Fields Institute Communications
Volume 23, 1999

Arbitrage with Price-Dependent Preferences: Equilibrium and Market Stability

Frank H. Page, Jr.
Department of Finance
University of Alabama
Tuscaloosa, Alabama 35487 USA
fpage@cba.ua.edu

Myrna H. Wooders
Department of Economics
University of Toronto
Toronto, Ontario, Canada M5S 1A1
mwooders@chass.utoronto.ca

Abstract. We introduce price-dependent preferences, a form of externalities, into a general equilibrium model of an economy with short sales. The condition of "no unbounded arbitrage," a similarity assumption on preferences and choice sets, is defined and shown to be sufficient for existence of an equilibrium, for balancedness of the economy, and for nonemptiness of the core. These conclusions rest on the result that no unbounded arbitrage is necessary and sufficient for compactness of the set of individually rational and feasible allocations. We also show that with additional assumptions on the model no unbounded arbitrage is necessary and sufficient for these properties to hold. Finally, we show that there is duality relationship between net trade conditions limiting arbitrage and price conditions limiting arbitrage. Since all conditions found in the literature fall into one of these two categories, our duality result allows us to clarify how the various conditions are related.

1 Introduction: arbitrage in models of economies

In an economy with short sales, unlike the standard Arrow-Debreu-MacKenzie general equilibrium model, there is no exogenous lower bound on consumption sets. Even when standard assumptions of convexity of preferences are satisfied, the potential for unbounded short sales poses problems for the existence of economic equilibrium. If agents have sufficiently dis-similar expectations about future

---

1991 *Mathematics Subject Classification.* Primary 90A09; Secondary 90A09, 90D12.

The authors are grateful to the Social Sciences and Humanities Research Council of Canada for support. We thank Erik Balder, Jonathan Borwein, Jean-Michel Grandmont, Birgit Grodal, Yakar Kannai, Donald Saari, and Norman Schofield for helpful comments. The authors are also grateful to two anonymous referees for helpful comments on a shortened version of this paper.

prices they may engage in arbitrarily large purchases of securities on one market for immediate resale on another market in order to profit from a price discrepancy; in other words, agents may engage in unbounded arbitrage. In a general equilibrium model, it is intuitvely clear that if the preferences for any two agents in the economy are sufficiently dissimilar, then there may be no set of prices at which the agents' willingnesses to trade can be equated. At any set of prices, one agent may be willing to sell an arbitrarily large amount of one commodity and another agent may be willing to buy an arbitrarily large amount. As discussed even by the classical economists, in order for **any** trade to take place there must be a mutual coincidence of wants. For the existence of an equilibrium, this coincidence of wants must be reconcilable by the trading process.

Using arbitrage conditions based on recession cones, a number of recent papers have addressed the question of existence of equilibrium in models of economies with short sales. Recession cones provide a very precise way of describing net trades that are arbitrarily large and utility non-decreasing. The recession cone of the preferred set of an agent describes the directions in which the agent will seek to make unbounded trades. If two agents have diametrically opposed preferences, then neither agent will *ever* cease to want to make yet further trades with the other agent; the potential for mutually advantageous trades cannot be exhausted. Hence there is no equilibrium. Since the possibility of unbounded utility increasing trades creates the possibly for nonexistence of equilibrium, the literature on models of economies with short sales has focused on conditions limiting arbitrage opportunities stated in terms of recession cones.

Arbitrage conditions characterizing competitive equilibrium have appeared in several economic contexts. These contexts include (a) temporary equilibrium models,[1] (b) general equilibrium models of asset markets,[2] and (c) general equilibrium models more broadly.[3] Many of these authors consider both necessary and sufficient conditions. Several papers also consider the core.[4].

In the context of asset market models, it is natural to have preferences depend on prices. Relative prices may well be taken as indicators of the desirability of assets and may convey information about future asset returns. In fact, in their models of asset markets, [**18**], [**16**], and [**35**] all allow price dependent preferences. In the case of the general equilibrium models noted above, however, with the exception of [**36**] and some brief remarks in [**51**] price dependent preferences are not permitted.

In this paper we extend the general equilibrium model of an economy to incorporate price-dependent preferences and price-dependent, convex, and possibly unbounded choice sets (or consumption sets). The fact that choice sets are arbitrary and price dependent introduces new complexities that do not appear in [**51**]. We establish that no unbounded arbitrage, an arbitrage condition introduced in [**35**], implies existence of equilibrium, compactness of the set of utility possibilities, "balancedness" of the economy, and nonemptiness of the core.[5] Since mutually compatible desires to trade can be equated by prices, we call economies satisfying

[1] For example, Green in [**13**] and Grandmont in [**14, 15**].

[2] For example, [**18**], [**16**], [**33**], [**34**], [**35**], [**37**], [**7**], [**25**], and [**24**].

[3] For example, [**9**], [**26**], [**36**], [**11**], [**51**], [**32**], [**6**], [**38, 39, 41**], [**8**], [**29, 30**], and [**42**].

[4] For example, [**25**] and [**41**].

[5] In fact, in the appendix, we show that no unbounded arbitrage is necessary and sufficient for the compactness of the set of individually rational and feasible *allocations*. Boundedness of the set of individually rational and feasible allocations underlies the existence of equilibrium in models of convex economies with unbounded short sales. Kousougeras in [**25**] shows that Page's condition in [**35**] of no unbounded arbitrage implies this boundedness.

our assumptions *reconcilable*. In "strictly reconcilable" economies,[6] including, for example, those in which agents have price dependent utility functions with indifference surfaces containing no half lines, no unbounded arbitrage is necessary and sufficient for these properties to hold.

In this paper, to limit the extent of arbitrage opportunities, we extend the condition of no unbounded arbitrage to a general equilibrium setting with price dependent preferences.[7] No unbounded arbitrage is a similarity assumption, ruling out situations where any two groups of agents have diametrically opposed preferences and choice sets. The condition implies that all opportunities for utility-increasing trades can be exhausted by trades bounded in size.

In addition to presenting results on the connections between no unbounded arbitrage, existence of equilibrium, compactness of utility possibilities, and nonemptiness of the core, we also present a careful discussion of the relationship between no unbounded arbitrage and other conditions limiting arbitrage found in the literature. Our discussion is not exhaustive. For another treatment, focused on general equilibrium models where preferences do not depend on prices, Dana, Le Van and Magnien in [**8**] provide an excellent discussion (see also [**29, 30, 31**]).

Conditions limiting arbitrage in the literature generally fall into two broad categories:

- conditions on net trades (e.g., [**9**], [**18**], [**35**], [**33**], [**32**], [**41, 38**], [**24**]);
- conditions on prices (e.g., [**13**], [**14, 15**], [**26**], [**16**], [**37, 36**], [**51**], [**6**]).

Here, using an elementary result due to ([**47**], Corollary 16.2.2), we show that there is duality relationship between net trade conditions limiting arbitrage and price conditions limiting arbitrage. Our duality result allows us to clarify how the various conditions are related. For example, we show that, in strictly reconcilable economies, Werner's condition [**51**] limiting arbitrage is a price condition dual to our condition of no unbounded arbitrage - a condition on net trades.

Necessary and sufficient conditions for nonemptiness of the core of a game have appeared in a number of papers (cf., [**5**], [**50**], [**19**], and [**23**]). The results of Bondareva and Shapley are immediately applicable to economies with quasi-linear utilities (where all agents have utility functions of the form $u_i(x, \xi) = \widehat{u}_i(x) + \xi$) while those of [**19**] apply to economies with coalition structures.[8] In the context of economies without short sales, it follows immediately from [**10**] that existence of equilibrium and nonemptiness of the core are equivalent. The study of the core in models with unbounded short sales was initiated in [**25**]. Page and Wooders in [**41, 38, 39**] demonstrate that no unbounded arbitrage is necessary and sufficient for nonemptiness of the core. In this paper we define the core relative to given prices; thus, in cooperative activities, individuals take prices as given.[9] Our main result for the core is that in a strictly reconcilable economy with price dependent preferences, no unbounded arbitrage is necessary and sufficient for nonemptiness.

[6] An economy is said to be strictly reconcilable if it satisfies two conditions. Roughly, these conditions are: the recession cones used to represent potential unbounded arbitrage opportunities must be invariant with respect to the endowments and prices, and indifference surfaces must contain no half lines.

[7] Page and Wooders in [**38, 39, 41**] use the same condition of no unbounded arbitrage as [**35**] but for the case where preferences do not depend on prices.

[8] Classes of economies corresponding to the class of games studied in [**23**] have not yet been characterized.

[9] This is reasonable in "large" economies where small coalitions are effective but cannot significantly influence aggregate outcomes, as in the literature on the f-core (cf., [**17**] and [**20**]).

But in fact we show more: no unbounded arbitrage is necessary and sufficient for the balancedness of the game derived from the economy. This implies nonemptiness of the "partnered core," introduced in [43] and [44, 45].[10]

Finally, before proceeding, we note that there have been some recent developments in the study of economies without price dependent preferences. These include [8], who introduce an arbitrage condition which implies existence of equilibrium but is weaker than compactness of the set of feasible and individually rational allocations. Their condition, however, implies compactness of the set of utility possibilities and existence of equilibrium. Another condition limiting arbitrage opportunities, called *inconsequential arbitrage*, was introduced in [42]. Inconsequential arbitrage is implied by compactness of the set of utility possibilities but it is unknown if the opposite implication holds. More recently, Le Van and Magnien [27] introduced a no arbitrage condition for economies with a continuum of agents. The arbitrage conditions of these papers have not been studied in situations with price-dependent preferences.

2 An economy with price-dependent preferences

Let $E = (X_i(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ denote an unbounded exchange *economy*. For each agent i there is given a set-valued mapping $p \to X_i(p)$ specifying, for each vector p of prices, the i^{th} agent's *choice set* $X_i(p) \subset \mathbf{R}^L$. The i^{th} agent's *endowment* is given by $\omega_i \in \mathbf{R}^L$. The i^{th} agent's preferences are specified via a *utility function*

$$u_i(\cdot,\cdot) : \mathbf{R}^L \times \mathbf{R}^L \longrightarrow \mathbf{R}.$$

We will maintain the following standard assumptions on the economy $E = (X_i(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ throughout the paper.

[A-1] For each agent $i = 1, ..., n$, the choice correspondence $p \to X_i(p)$ is zero-homogeneous (depends only on relative prices), convex-valued, lower hemicontinuous with a closed graph and the endowment is in the interior of the choice set $X_i(p)$ for each p.

- **[A-1.1]** $p \to X_i(p)$ is convex-valued and lower hemicontinuous with a closed graph,[11]
- **[A-1.2]** for each $p \in \mathbf{R}^L$ and $\lambda > 0$, $X_i(\lambda p) = X_i(p)$, and
- **[A-1.3]** for each $p \in \mathbf{R}^L$, $\omega_i \in intX_i(p)$, where "*int*" denotes interior.

[A-2] For each agent $i = 1, ..., n$, the utility function $u_i(\cdot,\cdot)$ is continuous, preferred sets are zero-homogeneous in prices, and preferences are locally non-satiated.

- **[A-2.1]** $u_i(\cdot,\cdot)$ is continuous on $\mathbf{R}^L \times \mathbf{R}^L$,
- **[A-2.2]** for each $(x,p) \in \mathbf{R}^L \times \mathbf{R}^L$, the preference set

$$P_i(x,p) = \{x' \in \mathbf{R}^L : u_i(x',p) > u_i(x,p)\}$$

is convex, and for $\lambda > 0$, $P_i(x,p) = P(x, \lambda p)$, and

- **[A-2.3]** for each $(x,p) \in \mathbf{R}^L \times \mathbf{R}^L$,

$$clP_i(x,p) = \{x' \in \mathbf{R}^L : u_i(x',p) \geq u_i(x,p)\}.$$

Remark 1. *On the consumption set and preferences.* Note that preferences are defined over all of $\mathbf{R}^L \times \mathbf{R}^L$ instead of over the consumption set $X_i(p)$ for each price p. While it is standard to define preferences only over consumption sets, in

[10] See also [1], [2], [3], [46], and [22] for related concepts, results and further discussion of partnership.

[11] $X_i(\cdot)$ is *lower hemicontinuous* if, for every open set $E \subset \mathbf{R}^L$, the set $\{p \in \mathbf{R}^L : X_i(p) \cap E \neq \emptyset\}$ is open. The graph of $p \to X_i(p)$ is given by $\Gamma(X_i(\cdot)) = \{(x,p) \in \mathbf{R}^L \times \mathbf{R}^L : x \in X_i(p)\}$.

our context there is no reason to do so. In fact, there may be legal or tax restrictions on consumption sets that make them different from the domain of preferences.

It follows from [A-1.2] and [A-2.2] that agents' preferences and choice sets are sensitive only to relative prices. Thus, without loss of generality we can restrict prices to be in the unit ball

$$\mathcal{B} := \{p \in \mathbf{R}^L : \|p\| \leq 1\}.$$

The local nonsatiation assumption [A-2.3] can be replaced by a weaker global nonsatiation assumption without changing the results in the paper. This can be accomplished via a technique introduced by Gale and Mas-Colell in [**12**] - see [**42**].

2.1 Individually rational and feasible allocations. The "at least as good as" set of the *ith* agent, $\widehat{P}_i(x_i, p)$, is the set of choices that are at least as desirable as his endowment. Formally,

$$\widehat{P}_i(x_i, p) = \{x' \in \mathbf{R}^L : u_i(x', p) \geq u_i(x_i, p)\}.$$

Given prices p and endowments $\omega = (\omega_1, ..., \omega_n)$, the set of all individually rational and feasible *allocations* for the economy $(X_i(\cdot), \omega_i, u_i(\cdot, \cdot))_{i=1}^n$ is given by

$$\begin{aligned} F(\omega, p) := \{(x_1, ..., x_n) \in \Pi_{i=1}^n X_i(p) : \sum_{i=1}^n x_i = \sum_{i=1}^n \omega_i \\ \text{and for each } i,\ x_i \in \widehat{P}_i(\omega_i, p)\}. \end{aligned} \tag{1}$$

Note that given [A-1.2] and [A-2.2], for all $p \in \mathbf{R}^L$ and for all $\lambda > 0$, $F(\omega, p) = F(\omega, \lambda p)$.

Let $\Gamma(F(\omega, \cdot))$ denote the graph of the individually rational allocation mapping $p \to F(\omega, p)$ where $p \in \partial\mathcal{B}$ and ∂ denotes the boundary, that is, $\partial\mathcal{B} := \{p' \in \mathbf{R}^L : \|p'\| = 1\}$. Thus,

$$\begin{aligned} \Gamma(F(\omega, \cdot)) = \{(x_1, ..., x_n, p) \in \mathbf{R}^L \times \cdots \times \mathbf{R}^L \times \partial B : \\ (x_1, ..., x_n) \in F(\omega, p)\ \} \end{aligned} \tag{2}$$

The corresponding set of *utility possibilities* is given by

$$\begin{aligned} U(F(\omega, \cdot)) = \{(u_1, ..., u_n) \in \mathbf{R}^n : \text{for some } (x_1, ..., x_n, p) \in \Gamma(F(\omega, \cdot)), \\ u_i = u_i(x_i, p) \text{ for each } i = 1, ..., n\}. \end{aligned}$$

2.2 Equilibrium. Given prices $p \in \mathbf{R}^L$, the cost of a consumption vector $x = (x_1, ..., x_L)$ is

$$\langle x, p\rangle = \sum_{\ell=1}^L p_\ell \cdot x_\ell.$$

Given prices p and endowment ω_i, the *budget set of the* i^{th} agent is defined by

$$B_i(\omega_i, p) = \{x \in \mathbf{R}^L : \langle x, p\rangle \leq \langle \omega_i, p\rangle\}.$$

[D-1] *Equilibrium.* An *equilibrium* for the economy $E = (X_i(\cdot), \omega_i, u_i(\cdot, \cdot))_{i=1}^n$ is an $(n+1)$-tuple of vectors $(x_1^*, ..., x_n^*, p^*)$ such that
$(x_1^*, ..., x_n^*) \in F(\omega, p^*)$ (the allocation is feasible);
$p^* \in \mathcal{B}\backslash\{0\}$ (prices are in the unit ball and not all prices are zero); and
for each agent i,

$\langle x_i^*, p^* \rangle = \langle \omega_i, p^* \rangle$ (budget constraints are satisfied), and $P_i(x_i^*, p^*) \cap X_i(p^*) \cap B_i(\omega_i, p^*) = \emptyset$ (there are no affordable preferred net trades).

2.3 Balanced economies and the core. Let C be a nonempty subset of $N = \{1, ..., n\}$, called a *coalition*. A collection of choice vectors $(z_i : i \in C)$ is a $C-$*allocation* given prices $p \in \mathbf{R}^L$ if, for each $i \in C$, it holds that $z_i \in \widehat{P}_i(\omega_i, p) \cap X_i(p)$ and $\sum_{i \in C} z_i = \sum_{i \in C} \omega_i$.

Let $F_C(\omega, p)$ denote the set of all $C-$allocations given prices p. Also, given prices p, for each nonempty coalition C define the set

$$V(C;p) = \{(u_1, ..., u_n) \in \mathbf{R}^n : \text{for some } C\text{-allocation} \\ (x_i : i \in C) \in F_C(\omega, p),\ u_i(x_i, p) \geq u_i \ \text{ for each } i \in C\ \}. \tag{3}$$

Expression (3) defines a set-valued mapping

$$V(\cdot;p) : 2^N \backslash \emptyset \to \text{ subsets of } \mathbf{R}^n.$$

If for each price vector $p \in \mathbf{R}^L$ and coalition C, $V(C;p)$ is nonempty and closed, and if $V(C;p)$ is bounded from above, then the pair $(N, V(\cdot;p))$ defines a game in characteristic function form (in the sense of [**48**]) corresponding to the economy $E = (X_i(\cdot), \omega_i, u_i(\cdot, \cdot))_{i=1}^n$ and given prices $p \in \mathbf{R}^L$.

[D-2] *Balanced economies.* Let β denote a balanced collection of subsets of N with balancing weights δ_C for $C \in \beta$ and let $\beta(i) = \{C \in \beta : i \text{ is contained in } C\}$. Thus, $\delta_C \geq 0$ for all $C \in \beta$ and for each $i \in M$

$$\sum_{C \in \beta(i)} \delta_C = 1\ .$$

Given a price vector p the economy E is *well-defined and balanced relative to* p if $(N, V(\cdot, p))$ is a well-defined game and for every balanced collection β of subsets of N it holds that

$$\bigcap_{C \in \beta} V(C;p) \subseteq V(N;p).$$

The economy is *well-defined and balanced* if for each $p \in \partial\mathcal{B}$ the economy is well-defined and balanced relative to p.

[D-3] *The core.* An allocation $(x_1^*, ..., x_n^*)$ is in the *core of the economy E given prices p* if there does not exist a coalition $C \subset \{1, ..., n\}$ of agents and an C-allocation $(z_i : i \in C) \in F_C(w, p)$ at p such that for all $i \in C$

$$u_i(z_i, p) > u_i(x_i, p).$$

We say that an economy *has a nonempty core* if for some $p \in \partial\mathcal{B}$ there exists an allocation $(x_1, ..., x_n) \in F(\omega, p)$ in the core of the economy given prices p.

Remark 2. Note that in the above definition of the core coalitions take prices as given. In large economies, if coalitions are relatively small and take prices as given, this assumption may be natural. (See [**17**] and [**20**] for more motivation for our approach to the core.) It would be interesting to consider a concept of the core where coalitions take into account the effect of their behaviour on economic aggregates, including prices. This is beyond the scope of the present paper.

Remark 3. It is immediate from Scarf's Theorem [**48**] that a balanced economy has a nonempty core. From [**44**] and [**38**] it follows that a balanced economy has a nonempty partnered core.

Remark 4. Note that the definition of the core above, standard in game theory, is different from the standard economic defintion, where it is only required that

$$\begin{array}{c} u_i(z_i,p) \geq u_i(x_i,p) \text{ for all } i \in C \text{ and} \\ \text{for at least one } i \text{ in } C,\ u_i(z_i,p) > u_i(x_i,p). \end{array} \quad (*)$$

In general these two definitions of the core are not equivalent. It is obvious that if a subset of agents can all be better off in an improving coalition then $(*)$ is satisfied. Thus, the core defined by $(*)$ contains the core defined by [D-3].

3 No unbounded arbitrage

In order to describe unbounded arbitrage for individual agents and for the economy, we will utilize the notions of recession cone and increasing cone.[12]

3.1 Recession cones, lineality spaces, and positive dual cones.

[**D-4**] *Directions of recession and recession cones.*
Given any convex subset S of $\mathbf{R}^L$, we say that $y \in \mathbf{R}^L$ is a *direction of recession* for S if $x + \lambda y \in S$ for all $\lambda \geq 0$ and for all $x \in S$. We shall denote by $\mathcal{R}(S)$ the set of all recession directions of S, and we shall refer to $\mathcal{R}(S)$ as the *recession cone* corresponding to S.

It is easy to show that $\mathcal{R}(S)$ is a convex cone containing the origin. Moreover, if S is closed then $\mathcal{R}(S)$ is closed and $x + \lambda y \in S$ for all $\lambda \geq 0$ and *some* $x \in S$ implies that $y \in \mathcal{R}(S)$.

[**D-5**] *Lineality spaces.*

Given any convex subset S of $\mathbf{R}^L$, we say that $y \in \mathbf{R}^L$ is a *direction of lineality* for S if $x + \lambda y \in S$ for all $\lambda \in \mathbf{R}$. We shall denote by $L(S)$ the set of all lineality directions of S. Given the definition of the recession cone, it is easy to see that for S convex, the *lineality space* $L(S)$ of S is given by

$$L(S) = -\mathcal{R}(S) \bigcap \mathcal{R}(S).$$

[**D-6**] *Positive dual cones.*

Given any convex subset S of $\mathbf{R}^L$, the positive dual cone corresponding to S is given by

$$\mathcal{D}(S) = \{p \in \mathbf{R}^L : \langle y,p \rangle > 0 \text{ for all } y \in S \backslash \{0\}\}.$$

If we think of the set S as a set of net trades and the corresponding positive dual cone $\mathcal{D}(S)$ as a set of prices, then any price vector $p \in \mathcal{D}(S)$ assigns a *positive* value to any nonzero net trade vector $y \in S$.

---

[12] Here we use the extended definition of the increasing cone, introduced in [**40**]. The increasing cone was first used in an asset market model to show that Page's [**37**] price conditon limiting arbitrage is necessary and sufficient for the existence of equilibrium (see also [**33**]). In [**41, 39**], the increasing cone is used to show that no unbounded arbitrage is necessary and sufficient for existence, compactness of utility possibilities, and nonemptiness of the core.

3.2 Arbitrage cones and strict increasing cones. Define

$$\mathcal{A}_i(x,p) := \mathcal{R}\left(X_i(p) \bigcap \widehat{P}_i(x,p)\right) \tag{4}$$

We shall refer to the recession cone $\mathcal{A}_i(x,p)$ as the i^{th} agent's *arbitrage cone* given prices p and choice vector x. Note that a vector of net trades y is contained in $\mathcal{A}_i(x,p)$ if and only if

1. $x + \lambda y \in X_i(p)$ for all $\lambda \geq 0$ and
2. $u_i(x + \lambda y, p) \geq u_i(x,p)$ for all $\lambda \geq 0$.

A cone closely related to the arbitrage cone is the increasing cone, $\mathcal{I}_i(x,p)$, defined as follows:

$$\mathcal{I}_i(x,p) := \left\{ \begin{array}{c} y \in \mathcal{A}_i(x,p) : \forall\lambda \geq 0, \exists\lambda' > \lambda \text{ such that} \\ \\ u_i(x + \lambda' y, p) > u_i(x + \lambda y, p) \end{array} \right\} \tag{5}$$

We shall refer to the positive dual cone $\mathcal{D}(\mathcal{A}_i(x,p))$ as the i^{th} agent's *no arbitrage price cone.*

3.3 Individual arbitrage opportunities. Define

$$H(p) = \{y \in \mathbf{R}^L : \langle y,p \rangle \leq 0\}.$$

Given prices p, $H(p)$ is the i^{th} agent's set of net trades that are at most costless. If $\langle y,p \rangle = 0$ then net trades y are costless to make and if $\langle y,p \rangle < 0$ then the net trades y generate a positive return.

[D-7] *Individual arbitrage opportunities.*

[D-7.1] From the perspective of the i^{th} agent, a nonzero vector of net trades $y \in \mathbf{R}^L$ represents *an opportunity for unbounded arbitrage given prices* $p \in \mathbf{R}^L$ *and choice vector* $x_i \in X_i(p)$ if and only if

$$y \in H(p) \cap \mathcal{A}_i(x_i,p).$$

[D-7.2] From the perspective of the i^{th} agent, a price vector $p \in \mathbf{R}^L$ *does not admit unbounded arbitrage at choice vector* x if and only if

$$p \in \mathcal{D}(\mathcal{A}_i(x_i,p)).$$

Thus, a vector of net trades y is an unbounded arbitrage for the i^{th} agent given prices p and choice vector $x_i \in X_i(p)$ if and only if the net trades y are at most costless to make and y is contained in the i^{th} agent's arbitrage cone $\mathcal{A}_i(x_i,p)$. Given a choice vector x_i, a price vector p admits no unbounded arbitrage if and only if p is a fixed point of the mapping $p \to \mathcal{D}(\mathcal{A}_i(x_i,p))$.

Note that if $y \in H(p) \cap \mathcal{A}_i(x,p)$ then $y \in H(\lambda p) \cap \mathcal{A}_i(x,\lambda p)$ for all $\lambda > 0$. Thus, arbitrages are determined by *relative* prices. Moreover, if $p \in \mathcal{D}(\mathcal{A}_i(x,p))$ then $\lambda p \in \mathcal{D}(\mathcal{A}_i(x,\lambda p))$ for all $\lambda > 0$.

3.4 No unbounded arbitrage. Given endowments and given any nonzero vector of prices, no unbounded arbitrage rules out the possibility that any agent can find a *mutually compatible* trading partner (or group of trading partners) with whom to engage in unbounded and utility nondecreasing trade.

[D-8] *No unbounded arbitrage.* Given endowments $(\omega_1, ..., \omega_n)$, the economy satisfies *no unbounded arbitrage* if and only if

$$\begin{array}{c} \text{for each } p \in \mathcal{B}\backslash\{0\} \text{ if } \sum_{i=1}^n y_i = 0 \text{ and } y_i \in \mathcal{A}_i(\omega_i,p) \text{ for all } i, \\ \text{then } y_i = 0 \text{ for all } i. \end{array} \tag{6}$$

In words, if for any price vector $p \in \mathcal{B}\backslash\{0\}$, there is a set of mutually compatible net trades (that is, if the y_i, $i = 1, ..., n$ are such that $\sum_{i=1}^{n} y_i = 0$) each of which is feasible and possibly utility increasing on any scale (that is, each y_i, $i = 1, ..., n$ is such that $y_i \in \mathcal{A}_i(\omega_i, p)$), then these mutually compatible net trades must be trivial (that is, $y_i = 0$ for all i).

It is important to note that no unbounded arbitrage does not rule out mutually compatible *bounded* arbitrages. Thus, even if no unbounded arbitrage is satisfied there may still be present in the economy arbitrages which can be exhausted via bounded trades. (See [**34**] and [**33**] for a detailed discussion of bounded and unbounded arbitrage in asset markets.)

In an investigation of the irreversibility assumption in general equilibrium models with production and with choice sets and preferences *independent* of prices, Debreu in [**9**] introduces a recession direction condition similar to our condition of no unbounded arbitrage, but applied to choice sets only. Translating Debreu's condition to our model, we have,

$$\text{for each } p \in \mathcal{B}\backslash\{0\} \text{ if } \sum_{i=1}^{n} y_i = 0 \text{ and } y_i \in \mathcal{R}(X_i(p)) \text{ for all } i,$$
$$\text{then } y_i = 0 \text{ for all } i.$$

Debreu's condition is essentially an assumption concerning the degree of similarity of agents' choice sets, $X_i(p)$. Note that, within the context of our model, Debreu's condition implies our condition.

In the analysis to follow, we will show that no unbounded arbitrage (6) is sufficient for the existence of equilibrium, compactness of the set of utility possibilities and nonemptiness of the core. Moreover, since we prove our results within the context of a general equilibrium model in which agents' preferences are price-dependent and consumption sets are price dependent and possibly unbounded, our model subsumes many of the models in the existing literature. Under additional conditions on the economy, we will also show that no unbounded arbitrage is necessary and sufficient for the existence of equilibrium, the compactness of the utility possibilities, and nonemptiness of the core. The additional conditions we will need are:

[A-3] *Extreme desirability.* Given any price vector $p \in \mathcal{B}\backslash\{0\}$ and choice vector $x \in \mathbf{R}^L$, each nonzero unbounded arbitrage opportunity $y \in \mathcal{A}_i(x,p)$ is eventually utility increasing. Thus,

$$\text{for all } p \in \mathcal{B}\backslash\{0\} \text{ and } x \in \mathbf{R}^L,$$
$$\mathcal{A}_i(x,p)\backslash\{0\} = \mathcal{I}_i(x,p)$$

[A-4] *Uniformity of arbitrage cones.* Each agent's utility function and choice correspondence is such that the set of unbounded arbitrage opportunities is invariant with respect to the price vector $p \in \mathcal{B}\backslash\{0\}$ and the choice vector $x \in \mathbf{R}^L$. Thus,

$$\mathcal{A}_i(x,p) = \mathcal{A}_i(x',p')$$
$$\text{for all } p \text{ and } p' \text{ in } \mathcal{B}\backslash\{0\} \text{ and for all } x \text{ and } x' \text{ in } \mathbf{R}^L.$$

We shall refer to an economy in which agents' utility functions satisfy assumptions [A-2]-[A-4] as a *strictly reconcilable economy.* (See [**41**].)

In Section 4 below, we construct an example of a strictly reconcilable asset exchange economy. Besides providing an example, this allows us to more easily discuss the conditions limiting arbitrage introduced by [**18**] and [**16**].

3.5 Characterizing no unbounded arbitrage: the duality of conditions on trades and conditions on prices. We begin by stating a general result on the connection between conditions on net trades limiting arbitrage (such as no unbounded arbitrage (6)) and conditions on prices limiting arbitrage. Theorem 1 below is obtained by specializing Corollary 16.2.2 in [**47**] to cover the exchange economy developed here.

From the perspective of unbounded arbitrage, the lineality space corresponding to the arbitrage cone $\mathcal{A}_i(x_i, p)$ is particularly noteworthy. Consider a vector of net trades $y \in L(\mathcal{A}_i(x_i, p))$. For this vector of net trades, $x_i + \lambda y \in X_i(p)$ for all $\lambda \in \mathbf{R}$ and more importantly, $u_i(x_i + \lambda y, p) = u_i(x_i, p)$ for all $\lambda \in \mathbf{R}$. Thus, given any unbounded arbitrage opportunity $y \in L(\mathcal{A}_i(x_i, p))$, trading in y direction or the $-y$ direction is feasible and utility constant. To save writing we will sometimes let

$$L(\mathcal{A}_i(x_i, p)) := \mathcal{L}_i(x_i, p) \tag{7}$$

Theorem ([47]) *The duality between conditions on trades and conditions on prices.* Let $E = (X_i(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ be an economy satisfying [A-1]-[A-2]. For any nonzero price vector $p \in \mathbf{R}^L$ and any $n-tuple$ of choice vectors $(x_1, \ldots, x_n) \in \mathbf{R}^L \times \cdots \times \mathbf{R}^L$ such that for all i, $\mathcal{A}_i(x_i, p) \backslash \mathcal{L}_i(x_i, p) \neq \emptyset$, the following statements are equivalent:

1. (*a condition on net trades*)

$$\sum_{i=1}^{n} y_i = 0 \text{ and } y_i \in \mathcal{A}_i(x_i, p) \text{ for all } i = 1, \ldots, n,$$

$$\text{then } y_i \in \mathcal{L}_i(x_i, p) \text{ for all } i = 1, \ldots, n.$$

2. (a condition on prices)

$$\cap_i \mathcal{D}(\mathcal{A}_i(x_i, p) \backslash \mathcal{L}_i(x_i, p)) \neq \emptyset.$$

Corollary to Rockafellar's Theorem *Characterization of no unbounded arbitrage as the nonempty intersection of the no arbitrage price cones.* Let $E = (X_i(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ be an economy satisfying [A-1]-[A-2]. For any nonzero price vector $p \in \mathbf{R}^L$ and any $n-tuple$ of choice vectors $(x_1, \ldots, x_n) \in \mathbf{R}^L \times \cdots \times \mathbf{R}^L$ such that for all i, $\mathcal{L}_i(x_i, p) = \{0\}$ and $\mathcal{A}_i(x_i, p) \backslash \mathcal{L}_i(x_i, p) \neq \emptyset$, the following statements are equivalent:

1. If $\sum_{i=1}^n y_i = 0$ and $y_i \in \mathcal{A}_i(x_i, p)$ for all i,then $y_i = 0$ for all i.
2. $\cap_i \mathcal{D}(\mathcal{A}_i(x_i, p)) \neq \emptyset$.

If in economy E agents utility functions are such that indifference surfaces contain no half lines, then for any nonzero price vector $p \in \mathbf{R}^L$ and any n-*tuple* of choice vectors $(x_1, \ldots, x_n) \in \mathbf{R}^L \times \cdots \times \mathbf{R}^L$ it will be true that for all i $\mathcal{L}_i(x_i, p) = \{0\}$. It follows from Theorem 1 in [**30**], that the no half lines condition is equivalent to [A-3] (extreme desirability). Thus, in view of the Corollary, in economies satisfying the no half lines condition, no unbounded arbitrage (6) can be stated equivalently as follows:

$$\text{for each } p \in \mathcal{B}\backslash\{0\}, \; \bigcap_i \mathcal{D}(\mathcal{A}_i(\omega_i, p)) \neq \emptyset. \tag{8}$$

Geometrically, $\cap_i \mathcal{D}(\mathcal{A}_i(\omega_i, p)) \neq \emptyset$ if and only if all of the arbitrage cones $\mathcal{A}_i(\omega_i, p)$, $i = 1, 2..., n$ lie *strictly* on one side of a hyperplane through the origin.

4 The main theorems

In this subsection we state our main results. We note that parts of some of the following Theorems are already proven in the literature for economies with unbounded short sales and without price-dependent preferences. We discuss the related results in a later section.

Theorem 1 *No unbounded arbitrage is sufficient for existence of equilibrium, compactness of the set of utility possibilities, balancedness of the economy, and nonemptiness of the core.* Let $E = (X_i(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ be an economy satisfying [A-1]-[A-2]. No unbounded arbitrage (6) implies the following:

1. E has an equilibrium.
2. The set of utility possibilities $U(F(\omega,\cdot))$ is compact.
3. E is well-defined and balanced.
4. E has a nonempty core.

In the context of a strictly reconcilable economy, existence of a competitive equilibrium, compactness of the set of utility possibilities, balancedness of the economy, and nonemptiness of the core are each necessary and sufficient for no unbounded arbitrage.[13]

Theorem 2 *In a strictly reconcilable economy no unbounded arbitrage is necessary and sufficient for the existence of equilibrium, compactness of the set of utility possibilities, balancedness of the economy, and nonemptiness of the core.* Let $E = (X_i(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ be an economy satisfying [A-1]-[A-4]. Then the following statements are equivalent:

1. E satisfies no unbounded arbitrage.
2. For each $p \in \mathcal{B}\backslash\{0\}$, $\cap_i \mathcal{D}\left(\mathcal{A}_i\left(\omega_i, p\right)\right) \neq \emptyset$.
3. E has an equilibrium.
4. The set of utility possibilities $U(F(\omega,\cdot))$ is compact.
5. E is well-defined and balanced.
6. E has a nonempty core.

Our next result shows that with fewer assumptions on the economy E no unbounded arbitrage is necessary for the existence of equilibrium.

Theorem 3 *A necessary conditions for the existence of equilibrium.* Let $E = (X_i(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ be an economy satisfying [A-1]-[A-3]. If $(x_1^*, ..., x_n^*, p^*)$ is an equilibrium for the economy E then the following conditions hold:

1. If $\sum_{i=1}^n y_i = 0$ and $y_i \in \mathcal{A}_i\left(x_i^*, p^*\right)$ for all i, then $y_i = 0$ for all i.
2. $p^* \in \cap_i \mathcal{D}\left(\mathcal{A}_i\left(x_i^*, p^*\right)\right)$.

Consider now the following weakening of assumption [A-4] concerning uniformity of arbitrage cones.

[A-5] *Uniformity of arbitrage cones with respect to choice.* Each agent's utility function and choice correspondence are such that for each price vector $p \in$

[13] In fact all the results for strictly reconcilable economies hold if only $n-1$ agents have strictly reconcilable preferences - see **[38]** .

$\mathcal{B}\backslash\{0\}$, the set of potential unbounded arbitrage opportunities is invariant with respect to the choice vector $x \in \mathbf{R}^L$. Thus,

$$\begin{gathered}\text{for } p \in \mathcal{B}\backslash\{0\} \\ \mathcal{A}_i(x,p) = \mathcal{A}_i(x',p) \\ \text{and for all } x \text{ and } x' \text{ in } \mathbf{R}^L.\end{gathered}$$

If, in addition to assuming [A-1]-[A-3], we assume [A-5], then we have the following Corollary to Theorem 3.

Corollary 1 *A stronger necessary condition for the existence of equilibrium.* Let $E = (X_i(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ be an economy satisfying [A-1]-[A-3] and [A-5]. If $(x_1^*, ..., x_n^*, p^*)$ is an equilibrium for the economy E then the following conditions hold:

1. If $\sum_{i=1}^n y_i = 0$ and $y_i \in \mathcal{A}_i(\omega_i, p^*)$ for all i,then $y_i = 0$ for all i.
2. $p^* \in \cap_i \mathcal{D}(\mathcal{A}_i(\omega_i, p^*))$.

Note that in the Corollary we have replaced $\mathcal{A}_i(x_i^*, p^*)$ with $\mathcal{A}_i(\omega_i, p^*)$. In a temporary equilibrium setting, the condition $p^* \in \cap_i \mathcal{D}(\mathcal{A}_i(\omega_i, p^*))$ is equivalent to the necessary condition given by [**13**]. (See also [**15**] and, for further discussion, [**16**], [**33**] and [**34**]).

5 Conditions limiting arbitrage in the literature

In this section we describe some of the other conditions limiting arbitrage found in the literature, and we relate these conditions to the condition of no unbounded arbitrage presented here. Our coverage is not exhaustive. We focus on the conditions of [**18**], [**16**], and [**51**]. For a more complete treatment of conditions in general equilibrium models, see [**8**].

In order to sensibly discuss the conditions limiting arbitrage introduced by [**18**] and [**16**], we begin by constructing an asset markets example.

5.1 An asset markets example of a strictly reconcilable economy. Consider an asset exchange economy $E = (X_i(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ in which n agents have expected utility over portfolios given by

$$u_i(x,p) = \int_{\mathbf{R}_+^L} v_i(\langle x, r\rangle) d\mu_i(dr \mid p)$$

and portfolio choice sets $X_i(p)$ for each price vector $p \in \mathbf{R}^L$. Here

- $\mathbf{R}_+^L$ denotes the nonnegative orthant,
- $v_i(\cdot) : \mathbf{R} \to \mathbf{R}$ is the i^{th} agent's utility function,
- $\mu_i(\cdot \mid p)$ is a probability measure defined on the Borel subsets $B(\mathbf{R}_+^L)$ of $\mathbf{R}_+^L$ and represents the i^{th} agent's probability beliefs concerning asset returns given prices p,
- $\langle x, r\rangle := \sum_{i=1}^L x_\ell \cdot r_\ell$ where, for $\ell = 1, ..., L$,
  x_ℓ is the number of shares of asset ℓ held in portfolio $x \in \mathbf{R}^L$, and
  r_ℓ is the gross return on one share of asset ℓ.

Let $S[\mu_i(\cdot \mid p)]$ denote the support of the probability measure $\mu_i(\cdot \mid p)$ and let

$$s_i(+) := \lim_{c \to \infty} \tfrac{dv_i(c)}{dc} \text{ and}$$

$$s_i(-) := \lim_{c \to -\infty} \tfrac{dv_i(c)}{dc}.$$

Thus, $s_i(+)$ and $s_i(-)$ are the asymptotic derivatives of the i^{th} agent's utility function.

Now consider the following list of assumptions concerning each agent's utility function and probability beliefs:

[a-1]: the choice correspondence $p \to X_i(p)$ satisfies [A-1],

[a-2]: $v_i(\cdot)$ is concave, increasing, and everywhere differentiable with $s_i(+) < s_i(-)$,

[a-3]: the mapping $p \to \mu_i(\cdot \mid p)$ is weakly continuous and positively homogeneous of degree zero,[14]

[a-4]: $S[\mu_i(\cdot \mid p)]$ is bounded,

[a-5]: if, given asset prices p, net trades y_i generate a nonnegative gross return with probability 1 (i.e., if y_i is such that $\mu_i(\{r \in \mathbf{R}^L_+ : \langle y_i, r\rangle \geq 0\} \mid p) = 1$), then $y_i \in \mathcal{R}(X_i(p))$,

[a-6]: assets are desirable, that is, $\mathbf{R}^L_{++} \cap kc\{S[\mu_i(\cdot \mid p)]\} \neq \emptyset$ (here, $\mathbf{R}^L_{++}$ denotes the positive orthant of $\mathbf{R}^L$and $kc\{S[\mu_i(\cdot \mid p)]\}$ denotes the convex cone generated by the support of the i^{th} agent's probability beliefs concerning asset returns),

[a-7]: there are no redundant assets, that is,

$$int\, kc\{S[\mu_i(\cdot \mid p)]\} \neq \emptyset,$$

("*int*" denotes interior),

[a-8]: the i^{th} agent is sufficiently risk averse. That is, either

$$\text{(a) for all } c \in \mathbf{R},\ 0 \leq s_i(+) < \frac{dv_i(c)}{dc} < s_i(-) \leq \infty,$$

or

$$\text{(b) } s_i(+) = 0 \text{ and/or } s_i(-) = +\infty,$$

[a-9]: the closed, convex cone $kc\{S[\mu_i(\cdot \mid p)]\}$ generated by the i^{th} agent's probability beliefs concerning asset returns is invariant with respect to $p \in \mathcal{B}\backslash\{0\}$.

If assumptions [a-1]-[a-6] hold, then the asset exchange economy E satisfies [A-1] and [A-2]. If assumptions [a-1]-[a-5] and assumptions [a-7] - [a-9], hold then the economy is strictly reconcilable (i.e., satisfies [A-1]-[A-4] - see [**33, 34, 35, 37**]). Note that since $v_i(\cdot)$ is concave, assumption [A-5] - uniformity of arbitrage cones with respect to (portfolio) choice - holds automatically.

5.2 Asset market models: the no unbounded arbitrage conditions of [18] and [16]. Having introduced the basic ingredients used in general equilibrium models of asset markets, we are now in a position to state the conditions limiting arbitrage introduced by [**18**] and [**16**].

Hart's asset exchange model is essentially identical to the model given in Section 4 under assumptions [a-1]- [a-6]. Hart in fact assumes that each agent's choice set is a closed, convex subset of $\mathbf{R}^L$ independent of asset prices. Thus, in Hart, $\mathcal{A}_i(\omega_i, p) = \mathcal{R}\left(X_i \cap \widehat{P}_i(\omega_i, p)\right)$. As shown in [**18**], the following no unbounded

[14]That is, for any bounded continuous function $f(\cdot) : \mathbf{R}^L_+ \to \mathbf{R}$, the function $p \to \int_{\mathbf{R}^L_+} f(r) d\mu_i(dr \mid p)$ is continuous, and for each Borel subset $E \in B(\mathbf{R}^L_+)$, $\mu_i(E \mid p) = \mu_i(E \mid \lambda p)$ for all $\lambda > 0$.

arbitrage condition, suffices for existence of equilibrium:

$$\begin{gathered}\text{for each } p' \in \mathcal{B}\backslash\{0\}, \text{ if } \sum_{i=1}^{n} y_i = 0 \text{ and } y_i \in \mathcal{A}_i(\omega_i, p') \text{ for all } i, \\ \text{then, for all } i \text{ and } p \text{ in some neighborhood of } p', \\ \mu_i(\{r \in \mathbf{R}_+^L : \langle y_i, r\rangle = 0\} \mid p) = 1.\end{gathered} \tag{9}$$

Note that because agents' utility functions $v_i(\cdot)$ are concave, [A-5] holds automatically. Thus,

$$\mathcal{A}_i(\omega_i, p) = \mathcal{A}_i(x_i, p) \text{ for all } x_i \in \mathbf{R}^L.$$

Note also that [a-5] and [a-6] together imply that for any nonzero price vector $p \in \mathbf{R}^L$ and any $n-tuple$ of choice vectors $(x_1, \ldots, x_n) \in \mathbf{R}^L \times \cdots \times \mathbf{R}^L$, $\mathcal{A}_i(x_i, p)\backslash\mathcal{L}_i(x_i, p) \neq \emptyset$.

Finally, note that if, given asset prices p, net trades y_i generate zero gross return with probability 1 (i.e., if y_i is such that $\mu_i(\{r \in \mathbf{R}_+^L : \langle y_i, r\rangle = 0\} \mid p) = 1$), then $y_i \in \mathcal{L}_i(x_i, p)$ for all $x_i \in \mathbf{R}^L$.

By Proposition 5.1 in [**35**], if there are no redundant assets (i.e., if [a-7] holds), then Hart's condition reduces to

$$\begin{gathered}\text{for each } p \in \mathcal{B}\backslash\{0\}, \text{ if } \sum_{i=1}^{n} y_i = 0 \text{ and } y_i \in \mathcal{A}_i(\omega_i, p) \text{ for all } i, \\ \text{then } y_i = 0 \text{ for all } i.\end{gathered}$$

Thus, under [a-7] Hart's condition reduces to our condition of no unbounded arbitrage (6).

If in addition to satisfying [a-1]-[a-5] and [a-7], the model also satisfies and [a-8] (sufficient risk aversion), then by Corollary 2 (see also [**33**], Proposition 5), Hart's condition is equivalent to

$$\text{for each } p \in \mathcal{B}\backslash\{0\}, \bigcap_i \mathcal{D}(\mathcal{A}_i(x_i, p)) \neq \emptyset. \tag{10}$$

Moreover, under [a-1]-[a-5] and [a-7]-[a-8], it follows from Lemma 5 in [**33**] that $\mathcal{A}_i(x_i, p)\backslash\{0\} = \mathcal{I}_i(x_i, p)$ - i.e., [A-3] holds. If [a-1]-[a-5] and [a-7] and [a-8](b) hold, then (10) reduces to Hammond's overlapping expectations condition given by

$$\text{for each } p \in \mathcal{B}\backslash\{0\} \bigcap_i int\, kc\{S[\mu_i(\cdot \mid p)]\} \neq \emptyset. \tag{11}$$

Thus, under assumptions [a-1]-[a-5] and [a-7] and [a-8](b), the conditions of Hart and Hammond are equivalent to our condition of no unbounded arbitrage. Moreover, if in addition assumption [a-9] holds, then it follows from our Theorem 2 that the conditions of Hart and Hammond are necessary and sufficient for the existence of equilibrium, compactness of utility possibilities, and nonemptiness of the core for the asset exchange economy E.

5.3 Abstract general equilibrium models: the no unbounded arbitrage conditions of Page [36], Werner [51], and Nielsen [32]. Werner in [**51**] considers a general equilibrium model, $E = (X_i, \omega_i, u_i(\cdot))_{1=1}^n$, without price dependent preferences in which agents' choice sets are arbitrary closed, convex sets. Thus, in Werner arbitrage cones are given by $\mathcal{A}_i(x, p) = \mathcal{R}\left(X_i \cap \widehat{P}_i(x)\right)$. With the exception that [**51**] requires *uniformity of arbitrage cones* ([A-4]), the conditions on his economic model are weaker than ours.[15] Let $\mathcal{A}_i$ denote the *ith*

[15]We do not require uniformity to show that no unbounded arbitrage is sufficient for existence. Uniformity means that the set of potential arbitrage opportunities available to the agent is the same no matter what the agent's starting point x and no matter what the prices p. See also Remark 5.

agent's arbitrage cone and let $\mathcal{L}_i$ denote the corresponding lineality space (under [A-4], $\mathcal{A}_i(x,p) = \mathcal{A}_i$ for all $(x,p) \in \mathbf{R}^L \times \mathbf{R}^L$). Werner's nonsatiation assumption is

$$\text{[W-1] for all } i,\ \mathcal{A}_i \backslash \mathcal{L}_i \neq \emptyset$$

(see assumption A.4 in [**51**]).[16] By our Theorem 4, Werner's price condition limiting arbitrage, given by

$$\bigcap\nolimits_i \mathcal{D}(\mathcal{A}_i \backslash \mathcal{L}_i) \neq \emptyset, \tag{12}$$

is weaker than no unbounded arbitrage. Finally, Werner's assumptions on endowments is given by

$$\text{[W-2] for all } i,\ \langle \omega_i, p\rangle \quad > \quad \inf\{\langle x, p\rangle : x \in X_i\} \text{ for all } p \in cl\left\{\bigcap\nolimits_i \mathcal{D}(\mathcal{A}_i \backslash \mathcal{L}_i)\right\},$$
$$\text{where "}cl\text{" denotes closure.}$$

To summarize, within the context of an exchange economy, $(X_i, \omega_i, u_i(\cdot))_{1=1}^n$, without price dependencies satisfying assumption [A-1.1], [A-2.1], [A-4] (uniformity), [W-1] ($\mathcal{A}_i \backslash \mathcal{L}_i \neq \emptyset$), and [W-2] (endowments), Werner shows that his price condition limiting arbitrage, $\cap_i \mathcal{D}(\mathcal{A}_i \backslash \mathcal{L}_i) \neq \emptyset$, is sufficient for the existence of an equilibrium. Werner also notes that his condition (12) is necessary for existence provided indifference curves contain no half lines. Since in the no half lines case $\mathcal{L}_i = \{0\}$, it follows from our Corollary 2 that for this case, Werner's condition (12) is equivalent to our condition of no unbounded arbitrage (6). Moreover, the no half lines assumption, along with Werner's assumption of uniformity implies the economy, $(X_i, \omega_i, u_i(\cdot))_{1=1}^n$, is strictly reconcilable.

Remark 5. In the conclusion of his paper, Werner in [**51**] remarks that his results can be generalized to treat price dependent preferences so long as the recession cones of the "at least as good as" sets do not depend on prices and utility functions depend continuously on prices. As he also notes, uniformity assumptions have a long tradition in the theory of concavifiable preference orderings (see [**21**]). As we have noted, our sufficiency results do not require uniformity – arbitrage cones may vary as prices vary.

Page in [**36**] considers the question of existence of equilibrium within the context of an exchange economy with price dependent preferences in which agents' choice sets are all of $\mathbf{R}^L$. In [**36**], the economic model is given by a triple, $(\mathbf{R}^L, \omega_i, u_i(\cdot,\cdot))_{i=1}^n$, satisfying [A-1] and [A-2]. Thus, in [**36**] arbitrage cones are given by $\mathcal{A}_i(\omega_i, p) = \mathcal{R}\left(\widehat{P}_i(\omega_i, p)\right)$. Page in [**36**] shows that the price condition

$$\text{for each } p \in \mathcal{B}\backslash\{0\},\ \bigcap\nolimits_i \mathcal{D}\left(\mathcal{R}\left(\widehat{P}_i(\omega_i, p)\right)\right) \neq \emptyset. \tag{13}$$

is sufficient for the existence of equilibrium.

For the sake of comparisons, Werner's condition (12) can be rewritten as

$$\bigcap\nolimits_i \mathcal{D}\left(\mathcal{R}\left(X_i \bigcap \widehat{P}_i(\omega_i)\right) \backslash L\left(\mathcal{R}\left(X_i \bigcap \widehat{P}_i(\omega_i)\right)\right)\right) \neq \emptyset. \tag{14}$$

---

[16] Recall that our results continue to hold even if we replace our local nonsatiation assumption [A-2.3] with a weaker global nonsatiation assumption. However, the relationship between the nonsatiation condition $\mathcal{A}_i \backslash \mathcal{L}_i \neq \emptyset$ and global nonsatiation in economic models satisfying uniformity is an open question.

If in [36] utility functions are *not* price dependent and if in [51], $X_i = \mathbf{R}^L$ and $L\left(\mathcal{R}\left(X_i \cap \widehat{P}_i(\omega_i)\right)\right) = \{0\}$ (e.g., if indifference surfaces contain no half lines), then the price conditions of [36] and [51] are equivalent.

In a general equilibrium model without price dependences in which agents' choice sets are arbitrary closed, convex sets, [32] shows that the condition

$$\text{if } \sum_{i=1}^{n} y_i = 0 \text{ and } y_i \in \mathcal{R}(X_i \bigcap \widehat{P}_i(\omega_i)) \text{ for all } i \\ \text{then } y_i = 0 \text{ for all } i, \tag{15}$$

is sufficient for the existence of equilibrium. Nielsen's economic model is similar to the model presented here, but without price dependencies. As Nielsen notes, the condition in [32] can be viewed as a variant of Page's condition in [35].

We close this section by noting that it follows from Corollary 2 that in a strictly reconcilable economy without price dependencies and choice sets given by all of $\mathbf{R}^L$, the conditions limiting arbitrage given by [36], [51], and [32] are all equivalent.

5.4 Comments on Green [13], and Grandmont [14, 15]. It follows from Lemma 2.5 in [13] that, in a temporary equilibrium model, overlapping expectations is necessary for the existence of equilibrium.[17] Also in a temporary equilibrium setting [14, 15] shows that if the economy satisfies conditions analogous to those needed to guarantee strict reconcilability in an asset market model, then the overlapping expectations condition is necessary and sufficient for the existence of a temporary equilibrium. Grandmont in [14, 15] makes an assumption similar to that of [51] for the case of price dependent preferences. It may be that our approach, using the result in [49], could be applied to temporary equilibrium models to treat a broader class of economies.

5.5 Comments on Kousougeras [25]. Kousougeras in [25] was the first to study the core in models with unbounded short sales. Kousougeras uses the no unbounded arbitrage condition of [35] to show that the set of individually rational and feasible allocations is bounded and then appeals to Scarf's Theorem [48] to prove that no unbounded arbitrage is sufficient for the nonemptiness of the core. This nonemptiness also follows from existence of a competitive equilibrium and is shown for the partnered core in [39] by appealing to the [46] result on the partnered core of a game without side payments.

6 Conclusions

We conclude this paper with some remarks on the core and the partnered core. Even in situations without externalities, the competitive equilibrium may be subject to asymmetric dependencies between groups of players.. If an allocation is in the core, it still may be the case that one player is dependent on another player – one player may need the cooperation of another player but the second player may not need the cooperation of the first. Similarly, a group of players may need the cooperation of another group. The partnered core, introduced in [43] and [46], is not subject to such asymmetric dependencies. Using the Reny and Wooders result, Page and Wooders in [39], and [38] show that the partnered core of a balanced economy is nonempty. From our result showing balancedness of an economy with

[17] The Green condition is sometimes referred to as common expectations (see [13], p. 1114).

price dependent preferences and satisfying no unbounded arbitrage it follows that the partnered core of the economy is nonempty.

In general, the competitive equilibrium payoff is not partnered. In strictly reconcilable economies, however, the partnership property of the competitive payoff was shown by [**3**]. It is immediate from our results in this paper and [**38, 39**] that in strictly reconcilable economies no unbounded arbitrage is necessary and sufficient for the nonemptiness of the partnered core and for existence of partnered competitive equilibrium.

7 Appendix: proofs

First, note that it follows from [A-2] that for each agent i the preference correspondences

$$(x,p) \to \widehat{P}_i(x,p) := \{x' \in \mathbf{R}^L : u_i(x',p) \geq u_i(x,p)\}$$
$$(x,p) \to P_i(x,p) := \{x' \in \mathbf{R}^L : u_i(x',p) > u_i(x,p)\}$$

exhibits the following continuity properties:

1. $(x,p) \to \widehat{P}_i(x,p)$ is continuous and, in particular, for each $x \in \mathbf{R}^L$, $p \to \widehat{P}_i(x,p)$ is continuous,
2. the graph of $P_i(\cdot,\cdot)$ is open relative to $(\mathbf{R}^L \times \mathcal{B}) \times \mathbf{R}^L$.

7.1 Preliminary lemmas. We begin with some preliminary lemmas. Lemma 1 is from [**35**].

Lemma 1 Let $\Lambda(\cdot)$ be a set-valued mapping defined on $\mathcal{B} \subset \mathbf{R}^L$ with nonempty convex values. Suppose $\Lambda(\cdot)$ is lower hemicontinuous (lhc) with a closed graph. Let $\{(p_k, x_k)\}_k \subset \mathcal{B} \times \mathbf{R}^L$ be any sequence with $p_k \to p^* \in \mathcal{B}$ and $x_k \in \Lambda(p_k)$ for all k. Further, let $\{\lambda_k\}_k$ be any sequence of positive real numbers such that $\lambda_k \downarrow 0$. Then any cluster point of the sequence $\{\lambda_k x_k\}_k$ is contained in the recession cone $\mathcal{R}(\Lambda(p^*))$.

The next Lemma can be proven using elementary facts concerning sequences.

Lemma 2 Let $\{x_{1k}, ..., x_{nk}\}_k$ be a sequence such that $\sum_{i=1}^n \|x_{ik}\| \to \infty$ as $k \to \infty$. Then for any cluster point $(y_1, ..., y_n)$ of the sequence $\{\lambda_k x_{1k}, ..., \lambda_k x_{nk}\}_k$ where $\lambda_k = \left(\sum_{i=1}^n \|x_{ik}\|\right)^{-1}$ it holds that $\sum_{i=1}^n y_i = 0$ and $\sum_{i=1}^n \|y_i\| = 1$.

7.2 Bounded economies. Let $\{r_k\}$ be a sequence of positive numbers such that $r_k \to \infty$ and let $S(k) \subset \mathbf{R}^L$ be a closed ball of radius r_k centered at the origin. The sequence $\{S(k)\}_k$ is an increasing sequence of closed balls. Suppose now that for each agent i, $\omega_i \in intS(k)$ for all k. Consider the k-bounded economy $E^k := (X_{ik}(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ where, for each price vector p,

$$X_{ik}(p) := S(k) \bigcap X_i(p).$$

Given prices p and endowments $\omega = (\omega_1, ..., \omega_n)$, the set of all rational allocations for the economy E^k is given by

$$F_k(\omega,p) := \{(x_1, ..., x_n) \in \prod_{i=1}^n X_{ik}(p) : \sum_{i=1}^n x_i = \sum_{i=1}^n \omega_i$$
$$\text{and for each } i,\ x_i \in \widehat{P}_i(\omega_i, p)\}.$$

Let $\Gamma(F_k(\omega,\cdot))$ denote the graph of the rational allocation mapping $p \to F_k(\omega,p)$ for prices $p \in \partial\mathcal{B}$. Thus,

$$\Gamma(F_k(\omega,\cdot)) = \{((x_1,...,x_n,p) \in \mathbf{R}^L \times \cdots \times \mathbf{R}^L \times \partial\mathcal{B} : \\ (x_1,...,x_n) \in F_k(\omega,p)\}.$$

Proposition 1 *No unbounded arbitrage is necessary and sufficient for compactness of the graph of the rational allocation mapping.* Let $E = (X_i(\cdot),\omega_i,u_i(\cdot,\cdot))_{i=1}^n$ be an economy satisfying [A-1] and [A-2]. The following statements are equivalent:

1. For each $p \in \mathcal{B}\backslash\{0\}$ if $\sum_{i=1}^n y_i = 0$ and $y_i \in \mathcal{A}_i(\omega_i,p)$ for all i, then $y_i = 0$ for all i (there is no unbounded arbitrage).
2. $\Gamma(F(\omega,\cdot))$ is compact.

Proof The graph $\Gamma(F(\omega,\cdot))$ of the rational allocation mapping $p \to F(\omega,p)$ is contained in the space

$$\mathbf{R}^L \times \cdots \times \mathbf{R}^L = \mathbf{R}^{L(n+1)}.$$

Equip $\mathbf{R}^{L(n+1)}$ with the norm $\|\cdot\|_E$ defined as follows: given

$$(x',p') = (x_1',...,x_n',p') \in \mathbf{R}^{L(n+1)} \text{ and } (x,p) = (x_1,...,x_n,p) \in \mathbf{R}^{L(n+1)},$$
$$\|(x',p') - (x,p)\|_E = \|p' - p\| + \textstyle\sum_{i=1}^n \|x_i' - x_i\|,$$

where $\|\cdot\|$ is the usual norm in $\mathbf{R}^L$. It is easy to check that $\Gamma(F(\omega,\cdot))$ is a closed subset of $\mathbf{R}^{L(n+1)}$ under the $\|\cdot\|_E$ norm. Moreover, for each k, $\Gamma(F_k(\omega,\cdot))$ is closed in $\mathbf{R}^{L(n+1)}$ in the topology induced by $\|\cdot\|_E$.

We will in fact prove the following: If the economy $E = (X_i(\cdot),\omega_i,u_i(\cdot,\cdot))_{i=1}^n$ satisfies [A-1] and [A-2], then the following are equivalent:

(i) For each $p \in \partial\mathcal{B}$, if $\sum_{i=1}^n y_i = 0$ and $y_i \in \mathcal{A}_i(\omega_i,p)$ for all i, then $y_i = 0$ for all i.

(ii) There exists an integer k_0 such that $\Gamma(F(\omega,\cdot)) = \Gamma(F_k(\omega,\cdot))$ for all $k \geq k_0$.

If (ii) holds then $\Gamma(F(\omega,\cdot))$ is $\|\cdot\|_E$-bounded and thus (i) holds. To see that (i) implies (ii), consider the following: Let $\{(\overline{x}_{1k},...,\overline{x}_{nk},\overline{p}_k)\}_k$ be a sequence contained in $\Gamma(F(\omega,\cdot))$ and satisfying the property that for each k, $(\overline{x}_{1k},...,\overline{x}_{nk},\overline{p}_k) \notin \Gamma(F_k(\omega,\cdot))$. Thus, for each k, there is some i such that $\overline{x}_{ik} \notin S(k)$ so that $\sum_{i=1}^n \|\overline{x}_{ik}\| \to \infty$ as $k \to \infty$.

Let $(\overline{y}_1,...,\overline{y}_n,\overline{p}) \in \mathbf{R}^L \times \cdots \times \mathbf{R}^L \times \partial\mathcal{B}$ be a cluster point of the sequence $\{(\lambda_k\overline{x}_{1k},...,\lambda_k\overline{x}_{nk},\overline{p}_k)\}_k$ where $\lambda_k = \left(\sum_{i=1}^n \|\overline{x}_{ik}\|\right)^{-1.}$ Note that the set-valued mapping $p \to X_i(p) \cap \widehat{P}_i(\omega_i,p)$ is convex-valued, lower hemicontinuous, and has a closed graph. Thus, since for each k, $\overline{x}_{ik} \in X_i(\overline{p}_k) \cap \widehat{P}_i(\omega_i,\overline{p}_k)$, it follows from Lemma 1 that $\overline{y}_i \in \mathcal{R}\left(X_i(\overline{p}) \cap \widehat{P}_i(\omega_i,\overline{p})\right) = \mathcal{A}_i(\omega_i,p)$ for each agent i. By Lemma 2, $\sum_{i=1}^n \overline{y}_i = 0$ and $\sum_{i=1}^n \|\overline{y}_i\| = 1$. Thus, for some agent i, $\overline{y}_i \neq 0$ and thus the no unbounded arbitrage condition fails to hold, a contradiction. □

Let $\Gamma(F_C(\omega,\cdot))$ denote the graph of the rational, C-allocation mapping $p \to F_C(\omega,p)$ for $p \in \partial\mathcal{B}$. The following is an easy corollary of Proposition 1.

Corollary Let $E = (X_i(\cdot),\omega_i,u_i(\cdot,\cdot))_{i=1}^n$ be an economy satisfying [A-1] and [A-2]. The following statements are equivalent:

1. For each $p \in \mathcal{B}\backslash\{0\}$ if $\sum_{i=1}^n y_i = 0$ and $y_i \in \mathcal{A}_i(\omega_i,p)$ for all i, then $y_i = 0$ for all i (there is no unbounded arbitrage).
2. For any coalition $C \in 2^N$, $\Gamma(F_C(\omega,\cdot))$ is compact.

An *equilibrium* for the $k-$bounded economy $(X_{ik}(\cdot),\omega_i,u_i(\cdot,\cdot))_{i=1}^n$ is an $(n+1)$-tuple of vectors $(x_1^*,...,x_n^*,p^*)$ such that

1. $(x_1^*, ..., x_n^*) \in F_k(\omega, p^*)$ (the allocation is feasible);
2. $p^* \in \mathcal{B}\backslash\{0\}$ (prices are in the unit ball and not all prices are zero); and
3. for each agent i,
   (a) $\langle x_i^*, p^* \rangle = \langle \omega_i, p^* \rangle$ (budget constraints are satisfied), and
   (b) $P_i(x_i^*, p^*) \cap X_{ik}(p^*) \cap B_i(\omega_i, p^*) = \emptyset$ (there are no affordable preferred net trades).

For each k and $(x, p) \in S(k) \times \mathcal{B}$ define the constraint correspondences

$$H_i(k, \omega_i, p) := X_i(p) \cap B_i(\omega_i, p) \cap S(k), \ i = 1, 2, ..., n.$$

and the preference correspondences

$$P_i(k, x, p) := P_i(x, p) \cap S(k), \ i = 1, ..., n \text{ and}$$
$$\widehat{P}_i(k, x, p) := \widehat{P}_i(x, p) \cap S(k), \ i = 1, ..., n.$$

Note that

$$H_i(\infty, \omega_i, p) := H_i(\omega_i, p) := X_i(p) \cap B_i(\omega_i, p),$$
$$\widehat{P}_i(\infty, x, p) := \widehat{P}_i(x, p)$$

and

$$P_i(\infty, x, p) := P_i(x, p).$$

The correspondences $H_i(k, \omega_i, \cdot)$, $P_i(k, \cdot, \cdot)$ and $\widehat{P}_i(k, \cdot, \cdot)$ have the following properties:

1. Given [A-1], for each $i = 1, ..., n$ and $k = 1, 2, ...$

$$p \to H_i(k, \omega_i, p)$$

   is continuous with nonempty, closed convex values.
2. Given [A-2], for each $i = 1, 2, ..., n$ and $k = 1, 2, ...$

$$(x, p) \to P_i(k, x, p)$$

   has nonempty, convex values in $S(k)$ and an open graph relative to $[S(k) \times \mathcal{B}] \times S(k)$. Moreover, for each $i = 1, 2, ..., n$ and $k = 1, 2, ...$

$$p \to \widehat{P}_i(k, \omega_i, p)$$

   is continuous.

Proposition 2 *Equilibrium for the k-bounded economies.*
Let $E = (X_i(\cdot), \omega_i, u_i(\cdot, \cdot))_{i=1}^n$ be an economy satisfying [A-1] and [A-2] and consider the sequence of k-bounded economies $E^k = (X_{ik}(\cdot), \omega_i, u_i(\cdot, \cdot))_{i=1}^n$, $k = 1, 2....$ Each $k-$bounded economy has an equilibrium $(x_{1k}^*, ..., x_{nk}^*, p_k^*)$ with $p_k^* \in \partial\mathcal{B}$.

Given observations 1. and 2. above, the proof of Proposition 2 follows directly from [**49**].

Now we have two important observations concerning the $k-$ bounded equilibria:

1. If $(x_{1k}^*, ..., x_{nk}^*, p_k^*)$ is an equilibrium for the k-bounded economy E^k, then for each agent $i = 1, 2, ..., n$, $x_{ik}^* \in X_i(p_k^*) \cap \widehat{P}_i(\omega_i, p_k^*)$.
2. If $(x_{1k}^*, ..., x_{nk}^*, p_k^*)$ is an equilibrium for the k-bounded economy E^k such that for each agent $i = 1, ..., n$, $x_{ik}^* \in intS(k)$, then $(x_{1k}^*, ..., x_{nk}^*, p_k^*)$ is an equilibrium for the economy $E = (X_i(\cdot), \omega_i, u_i(\cdot, \cdot))_{i=1}^n$.

7.3 Proof of Theorem 1. Consider an economy $E = (X_i(\cdot), \omega_i, u_i(\cdot,\cdot))_{i=1}^n$ satisfying [A-1] and [A-2].

1. *No unbounded arbitrage implies the existence of equilibrium.* Consider a sequence $\{(x_{1k}^*, ..., x_{nk}^*, p_k^*)\}_{k=1}^\infty$ of equilibria for the bounded economies E^k, with $p_k^* \in \partial\mathcal{B}$. Suppose that for each k, $x_{ik}^* \notin intS(k)$ for some agent i. Thus, $\sum_{i=1}^n \|x_{ik}^*\| \to \infty$ as $k \to \infty$. Let $(y_1^*, ..., y_n^*, p^*) \in \mathbf{R}^L \times \cdots \mathbf{R}^L \times \partial\mathcal{B}$ be a cluster point of the sequence $\{(\lambda_k x_{1k}^*, ..., \lambda_k x_{nk}^*, p_k^*)\}_k$ where $\lambda_k = (\sum_{i=1}^n \|x_{ik}^*\|)^{-1}$. Again, note that the set-valued mapping $p \to X_i(p) \cap \widehat{P}_i(\omega_i, p_k^*)$ is convex-valued, lower hemicontinuous, and has a closed graph. Thus, since for each k and i, $x_{ik}^* \in X_i(p_k^*) \cap \widehat{P}_i(\omega_i, p_k^*)$, it follows from Lemma 1 that for each agent i, $y_i^* \in \mathcal{R}\left(X_i(p^*) \cap \widehat{P}_i(\omega_i, p^*)\right) = \mathcal{A}_i(\omega_i, p^*)$. By Lemma 2, $\sum_{i=1}^n y_i = 0$ and $\sum_{i=1}^n \|y_i^*\| = 1$. Thus, for some agent i, $y_i^* \neq 0$, contradicting the condition of no unbounded arbitrage.
2. *No unbounded arbitrage implies that $U(F(\omega,\cdot))$ is compact.* By Proposition 1, no unbounded arbitrage implies that $\Gamma(F(\omega,\cdot))$ is compact. The compactness of $U(F(\omega,\cdot))$ then follows directly from assumption [A-2.1] that the utility functions $u_i(\cdot,\cdot)$ are continuous.
3. *No unbounded arbitrage implies that E is well-defined and balanced.* It follows from the definition (see expression 3) that $V(C,p)$ is nonempty for any price vector $p \in \partial\mathcal{B}$ and any coalition $C \in 2^N$. By [A-2.2] (the quasi-concavity of preferences), $V(C,p)$ is convex for any $C \in 2^N$ and $p \in \partial\mathcal{B}$. Finally, since no unbounded arbitrage implies that $\Gamma(F_C(\cdot,\omega))$ is compact for any coalition C, $V(C,p)$ is closed and bounded from above for any coalition C and $p \in \partial\mathcal{B}$. Thus, for each $p \in \partial\mathcal{B}$ the characteristic form game $(N, V(\cdot,p))$ corresponding to the economy E is well-defined.

   To see that the game is balanced, consider the following: Let β be a balanced collection of nonempty subsets of N with balancing weights δ_C for $C \in \beta$. Thus, $\delta_C \geq 0$ for all $C \in \beta$ and, for each $i \in N$,

   $$\sum_{C \in \beta(i)} \delta_C = 1.$$

   Given $p \in \partial\mathcal{B}$, let $(u_1, ..., u_n) \in \cap_{C\in\beta} V(C,p)$. It follows that for each $C \in \beta$ there is a individually rational C-allocation $(x_i^C : i \in C) \in F_C(\omega, p)$ such that

   $$u_i(x_i^C, p) \geq u_i \text{ for each } i \in C. \quad (*)$$

   For each $i \in N$, let

   $$x_i = \sum_{C\in\beta(i)} \delta_C x_i^C,$$

   and observe that

   $$\sum_{i\in N} x_i = \sum_{i\in N} \sum_{C\in\beta(i)} \delta_C x_i^C = \sum_{i\in N} \omega_i.$$

   By inequality $(*)$ and the quasi-concavity of $u_i(\cdot,p)$ we have for each $i \in N$

   $$u_i = \sum_{C\in\beta(i)} \delta_C u_i \leq \sum_{C\in\beta(i)} \delta_C u_i(x_i^C, p) \leq u_i(x_i, p).$$

Moreover, since $(x_i^C : i \in C) \in F_C(\omega, p)$, we have for each $i \in N$,

$$u_i(\omega_i, p) = \sum_{C \in \beta(i)} \delta_C u_i(\omega_i, p) \leq \sum_{C \in \beta(i)} \delta_C u_i(x_i^C, p) \leq u_i(x_i, p).$$

Thus, $(x_1, ..., x_n) \in F(w, p)$ and we can conclude that $(u_1, ..., u_n) \in V(N, p)$. This implies that the game $(N, V(\cdot, p))$ corresponding to the economy is well-defined and balanced for each $p \in \partial\mathcal{B}$.

4. *No unbounded arbitrage implies that E has a nonempty core.* By part 1, no unbounded arbitrage implies that E has an equilibrium, say $(x_1^*, ..., x_n^*, p^*)$, with $p^* \in \partial\mathcal{B}$. We will show that $(x_1^*, ..., x_n^*)$ is in the core given p^*. Suppose not. Then for some nonempty coalition C there is a rational C-allocation $(x_i : i \in C) \in F_C(\omega, p^*)$ such that

$$u_i(x_i, p^*) > u_i(x_i^*, p^*) \text{ for all agents } i \in C. \; (**)$$

Since $\sum_{i \in C} x_i = \sum_{i \in C} \omega_i$, it holds that $\left\langle \sum_{i \in C} x_i, p^* \right\rangle = \left\langle \sum_{i \in C} \omega_i, p^* \right\rangle$. However, given (**), $\langle x_i^*, p^* \rangle > \langle \omega_i, p^* \rangle$ for all $i \in C$. Therefore

$$\left\langle \sum_{i \in C} x_i, p^* \right\rangle > \left\langle \sum_{i \in C} \omega_i, p^* \right\rangle,$$

a contradiction. □

7.4 Proof of Theorem 2. Consider a strictly reconcilable economy $E = (X_i(\cdot), \omega_i, u_i(\cdot, \cdot))_{i=1}^n$. Thus, in addition to satisfying [A-1] and [A-2] the economy satisfies [A-3] and [A-4]. First, note that the equivalence of (1) and (2) follows immediately from the Corollary to Rockafellar's Theorem.

1. *The existence of an equilibrium implies no unbounded arbitrage.*
   Let $(x_1^*, ..., x_n^*, p^*)$ be an equilibrium for the economy E. If no unbounded arbitrage fails to hold then by the Corollary to Rockafellar's Theorem we have for some $p' \in \partial\mathcal{B}$, $\cap_i \mathcal{D}(\mathcal{A}_i(\omega_i, p')) = \emptyset$. This implies via assumption [A-4] (the uniformity of arbitrage cones) that $\cap_i \mathcal{D}(\mathcal{A}_i(x_i^*, p^*)) = \emptyset$. Thus, for some agent i', there is a nonzero vector of net trades $y_{i'} \in \mathcal{A}_i(x_{i'}^*, p^*)$ such that $\langle y_{i'}, p^* \rangle \leq 0$. By assumption [A-3] (extreme desirability), for some $\lambda' > 0$ $u_{i'}(x_{i'}^* + \lambda' y_{i'}, p^*) > u_{i'}(x_{i'}^*, p^*)$. Thus for λ',

$$x_{i'}^* + \lambda' y_{i'} \in P_{i'}(x_{i'}^*, p^*) \cap X_{i'}(p^*) \cap B_{i'}(\omega_{i'}, p^*),$$

   contradicting the fact that $(x_1^*, ..., x_n^*, p^*)$ is an equilibrium.
2. *Compactness of the utility possibilities set $U(F(\omega, \cdot))$ implies no unbounded arbitrage.* Suppose that $U(F(\omega, \cdot))$ is compact and no unbounded arbitrage fails to hold. Then for some $p' \in \partial\mathcal{B}$, there exists $(y_1', ..., y_n') \neq (0, ..., 0)$ such that

$$\sum_{i=1}^n y_i' = 0 \text{ and } y_i' \in \mathcal{A}_i(\omega_i, p') \text{ for all } i.$$

   By the uniformity assumption [A-4], for any $(x_1, ..., x_n, p) \in \Gamma(F(\omega, \cdot))$, $y_i' \in \mathcal{A}_i(x_i, p)$ for all i and $(x_1 + \lambda y_1', ..., x_n + \lambda y_n', p) \in \Gamma(F(\omega, \cdot))$ for all $\lambda > 0$. Now let $\delta = (\delta_1, ..., \delta_n)$ be a vector in $\mathbf{R}^n$ with strictly positive components and let $\overline{u} = (\overline{u}_1, ..., \overline{u}_n) \in U(F(\omega, \cdot))$ satisfy

$$\langle \delta, \overline{u} \rangle = \max\{\langle \delta, u \rangle : u \in U(F(\omega, \cdot))\}.$$

Since $U(F(\omega,\cdot)$ is compact such a $\overline{u}$ exists. Suppose $(\overline{x}_1,...,\overline{x}_n,\overline{p})\in\Gamma(F(\omega,\cdot))$ is such that $\overline{u}_i = u_i(\overline{x}_i,\overline{p})$ for each i. By the extreme desirability assumption [A-3], we have for some $\lambda' > 0$,

$$\sum_{i=1}^{n}\delta_i u_i(\overline{x}_i + \lambda' y_i',\overline{p}) > \sum_{i=1}^{n}\delta_i u_i(\overline{x}_i,\overline{p}) = \sum_{i=1}^{n}\delta_i\overline{u}_i = \langle\delta,\overline{u}\rangle$$
$$= \max\{\langle\delta,u\rangle : u \in U(F(\omega,\cdot))\}.$$

But since $(u_1(\overline{x}_1 + \lambda y_1',\cdot),...,u_n(\overline{x}_n + \lambda y_n',\cdot)) \in U(F(\omega,\cdot)$ for all $\lambda > 0$, we have a contradiction. □

3. *Well-definedness and balancedness imply no unbounded arbitrage.* If the economy E is well-defined and balanced then E has a nonempty core. Let $(\overline{x}_1,...,\overline{x}_n,\overline{p}) \in \Gamma(F(\omega,\cdot))$ be such that $(\overline{x}_1,...,\overline{x}_n)$ is in the core. Suppose no unbounded arbitrage fails to hold. Then for some $p' \in \partial\mathcal{B}$, there exists $(y_1',...,y_n') \neq (0,...,0)$ such that
$$\sum_{i=1}^{n} y_i' = 0 \text{ and } y_i' \in \mathcal{A}_i\left(\omega_i,p'\right) \quad \text{for all } i.$$
By the uniformity assumption (i.e., [A-4]), $y_i' \in \mathbf{A}_i(\overline{x}_i,\overline{p}_i)$ for all i and $(\overline{x}_1 + \lambda y_1',...,\overline{x}_n + \lambda y_1',\overline{p}) \in \Gamma(F(\omega,\cdot))$ for all $\lambda > 0$. By the extreme desirability assumption (i.e., [A-3]), there exists $\lambda' > 0$ such that $u_i(\overline{x}_i+\lambda' y_i',\overline{p}) > u_i(\overline{x}_i,\overline{p})$ for any agent i such that $y_i' \neq 0$. Thus, the individually rational allocation $(\overline{x}_1 + \lambda' y_1',...,\overline{x}_n + \lambda' y_1') \in F(\omega,\overline{p})$ Pareto dominates the individually rational allocation $(\overline{x}_1,...,\overline{x}_n) \in F(\omega,\overline{p})$, contradicting the supposition that $(\overline{x}_1,...,\overline{x}_n)$ is in the core.
4. *Nonemptiness of the core implies no unbounded arbitrage.* The proof follows from the arguments given in (3) above.

7.5 Proof of Theorem 3. Let $(X_i(\cdot),\omega_i,u_i(\cdot,\cdot))_{i=1}^{n}$ be an economy satisfying [A-1]- [A-3] and let $(x_1^*,...,x_n^*,p^*)$ be an equilibrium for E. Suppose $p^* \notin \cap_i\mathcal{D}\left(\mathcal{A}_i\left(x_i^*,p^*\right)\right)$. Thus, for some agent i', there is a nonzero vector of net trades $y_{i'} \in \mathcal{A}_{i'}\left(x_{i'}^*,p^*\right)$ such that $\langle y_{i'},p^*\rangle \leq 0$. By assumption [A-3] (extreme desirability), $u_{i'}(x_{i'}^* + \lambda' y_{i'},p^*) > u_{i'}(x_{i'}^*,p^*)$ for some $\lambda' > 0$. Thus, for any λ',
$$x_{i'}^* + \lambda' y_{i'} \in P_{i'}(x_{i'}^*,p^*) \cap X_{i'}(p^*) \cap B_{i'}(\omega_{i'},p^*),$$
contradicting the fact that $(x_1^*,...,x_n^*,p^*)$ is an equilibrium. The fact that the condition
$$\text{if } \sum_{i=1}^{n} y_i = 0 \text{ and } y_i \in \mathcal{A}_i\left(x_i^*,p^*\right) \text{ for all } i, \text{then } y_i = 0 \text{ forall } i,$$
holds if $(x_1^*,...,x_n^*,p^*)$ is an equilibrium now follows from the Corollary to Rockafellar's Theorem. □

References

[1] Albers, W. [1979], *Core and kernel variants based on imputations and demand profiles*, in Game Theory and Related Topics (O. Moeschlin and D. Pallaschke, eds.), North Holland, Amsterdam.

[2] Bennett, E. [1983], *The aspiration approach to predicting coalition formation and payoff distribution in sidepayment games*, International Journal of Game Theory **12**, 1-28.

[3] Bennett, E. and Zame, W. R. [1988], *Bargaining in cooperative games*, International Journal of Game Theory **17**, 279-300.

[4] Berge, C. [1963], *Topological Spaces*, Macmillan, New York.

[5] Bondareva, O. [1962], *Theory of the core in an n -Person Game*, Vestnik, LGU13, 141-142 (in Russian), (Leningrad State University, Leningrad).

[6] Brown, J. D. and Werner, J. [1995], *Arbitrage and existence of equilibrium in infinite asset markets*, Review of Economic Studies **62**, 101-114.

[7] Cheng, H. C. C. [1991], *Asset Market Equilibrium in Infinite Dimensional Complete Markes*, Journal of Mathematical Economics **20**, 137-152.

[8] Dana, R. A., Le Van, C. and Magnien, F. [1996], *On different notions of arbitrage and existence of equilibrium*, Journal of Economic Theory (to appear).

[9] Debreu, G. [1962], *New concepts and techniques for equilibrium analysis*, International Economic Review **3**, 257-273.

[10] Debreu G. and Scarf, H. [1963], *A limit theorem on the core of an economy*, International Economic Review **4**, 235-246.

[11] Duffie, D. [1986], *Competitive equilibrium in general choice spaces*, Journal of Mathematical Economics **14**, 9-16.

[12] Gale, D. and Mas-Colell, A. [1975], *An equilibrium existence theorem for a general model without ordered preferences*, Journal of Mathematical Economics **2**, 9-16.

[13] Green, J. R. [1973], *Temporary general equilibrium in a sequential trading model with spot and futures transactions*, Econometrica **41**, 1103-1123.

[14] Grandmont, J. M. [1977], *Temporary general equilibrium theory*, Econometrica **45**, 535-572.

[15] Grandmont, J. M. [1982], *Temporary general equilibrium theory*, Handbook of Mathematical Economics, Volume II, North Holland.

[16] Hammond, P. J. [1983], *Overlapping expectations and Hart's condition for equilibrium in a securities model*, Journal of Economic Theory **31**, 170-175.

[17] Hammond, P., Kaneko, M. and Wooders M. H. [1989], *Continuum economies with finite coalitions: Core, equilibria, and widespread externalities*, Journal of Economic Theory **49**, 113-134.

[18] Hart, O. D. [1974], *On the Existence of equilibrium in a securities model*, Journal of Economic Theory **9**, 293-311.

[19] Kaneko, M. and Wooders, M. H. [1982], *Cores of partitioning games*, Mathematical Social Sciences **3**, 313-327.

[20] Kaneko, M. and Wooders, M. H. [1989], *The core of a continuum economy with widespread externalities and finite coalitions: From finite to continuum economics*, Journal of Economic Theory **49**, 135-168.

[21] Kannai, Y. [1977], *Concavifability and constructions of concave utility functions*, Journal of Mathematical Economics **4**, 1-56.

[22] Kannai, Y. and Wooders, M. H. [1996], *A further extension of the KKMS Theorem*, University of Bielefeld IMW Working Paper 251.

[23] Keiding, H. and Thorlund-Petersen, L. [1987], *The core of a cooperative game without side payments*, Journal of Optimization Theory **54**, 273-288.

[24] Kim, C. [1998], *Stochastic dominance, Pareto optimality, and equilibrium asset pricing*, Review of Economic Studies **65**, 341-356.

[25] Kousougeras, L. [1992], *A two-stage core with applications to asset market and differential information economies*, Revised as Economic Theory **11(3)**, 563-584, 4/1998.

[26] Kreps, D. [1981], *Arbitrage and equilibrium in economies with infinitely many commodities*, Journal of Mathematical Economics **8**, 15-35.

[27] Le Van, C. and Magnien, F. [1998], *No arbitrage condition and existence of equilibria in asset markets with a continuum of traders*, presented at the 1998 Association for Public Economic Theory Conference, University of Alabama May 1998.

[28] Monteiro, P., Page, F. and Wooders, M. H. [1998], *Increasing cones, global cones and recession cones*, Optimization (to appear).

[29] Monteiro, P., Page, F. and Wooders, M. H. [1999], *Arbitrage and global cones; another counterexample*, Social Choice and Welfare **16**, 337-346.

[30] Monteiro, P., Page, F. and Wooders, M. H. [1997], *Arbitrage, equilibrium and gains from trade; A counterexample*, Journal of Mathematical Economics **28**, 481-501.

[31] Monteiro, P., Page, F. H. Jr. and Wooders, M. H. [1995], *Arbitrage and global cones; Global cones are not open*, Department of Finance University of Alabama Working Paper 254.

[32] Nielsen, L. [1989], *Asset Market Equilibrium with Short Selling*, Review of Economic Studies **56**, 467-474.

[33] Page, F. H., Jr. [1996], *Arbitrage and Asset Prices*, Mathematical Social Sciences **31**, 183-208.

[34] Page, F. H., Jr. [1989], *Utility arbitrage and asset prices*, Graduate School of Business, Indiana University Working Paper No. 401.

[35] Page, F. H., Jr. [1987], *On equilibrium in Hart's securities exchange model*, Journal of Economic Theory **41**, 392-404.

[36] Page, F. H., Jr. [1984], *Equilibria in Unbounded Economies*, Department of Finance, The University of Texas at Austin, Working Paper 83/84-2-19.

[37] Page, F. H., Jr. [1982], *Information, arbitrage, and equilibrium*, Department of Finance, The University of Texas at Austin, Working paper 81/82-2-51.

[38] Page, F. H., Jr. and Wooders, M. H. [1996a], *The partnered core and the partnered competitive equilibrium*, Economics Letters **52**, (1996) 143-152.

[39] Page, F. H., Jr. and Wooders, M. H. [1996b], *A necessary and sufficient condition for compactness of individually rational and feasible outcomes and existence of an equilibrium*, Economics Letters **52**, (1996) 153-162.

[40] Page, F. H., Jr. and Wooders, M. H. [1995], *The partnered core and the partnered competitive equilibrium*, University of Alabama Department of Finance Working Paper No. 250 (Revised November 1995).

[41] Page, F. H., Jr. and Wooders, M. H. [1993], *Arbitrage in Markets with unbounded short sales: Necessary and sufficient conditions for nonemptiness of the core and existence of equilibrium*, University of Toronto Working Paper No. 9409.

[42] Page, F. H., Jr., Wooders, M. H. and Monteiro, P. [1997], *Inconsequential arbitrage*, Presented at the Fall 1997 Meetings of the Midwest Math Econ Association and at the 1998 Winter Meetings of the Econometric Society.

[43] Reny, P. J., Winter, E. and Wooders, M. H. [1993], *The partnered core of a game with side payments*, Hebrew University Center for Rationality and Interactive Decision Theory, Discussion Paper 33.

[44] Reny, P. J. and Wooders, M. H. [1996], *The partnered core of a game without side payments*, Journal of Economic Theory **70**, 298-311.

[45] Reny, P. J. and Wooders, M. H. [1996], *Credible threats of secesion, partnership, and commonwealths*, in Understanding Strategic Interaction; Essays in Honor of Reinhard Selten, (W. Gueth, P. Hammerstein, B. Moldovanu, E. van Damme, eds.), Springer Verlag Berlin/Heidelberg/New York/Tokyo 305-312.

[46] Reny, P. J. and Wooders, M. H. [1998], *An extension of the KKMS Theorem*, Journal of Mathematical Economics **29**, 125-134.

[47] Rockafellar, R. T. [1970], *Convex Analysis,* Princeton University Press.

[48] Scarf, H. E. [1967], *The Core of an n-person Game*, Econometrica **35**, 50-67.

[49] Shafer, W. J. and Sonnenschein, H. [1975], *Equilibrium in abstract economies without ordered preferences*, Journal of Mathematical Economics **2**, 345-348.

[50] Shapley, L. S. [1967], *On balanced sets and cores*, Naval Research Logistics Quarterly **14**, 453-460.

[51] Werner, J. [1987], *Arbitrage and existence of competitive equilibrium*, Econometrica **55**, 1403-1418.