtain necessary and sufficient conditions when more than one agent has half lines in indifference surfaces.

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Appendix

Proposition 1 (Utility functions satisfy assumptions C1).

(a) *Agent 1's utility function given in* (9) *is concave, strictly increasing, and twice continuously differentiable, with* $u_1(0) = 0$. *Moreover, agent 1's utility function satisfies the no-half-lines condition.*

(b) *Agent 2's expected utility function given in* (10) *is concave, non-decreasing, and twice continuously differentiable, with* $u_2(0) = 0$.

Proof. The conclusions of part (b) follow and are immediate. To prove part (a) we first calculate the derivatives:

$$\frac{\partial u_1(x_1, x_2)}{\partial x_1} = 1 - \frac{x_1 - x_2}{\sqrt{(x_1 - x_2)^2 + 4}} > 0, \tag{13}$$

$$\frac{\partial u_1(x_1, x_2)}{\partial x_2} = 1 - \frac{x_2 - x_1}{\sqrt{(x_1 - x_2)^2 + 4}} > 0, \tag{14}$$

$$\frac{\partial^2 u_1(x_1, x_2)}{\partial x_1^2} = \frac{\partial^2 u_1(x_1, x_2)}{\partial x_2^2} = \frac{-4}{\left((x_1 - x_2)^2 + 4\right)^{3/2}}. \tag{15}$$

Finally,

$$\frac{\partial^2 u_1(x_1, x_2)}{\partial x_1 \partial x_2} = \left| \frac{\partial^2 u_1(x_1, x_2)}{\partial x_1^2} \right|. \tag{16}$$

Therefore, $u_1(\cdot)$ is concave, strictly increasing and C^∞. Moreover, it follows from (13) and (14) that $u_1(\cdot)$ satisfies the no-half-lines condition. □Q.E.D.

Since both agents have concave utility functions, for each agent j we have

$$R\big(U_j(x)\big) = R\big(U_j(\bar{x})\big) \quad \text{for all } x \text{ and } \bar{x} \text{ in } R^L$$

(i.e. uniformity is satisfied).

Proposition 2 (The evaluation of limits).

$$\lim_{\lambda \to \infty} u_1((x_1, x_2) + \lambda(y_1, y_2))$$
$$= \begin{cases} \infty, & \text{if } y_1 > 0,\ y_2 > 0, \\ \textit{finite}, & \text{if } (y_1, y_2) = (1, 0) \textit{ or } (y_1, y_2) = (0, 1), \\ -\infty, & \textit{otherwise}. \end{cases}$$

Proof. It is clear that $\lim_{\lambda \to \infty} u_1((x_1, x_2) + \lambda(y_1, y_2)) = \infty$ if $y_1 > 0$ and $y_2 > 0$. Now let us suppose $(y_1, y_2) = (1, 0)$. The case $(y_1, y_2) = (0, 1)$ is similar. If $(y_1, y_2) = (1, 0)$, then $u_1((x_1, x_2) + \lambda(y_1, y_2)) = u_1(x_1 + \lambda, x_2)$. We observe that

$$u_1(x_1 + \lambda, x_2)$$
$$= x_1 + x_2 + 2 + \frac{\lambda - \sqrt{(x_1 - x_2 + \lambda)^2 + 4}}{\lambda + \sqrt{(x_1 - x_2 + \lambda)^2 + 4}} \left(\lambda + \sqrt{(x_1 - x_2 + \lambda)^2 + 4}\right)$$
$$= x_1 + x_2 + 2 + \frac{-(x_1 - x_2)^2 - 2\lambda(x_1 - x_2) - 4}{\lambda + \sqrt{(x_1 - x_2 + \lambda)^2 + 4}}.$$

Therefore, $\lim_{\lambda \to \infty} u_1(x_1 + \lambda, x_2) = 2x_2 + 2$.

Now let us suppose (y_1, y_2) is not contained in the non-negative orthant. We consider the following cases:

(i) If $y_1 + y_2 < 0$, then $u_1(x_1 + \lambda y_1, x_2 + \lambda y_2) \le x_1 + x_2 + 2 + \lambda(y_1 + y_2)$.

Thus,

$$\lim_{\lambda \to \infty} u_1(x_1 + \lambda y_1, x_2 + \lambda y_2) = -\infty.$$

(ii) If $y_1 + y_2 = 0$, then

$$u_1(x_1 + \lambda y_1, x_2 + \lambda y_2) = x_1 + x_2 + 2 + \lambda \cdot 0 - \sqrt{(x_1 - x_2 + 2\lambda y_1)^2 + 4}.$$

Thus, $\lim_{\lambda \to \infty} u_1(x_1 + \lambda y_1, x_2 + \lambda y_2) = -\infty$, since $y_1 \neq 0$.

(iii) Let $y_1 + y_2 > 0$. We have

$$\begin{aligned} & u_1(x_1 + \lambda y_1, x_2 + \lambda y_2) \\ & \quad = x_1 + x_2 + 2 \\ & \qquad + \left(\frac{\lambda(y_1 + y_2) - \sqrt{(x_1 + \lambda y_1 - x_2 - \lambda y_2)^2 + 4}}{\lambda(y_1 + y_2) + \sqrt{(x_1 + \lambda y_1 - x_2 - \lambda y_2)^2 + 4}} \right) \\ & \qquad \times \left(\lambda(y_1 + y_2) + \sqrt{(x_1 + \lambda y_1 - x_2 - \lambda y_2)^2 + 4} \right) \\ & \quad = x_1 + x_2 + 2 + \frac{\lambda^2(y_1 + y_2)^2 - (x_1 + \lambda y_1 - x_2 - \lambda y_2)^2 - 4}{\lambda(y_1 + y_2) + \sqrt{(x_1 + \lambda y_1 - x_2 - \lambda y_2)^2 + 4}}. \end{aligned}$$

Thus we have

$$\begin{aligned} & u_1(x_1 + \lambda y_1, x_2 + \lambda y_2) \\ & \quad = x_1 + x_2 + 2 + \frac{\lambda^2(y_1 + y_2)^2 - (x_1 + \lambda y_1 - x_2 - \lambda y_2)^2 - 4}{\lambda(y_1 + y_2) + \sqrt{(x_1 + \lambda y_1 - x_2 - \lambda y_2)^2 + 4}}. \end{aligned}$$

Simplifying the numerator in the last term in the expression above, we obtain:

$$\begin{aligned} & \lambda^2(y_1 + y_2)^2 - (x_1 + \lambda y_1 - x_2 - \lambda y_2)^2 - 4 \\ & \quad = 4\lambda^2 y_1 y_2 - (x_1 + x_2)^2 - 2\lambda(x_1 + x_2)(y_1 - y_2). \end{aligned}$$

Thus,

$$\begin{aligned} & u_1(x_1 + \lambda y_1, x_2 + \lambda y_2) \\ & \quad = x_1 + x_2 + 2 + \frac{4\lambda^2 y_1 y_2 - (x_1 + x_2)^2 - 2\lambda(x_1 + x_2)(y_1 - y_2)}{\lambda(y_1 + y_2) + \sqrt{(x_1 + \lambda y_1 - x_2 - \lambda y_2)^2 + 4}}. \end{aligned}$$

Since the λ^2 term dominates and since $4\lambda^2 y_1 y_2 < 0$, we conclude that

$$\lim_{\lambda \to \infty} u_1(x_1 + \lambda y_1, x_2 + \lambda y_2) = -\infty$$

for the case $y_1 + y_2 > 0$, (y_1, y_2) not contained in the non-negative orthant. □ Q.E.D.

Proof of Theorem 3. Given agent 2's utility function (see (10)) it is immediate that agent 2's global cone is given by

$$A_2(x) = \{y \in R^2 : y = (y_1, y_2) \text{ with } y_1 > 0\} \text{ for all } x \in R^2,$$

while agent 2's recession cone is given by

$$R(U_2(x)) = \{y \in R^2 : y = (y_1, y_2) \text{ with } y_1 \geq 0\} \text{ for all } x \in R^2.$$

Given agent 1's utility function (see (9)) it is also easy to see that agent 1's recession cone is given by

$$R(U_1(x)) = \{y \in R^2 : y = (y_1, y_2), \text{ with } y_1 \geq 0 \text{ and } y_2 \geq 0\}$$
$$\text{for all } x \in R^2.$$

Since agent 1's utility function satisfies the no-half-lines condition, it follows from Theorem 1 that

$$I_1(x) = \{y \in R^2 : y = (y_1, y_2) \neq 0, \text{ with } y_1 \geq 0 \text{ and } y_2 \geq 0\}$$
$$\text{for all } x \in R^2.$$

It only remains to show that agent 1's global cone is given by

$$A_1(x) = \{y \in R^2 : y = (y_1, y_2), \text{ with } y_1 > 0 \text{ and } y_2 > 0\} \text{ for all } x \in R^2.$$

This conclusion now follows from Proposition 2. Q.E.D.

Proposition 3 (Utility functions satisfy assumption C2).

(a) *Agent* 1'*s utility function given in* (9) *has indifference curves bounded from below.*

(b) *Agent* 2'*s utility function given in* (10) *has straight line indifference curves.*

Proof. The proof of part (b) is immediate. The conclusions of part (a) follow from the fact that agent 1's recession cone is the non-negative orthant. Q.E.D.

Proof of Theorem 2. It only remains to verify that the utility functions of agents 1 and 2 satisfy C3 (i.e. the norm of the gradient condition) and C4. It is immediate that agent 2's utility function satisfies C3 and C4. The proof that this is true for agent 1's utility function follows by inspection of the derivatives given in the proof of Proposition 1. Q.E.D.

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