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Axiomatization of ratio equilibria in public good economies

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Abstract. Using consistency properties, we characterize the cost-sharing scheme arising from the ratio equilibrium concept for economies with public goods. The characterization turns out to be surprisingly simple and direct. In contrast to most axiomatic characterizations based on reduced games and consistency properties, our characterization requires that in the reduced game, the players take as given the *proportions* of the costs paid by the members of the complementary player set, rather than their utility levels.

1 Introduction

A (pure) public good is a commodity that can be consumed in its entirety by all members of an economy; consumption of the good by an additional agent does not decrease the amount available to the other members of the society. Thus, unlike the situation for private goods, cost-sharing rules for public goods cannot be determined by competition between agents for the available supplies of the commodity.

Various solutions to the problem of allocation of costs of public good provision have been proposed. The most well-known is perhaps the Lindahl equilibrium, introduced in Lindahl (1919) and formalized in Samuelson (1954) and Johansen (1963). As formalized by Samuelson, the Lindahl equilibrium permits individuals to pay personalized prices for public goods. In equilibrium,

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these personalized prices have the property that all individuals demand the same quantities of public goods. Kaneko's (1977a,b) formalization of Lindahl's concept as the ratio equilibrium, in keeping with the spirit of Lindahl (1919), requires agents to pay personalized proportions of the total costs of public good provision. In the current paper we axiomatize the ratio equilibrium cost-sharing rule by means of consistency properties.[1] The consistency property that we use is, as we will argue, very much in the spirit of Lindahl's original work and Kaneko's ratio equilibrium.[2]

As documented by Aumann and Maschler (1985), consistency was already used in problems of cost sharing some 2000 years ago. The consistency principle dictates that methods of reaching agreements should be consistent whatever the group of agents involved. More precisely, whenever the members of a group, using some particular method of making a decision, have all accepted an agreement, no subgroup of agents, given the acceptance of the complementary coalition and using the same method, has an incentive to reach a different agreement. The consistency principle has been applied to a number of game theoretic and economic solution concepts.[3] In addition to consistency, we also use a property of converse consistency, first examined by Peleg (1986).

The outline of the paper is as follows. We introduce the model of a public good economy in Sect. 2 and in Sect. 3 we provide the definition of the ratio equilibrium. In Sect. 4 we introduce consistency and discuss the consistency properties of the cost-sharing rule induced by the ratio equilibrium. We introduce two additional properties in Sect. 5, namely converse consistency and one-person rationality, and prove that the ratio equilibrium cost-sharing rule is the unique cost-sharing rule that satisfies consistency, converse consistency, and one-person rationality. The last section, Sect. 6, concludes the paper.

2 Public good economies

In this section we provide formal definitions of a public good economy and of some associated concepts. Throughout the paper, we restrict discussion to economies with one public good and one private good. Our results, however, extend to public good economies with any finite number of public goods. We choose not to consider this broader framework in order to avoid complicated notation and distracting technical matters.

A *public good economy* (with one private good and one public good) is a list $E = \langle N; (w^i)_{i\in N}; (u^i)_{i\in N}; f\rangle$, where N (sometimes denoted $N(E)$) is a non-

[1] An interesting axiomatization that takes another approach can be found in Diamantaras (1992).

[2] Note that by "consistency" we mean consistency with respect to reduced economies, as in the economics and social choice literature on axiomatizations rather than the general notion of consistency from mathematics.

[3] A more complete discussion of the literature on consistency is provided in Thomson (1990).

empty finite set of agents, $w^i \in \mathbb{R}_+ = (0, \infty)$ is the positive endowment of agent $i \in N$ of a private good, $u^i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is the utility function of agent $i \in N$, and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is the non-decreasing cost function for the production of a public good. If agent $i \in N$ consumes an amount x^i of the private good and an amount y of the public good, then agent i enjoys utility $u^i(x^i, y)$. We assume that u^i is strictly increasing in both private and public good consumption. If each agent i contributes an amount z^i of the private good toward the production of the public good, then some bundle y of the public good can be provided. The bundle y must satisfy the feasibility condition $f(y) \leq \sum_{i \in N} z^i$. The family of all public good economies is denoted by $\mathscr{E}$.

A *configuration* in a public good economy $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; f \rangle$ is a vector $(\mathbf{x}, y) = ((x^i)_{i \in N}, y)$, where $x^i \in \mathbb{R}_+$ is the consumption of the private good by agent i for each $i \in N$ and $y \in \mathbb{R}_+$ is the level of public good provided.

3 The ratio equilibrium cost-sharing scheme

A ratio equilibrium consists of a set of ratios – one for each agent in the economy – and a configuration. The ratios reflect the way agents share the cost of public good production; if an agent $i \in N$ has a ratio r^i, then agent i pays the share r^i of the cost of public good production. A set of ratios and a configuration constitute a ratio equilibrium if every agent can afford his consumption bundle and if, given his share of the costs, no agent can afford to consume a consumption bundle that gives him a higher utility. Moreover, the level of public good consumption must be the same for each agent. Hence, in a ratio equilibrium, agents agree on *both* the cost shares arising from their personalized ratios and on the level of public good production; agreement on the ratios determining cost shares and the level of public good go hand in hand. Imagine that first ratios are proposed and then agents state their optimizing quantities of public goods. The outcome is an equilibrium only if at the given ratios the quantities demanded of the public good by all agents are equated. Thus, agreement about the quantity of the public good is inextricably linked to agreement about the ratios.

Formally, for a public good economy $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; f \rangle$, a set of ratios and a configuration $(\mathbf{r}, (\mathbf{x}, y))$ is a *ratio equilibrium* if

$$\mathbf{r} = (r^i)_{i \in N} \in \Delta^N = \left\{ (q^i)_{i \in N} \in \mathbb{R}^N \mid \sum_{i \in N} q^i = 1 \right\}$$

and, for each $i \in N$,

$$r^i f(y) + x^i = w^i, \text{ and,}$$

for all $(\bar{x}^i, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}_+$ satisfying $r^i f(\bar{y}) + \bar{x}^i = w^i$,
it holds that $u^i(x^i, y) \geq u^i(\bar{x}^i, \bar{y})$.

The set of configurations associated with ratio equilibria of an economy $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; f \rangle$ is denoted $R(E)$ and defined by

$$R(E) = \{(\mathbf{x}, y) \in \mathbb{R}_+^N \times \mathbb{R}_+ \mid \text{there exists an } \mathbf{r} \in \Delta^N \text{ such that } (\mathbf{r}, (\mathbf{x}, y)) \text{ is a ratio equilibrium of } E\}.$$

We refer to R as the *ratio correspondence*. It is apparent that R is a mapping that assigns to each public good economy $E \in \mathscr{E}$ a set of configurations $\phi(E) \subseteq \mathbb{R}_+^{N(E)} \times \mathbb{R}_+$. We will call such a mapping a *solution* on $\mathscr{E}$.

4 Consistency

Consistency dictates that methods of reaching agreements should lead to the same agreements when applied to subgroups of agents as when applied to the group consisting of all agents. The scenario we have in mind is the following. Suppose that the agents in some set N agree on a set of ratios of cost-shares and on an amount of public good to be produced. The agreement is acceptable only when it has the property that, if the agents in any (strict) subset S were to withdraw from the decision-making process and then the remaining agents – those in $N \backslash S$ – were to reconsider, taking the ratios for the members of S as fixed, the agents in $N \backslash S$ would not arrive at any different set of ratios for themselves nor at a different level of public good.

Note that the treatments of the amount of public good and the cost shares given by the ratios are asymmetric. If the members of S were to withdraw, they would leave with their ratios fixed but with the amount of public good they consume open to reconsideration by $N \backslash S$. Since our purpose is to axiomatize cost-sharing schemes rather than allocations of commodities, this is appropriate. It is precisely the fact that the members of $N \backslash S$ would not choose to change the level of public good provision that makes the cost shares given by the ratios those of the ratio equilibrium.

Our notions of reduced economies differ fundamentally from those typically studied in the literature – see, for example, Thomson (1988, 1990). In that literature, the reduced economy for a coalition $N \backslash S$ is defined under the assumption that the physical consumption bundles, or at least the utility levels (payoffs), of the members of the "departing" coalition S are fixed at the original solution outcome. This means that the members of S leave the scene (allowing the members of $N \backslash S$ to change the allocation among themselves) only when such a reallocation in $N \backslash S$ has no effect on the utilities of the outside agents. Our approach is quite distinct – the ratios for those members leaving the scene are held fixed but their utility levels are not guaranteed. Indeed, it appears that our axiomatization places a heavy burden on the cost shares since, even though only these cost shares are held constant, it turns out that the utilities of the departing agents will remain unchanged. Since our motivation is the axiomatization of cost-sharing schemes rather than allocations and since, as we argue below, our approach is very much in the spirit of the original work of Lindahl, we choose to deviate from the typical approach in defining the reduced economy.

Our consistency property is quite closely related in spirit to the original description of equilibrium in Lindahl (1919). In his seminal paper, Lindahl writes:

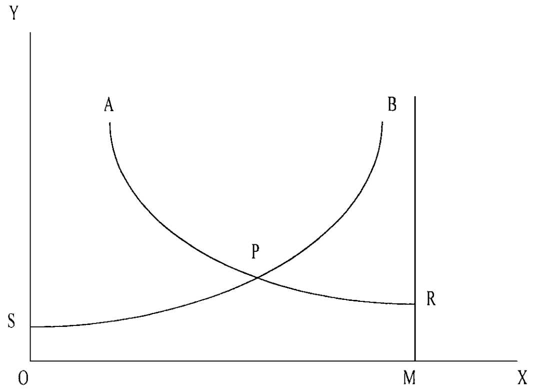

Fig. 1. Lindahl's diagram

> "One party's demand for certain collective goods appears from the other party's point of view as a supply of these goods at a price corresponding to the remaining part of total cost; for collective activity can only be undertaken if the sum of the prices paid is just sufficient to cover the cost. In fact, however, the demand and supply do not concern the collective goods themselves *but only shares therein*."

(The emphasis is ours.) To further explain, Lindahl introduces the diagram in Fig. 1, showing the total costs of parties A and B on the vertical axes and the shares of total cost on the horizontal axis. The value $x \in [0, M]$, where $M \leq 1$, indicates the percentage of the total cost paid by party A while $(1 - x)$ indicates the total cost paid by B. At O, party A pays nothing and the entire total cost, denoted by S, is borne by B. At M, the total cost R is borne by A. The curves SB and AR show the amount of public expenditure each party is prepared to sanction at the various ratios in $[0, M]$. The intersection of the two curves indicates the only distribution of costs at which both parties agree to the level of total costs and associated public good provision.[4] Notice that the shares of the total costs are the primary objects and when these shares are "in equilibrium" then total revenues can cover the total costs for public good provision.

To formally introduce reduced economies we have to extract ratio's from configurations. Every configuration $(\mathbf{x}, y) = ((x^i)_{i \in N}, y) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ has a set of ratios $\mathbf{r}(\mathbf{x}) = (r^i(\mathbf{x}))_{i \in N}$ associated with it defined by

$$r^i(\mathbf{x}) = \begin{cases} \frac{w^i - x^i}{\sum_{j \in N}(w^j - x^j)} & \text{if } \sum_{j \in N}(w^j - x^j) \neq 0 \\ \frac{1}{|N|} & \text{if } \sum_{j \in N}(w^j - x^j) = 0. \end{cases}$$

4 This is also emphasized by Johansen (1963), see his point 6, p. 350.

The definition of the ratios in the case that $\sum_{j \in N}(w^j - x^j) = 0$ is arbitrary. We will not actually encounter this case, so when $\sum_{j \in N}(w^j - x^j) = 0$ we could define the ratios any way we wish (under the restriction that they add up to 1); the ratios need only be well-defined. The following lemma shows that for any ratio-equilibrium configuration $(\mathbf{x}, y) \in R(E)$ the associated ratios $r^i(\mathbf{x})$ are the unique ratios that together with the configuration constitute a ratio equilibrium.

Lemma 1. *Let* $(\mathbf{r}, (\mathbf{x}, y))$ *be a ratio equilibrium of an economy* $E \in \mathscr{E}$. *Then* $w^i - x^i > 0$ *and* $r^i = r^i(\mathbf{x}) = \frac{w^i - x^i}{\sum_{j \in N}(w^j - x^j)}$ *for each* $i \in N$.

Proof. Let $i \in N$. Then $u^i(x^i, y) \geq u^i(\bar{x}^i, \bar{y})$ for all $(\bar{x}^i, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}_+$ satisfying $r^i f(\bar{y}) + \bar{x}^i = w^i$. Also, f is non-decreasing and u^i is strictly increasing in public good consumption. Hence, it follows that $r^i > 0$ must hold. Also, $r^i f(y) + x^i = w^i$, which implies that $w^i - x^i > 0$ and $r^i = \frac{w^i - x^i}{f(y)}$ (notice that $f(y) \in \mathbb{R}_+$). Clearly, $f(y) = \sum_{j \in N} r^j f(y) = \sum_{j \in N} \frac{w^j - x^j}{f(y)} f(y) = \sum_{j \in N}(w^j - x^j) > 0$. This shows that $r^i = \frac{w^i - x^i}{\sum_{j \in N}(w^j - x^j)} = r^i(\mathbf{x})$. ■

We can now formally introduce reduced economies. Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; f \rangle$ be a public good economy and let $S \subseteq N$, $S \neq \emptyset$, and let $(\mathbf{x}, y)$ be a configuration in E. The *reduced economy* of E with respect to S and $(\mathbf{x}, y)$ is

$$E^{S, (\mathbf{x}, y)} = \langle S; (w^i)_{i \in S}; (u^i)_{i \in S}; h \rangle,$$

where

$$h(\bar{y}) = \left[\sum_{i \in S} r^i(\mathbf{x}) \right] f(\bar{y})$$

for all $\bar{y} \in \mathbb{R}_+$. The interpretation of the reduced economy is the following. Suppose all the agents agree on the configuration $(\mathbf{x}, y)$. This implies that they agree on a level of public good production and on a cost-sharing scheme corresponding to the ratios $r^i(\mathbf{x})$. Then, if the agents in $N \backslash S$ withdraw from the decision-making process, the agents in S can reconsider both the way in which they are going to share the costs among themselves and the level of public good to be produced. When they reconsider, they assume that the agents in $N \backslash S$ will pay the share $\sum_{i \in N \backslash S} r^i(\mathbf{x})$ of the cost of producing the public good. Hence, when reconsidering the cost-sharing scheme, the agents in S face the cost function h.

The consistency property is based on reduced economies. A solution ϕ on $\mathscr{E}$ is *consistent* (CONS) if it satisfies the following condition.

If $E \in \mathscr{E}, (\mathbf{x}, y) \in \phi(E)$, and $S \subseteq N(E)$, $S \neq \emptyset$,
then $E^{S, (\mathbf{x}, y)} \in \mathscr{E}$ and $(\mathbf{x}^S, y) \in \phi(E^{S, (\mathbf{x}, y)})$.

Hence, for a consistent solution it holds that once an agreement is reached, the withdrawal of some agents from the decision-making process and the subsequent reconsideration by the remaining agents does not change the outcome of the process. It is shown in the following lemma that the ratio correspondence is a consistent solution.

Lemma 2. *The ratio correspondence on the family $\mathscr{E}$ of public good economies is consistent.*

Proof. Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; f \rangle \in \mathscr{E}$ be a public good economy, let $(\mathbf{x}, y) \in R(E)$, and let $S \subseteq N$, $S \neq \varnothing$. Then by Lemma 1 it follows that $w^i - x^i > 0$, so that $r^i(\mathbf{x}) = \frac{w^i - x^i}{\sum_{j \in N}(w^j - x^j)} > 0$ for each $i \in N$. Let h be the cost function of the reduced economy $E^{S,(\mathbf{x},y)}$; that is, $h(\bar{y}) = \left[\sum_{i \in S} r^i(\mathbf{x})\right] f(\bar{y}) > 0$ for every $\bar{y} \in \mathbb{R}_+$. Note that this implies that $E^{S,(\mathbf{x},y)} \in \mathscr{E}$. Further, $r^i(\mathbf{x}) f(\bar{y}) = r^i(\mathbf{x}) \frac{\sum_{j \in S} r^j(\mathbf{x})}{\sum_{j \in S} r^j(\mathbf{x})} f(\bar{y}) = \frac{r^i(\mathbf{x})}{\sum_{j \in S} r^j(\mathbf{x})} h(\bar{y})$ for all $\bar{y} \in \mathbb{R}_+$. Define $\bar{r}^i = \frac{r^i(\mathbf{x})}{\sum_{j \in S} r^j(\mathbf{x})}$ for each $i \in S$. Since $(\mathbf{x}, y) \in R(E)$, $(\mathbf{r}(\mathbf{x}), (\mathbf{x}, y))$ is a ratio equilibrium of E by Lemma 1. Hence, for all $i \in S$ it holds that $r^i(\mathbf{x}) f(y) + x^i = w^i$ and $u^i(x^i, y) \geq u^i(\bar{x}^i, \bar{y})$ for all $(\bar{x}^i, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}_+$ satisfying $r^i(\mathbf{x}) f(\bar{y}) + \bar{x}^i = w^i$. Now it easily follows that $((\bar{r}^i)_{i \in S}, (\mathbf{x}^S, y))$ is a ratio equilibrium of the reduced economy $E^{S,(\mathbf{x},y)}$. This proves that $(\mathbf{x}^S, y) \in R(E^{S,(\mathbf{x},y)})$. ■

5 An axiomatization using consistency

In this section, we use consistency to provide an axiomatic characterization of the allocations corresponding to the ratio equilibrium, thereby characterizing the cost-sharing rules corresponding to this equilibrium concept and thus providing more insight into the ratio equilibrium. Our axiomatic characterization uses two additional axioms, converse consistency and one-person rationality.

Converse consistency states that if a configuration constitutes an acceptable solution for all subgroups of agents, then it also constitutes an acceptable solution for the group as a whole. Formally, a solution ϕ on $\mathscr{E}$ is *converse consistent* (COCONS) if, for every $E \in \mathscr{E}$ with at least two agents ($|N(E)| \geq 2$) and for every configuration $(\mathbf{x}, y) \in \mathbb{R}_+^{N(E)} \times \mathbb{R}_+$, the following condition is satisfied.

> If $E \in \mathscr{E}$ and for every $S \subseteq N(E)$ with $S \notin \{\varnothing, N(E)\}$ it holds that $E^{S,(\mathbf{x},y)} \in \mathscr{E}$ and $(\mathbf{x}^S, y) \in \phi(E^{S,(\mathbf{x},y)})$, then $(\mathbf{x}, y) \in \phi(E)$.

The ratio correspondence satisfies converse consistency, as is proven in the following lemma.

Lemma 3. *The ratio correspondence on the family $\mathscr{E}$ of public good economies satisfies converse consistency.*

Proof. Let $E = \langle N; (w^i)_{i \in N}; (u^i)_{i \in N}; f \rangle \in \mathscr{E}$ with $|N| \geq 2$ and let $(\mathbf{x}, y) \in \mathbb{R}_+^N \times \mathbb{R}_+$ such that, for every $S \subseteq N$ with $S \notin \{\varnothing, N\}$, it holds that $E^{S,(\mathbf{x},y)} \in \mathscr{E}$ and $(\mathbf{x}^S, y) \in R(E^{S,(\mathbf{x},y)})$. Then $(x^i, y) \in R(E^{\{i\},(\mathbf{x},y)})$ for each $i \in N$. Let $i \in N$ and let h be the cost function of the reduced economy $E^{\{i\},(\mathbf{x},y)}$; that is, $h(\bar{y}) = r^i(\mathbf{x}) f(\bar{y})$ for all $\bar{y} \in \mathbb{R}_+$. Since $(x^i, y) \in R(E^{\{i\},(\mathbf{x},y)})$, we know that $h(y) + x^i = w^i$ and $u^i(x^i, y) \geq u^i(\bar{x}^i, \bar{y})$ for all $(\bar{x}^i, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}_+$ satisfying $h(\bar{y}) + \bar{x}^i = w^i$. Knowing that $h(\bar{y}) = r^i(\mathbf{x}) f(\bar{y})$ for all $\bar{y} \in \mathbb{R}_+$ and noting that we can make similar derivations for every $i \in N$, we find that $(\mathbf{r}(\mathbf{x}), (\mathbf{x}, y))$ is a ratio equilibrium of E, and so $(\mathbf{x}, y) \in R(E)$. ■

In order to characterize the configurations associated with ratio equilibria using consistency and converse consistency, we need a "starting point" – we need to say something about the solution at the level of one-person economies. Thus, we introduce the following property. A solution ϕ on $\mathscr{E}$ satisfies *one-person rationality* (OPR) if, for every one-agent public good economy $E = \langle \{i\}; w^i; u^i; f \rangle \in \mathscr{E}$, it holds that

$$\phi(E) = \{(x^i, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid f(y) + x^i = w^i \text{ and } u^i(x^i, y) \geq u^i(\bar{x}^i, \bar{y}) \text{ for all } (\bar{x}^i, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}_+ \text{ satisfying } f(\bar{y}) + \bar{x}^i = w^i\}.$$

The one-person rationality axiom can be interpreted as dictating that the individual agent maximizes his utility given his endowment of the private good and the cost of producing certain amounts of the public good (which is in this case like a private good to the agent). Such rationality assumptions prevail throughout economics and therefore this property does not set our work apart from other work.

The following lemma shows how the three axioms consistency, converse consistency, and one-person rationality interact.

Lemma 4. *Let ϕ and ψ be two solutions on $\mathscr{E}$ that both satisfy OPR. If ϕ is consistent and ψ is converse consistent, then it holds that $\phi(E) \subseteq \psi(E)$ for all $E \in \mathscr{E}$.*

Proof. We will prove the lemma by induction on the number of agents. If $E \in \mathscr{E}$ is a one-agent economy – $|N(E)| = 1$ – then it follows from OPR of ϕ and ψ that $\phi(E) = \psi(E)$.

Now, let $E \in \mathscr{E}$ be an economy with n agents and suppose that it has already been proven that $\phi(E) \subseteq \psi(E)$ for all economies with less than n agents. Let $(\mathbf{x}, y) \in \phi(E)$. Then, by CONS of ϕ, we know that $E^{S,(\mathbf{x},y)} \in \mathscr{E}$ and $(\mathbf{x}^S, y) \in \phi(E^{S,(\mathbf{x},y)})$ for all $S \subseteq N(E)$, $S \notin \{\emptyset, N(E)\}$. Hence, it follows from the induction hypothesis that $(\mathbf{x}^S, y) \in \psi(E^{S,(\mathbf{x},y)})$ for all $S \subseteq N(E)$, $S \notin \{\emptyset, N(E)\}$. So, by COCONS of ψ, we know that $(\mathbf{x}, y) \in \psi(E)$. We conclude that $\phi(E) \subseteq \psi(E)$. ■

Using Lemma 4, the proof of Theorem 1 follows directly.

Theorem 1. *The ratio correspondence is the unique solution on $\mathscr{E}$ that satisfies OPR, CONS, and COCONS.*

Proof. In Lemmas 2 and 3 we proved that the ratio correspondence satisfies CONS and COCONS. To show that the ratio correspondence satisfies OPR, let $E = \langle \{i\}; w^i; u^i; f \rangle \in \mathscr{E}$ be a one-agent public good economy. Note that in a one-agent economy, the single agent present will have to pay fully for each level of "public good" that he wants to have available. Hence, the set of ratio equilibria of economy E is $\{(1, (x^i, y)) \in \Delta^1 \times \mathbb{R}_+ \times \mathbb{R}_+ \mid f(y) + x^i = w^i$ and $u^i(x^i, y) \geq u^i(\bar{x}^i, \bar{y})$ for all $(\bar{x}^i, \bar{y}) \in \mathbb{R}_+ \times \mathbb{R}_+$ satisfying $f(\bar{y}) + \bar{x}^i = w^i\}$. This proves that the ratio correspondence satisfies OPR.

To prove unicity, assume that ϕ is a solution on $\mathscr{E}$ that also satisfies the three foregoing axioms. Let $E \in \mathscr{E}$ be arbitrary. Then, Lemma 4 shows that $\phi(E) \subseteq R(E)$ by CONS of ϕ and COCONS of the ratio correspondence, and that $R(E) \subseteq \phi(E)$ by CONS of the ratio correspondence and COCONS of ϕ. Hence, $\phi(E) = R(E)$. ■

The ratio equilibrium reflects the equilibrium notion of Lindahl in that agents take as given their shares of the total costs of public good provision. Our notion of consistency is thus, as we argued in Sect. 4, very much in the spirit of Lindahl's original notion. Theorem 1 is a two-edged knife. On the one hand, it states that there is a unique solution satisfying consistency, converse consistency, and one-person rationality. This indicates that there is a unique solution that formalizes Lindahl's ideas while at the same time adhering to the rationality requirements that are basic to most of economics. On the other hand, Theorem 1 identifies the ratio correspondence to be this unique solution.

We conclude this section with the remark that the three axioms used to characterize the ratio correspondence in Theorem 1 are logically independent. This is easily seen by considering the following three solutions on $\mathscr{E}$. The solution ϕ on $\mathscr{E}$ defined by $\phi(E) = \{(\mathbf{x}, y) \mid x^i = w^i$ for each $i \in N(E)$ and $y = 1\}$ satisfies CONS and COCONS, but fails to satisfy OPR. The solution χ on $\mathscr{E}$ defined by $\chi(E) = R(E)$ if $|N(E)| = 1$ and $\chi(E) = \emptyset$ if $|N(E)| > 1$ satisfies OPR and CONS, but does not satisfy COCONS. Finally, the solution ψ on $\mathscr{E}$ defined by $\psi(E) = R(E)$ if $|N(E)| = 1$ and $\psi(E) = \{(\mathbf{x}, y) \in \mathbb{R}_+^{N(E)} \times \mathbb{R}_+ \mid x^i \leq w^i$ for all $i \in N(E)$ and $\sum_{i \in N(E)} (w^i - x^i) = f(y)\}$ if $|N(E)| > 1$ satisfies OPR and COCONS, but does not satisfy CONS.

6 Conclusions

In this paper we provide an axiomatic characterization for the ratio equilibrium cost-sharing rule by means of consistency properties that are in the spirit of Lindahl's original work. This adds to earlier work by Kaneko (1977b), who defined the ratio equilibrium and characterized the ratio correspondence through cores of cooperative games.

In addition, our work reveals further parallels between the theories of public and private goods provision. Our axioms are remarkably similar to those for the Walrasian equilibrium, first given by van den Nouweland et al. (1996). Our characterization of the ratio correspondence has one less axiom than the characterization of the Walrasian equilibrium. The extra axiom in the axiomatization of the Walrasian equilibrium treats two-person economies and has the consequence that all individuals face the same prices for the same commodities. In the axiomatization of the ratio correspondence we do not need such an axiom because the ratios are individualized.

In this paper we have considered a special model in the sense that there is only one private good and one public good. Adding more private goods does not appear to shed light on our study of cost shares[5] – we arrive at the prob-

[5] Of course from the perspective of general equilibrium theory, allowing multiple private goods may be illuminating. Our concern here, however, is to shed light on cost sharing schemes for public goods that take prices for private goods as given.

lem of axiomatizing private goods economies (found in van den Nouweland et al. (1996)) as well as additional problems of public goods economies without reaching any new conclusions.

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