Clubs, Near Markets and Market Games*

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Abstract

In this paper we develop a new model of a replica economy with clubs, with crowding only in production, and with multiple private and public goods. Each agent may belong to many clubs. Our main result is that under apparently mild conditions on the model, economies generate games satisfying per capita boundedness (finiteness of the supremum of per capita payoff). Thus, the results of Wooders (1979,1980b) can be applied to show that (a) for all sufficiently large economies, the game generated by the economy is approximately a market game, and (b) approximate cores are nearly equal treatment in the sense that most agents who are identical are treated approximately identically. When *all* gains to collective activities can be exhausted by groups of agents bounded in size, then stronger results are obtained: (c) for all sufficiently large games, all payoffs in the core have the equal treatment property and (d) approximate cores converge to cores of associated "balanced cover" games. In addition, a discussion of the issues involved in modeling economies as markets is provided and, since much related research has been carried out since the current paper appeared as a 1982 Cowles Foundation Discussion Paper, we provide some relationships to the more recent literature.

*This paper originally appeared as Cowles Discussion Paper No. 657 (1982c). The nonemptiness result is published in Shubik and Wooders (1986). The results reported in the current version of the paper are identical to those in the 1982 Discussion Paper. We have updated references and revised and shortened some of the exposition. Also, proofs from Wooders (1979,1980b), available elsewhere (cf. Wooders 1994a,b or the original papers, available on the internet), have been removed.

1. Introduction

1.1. Market Games

The market game, both in its sidepayment and nosidepayment versions, has provided a valuable tool for the utilization of game theoretic analysis for the study of exchange economies. It is known that exchange economies map into totally balanced games[1] for both sidepayments and nosidepayments. In addition, totally balanced sidepayment games map into exchange economies and, with special assumptions on consumption sets, totally balanced nosidepayment games also map into exchange economies.[2] Clearly the mappings are not one to one in both directions, as the market game contains far less information than does the exchange economy.

The intimate relationship between exchange economies and totally balanced games gives no intimation of the relationship of the competitive equilibria to limit points of the core. In order to study this property we need to be able to define a sequence of exchange economies and a related sequence of totally balanced games. The result that we can define a replication sequence of exchange economies and associate with it a sequence of totally balanced games for which cores converge to competitive payoffs is an economic result arising from having utilized the economic data to construct the sequence of games.

1.2. The purpose of this paper

A natural question to ask is, are there other economic phenomena which give rise to market games or "near-market games" and from which one can construct a sequence of games which exhibits core convergence. In this paper we consider the relationship of market games and near-market games to economies with complexities beyond that of the exchange economy. We show that a broad class of economies generate near-market games, including private goods economies with non-convexities, coalition production economies, and economies with local public goods.[3] We also illustrate that economies with pure public goods do not, without special restrictions, give rise to near-market games.

To make this more precise, some terminology is required. A *sequence of replica games* is a sequence of games with a fixed distribution of a finite number of types. The r^{th} replica game contains r times as many players of each type as the first game in the sequence. The sequence satisfies *minimum efficient scale* (or, equivalently, exhaustion of gains to scale by coalitions bounded in size) if there is some given bound satisfying the condition that all returns to coalition size can be exhausted by coalitions containing fewer members than the bound. A sequence of replica games is *a sequence of near-market games* if the games are superadditive and the sequence satisfies a "near - minimum efficient scale for coalitions" property – for large bounds, almost all returns to coalition size are nearly exhausted by coalitions bounded in

[1] We note that every subgame of a totally balanced game has a non-empty core

[2] See Billera and Bixby (1974) and references therein.

[3] See also Wooders (1979,1980a,b).

size. Equivalently, the sequence satisfies *per capita boundedness*, the property that the supremum of per capita payoffs is finite.

Wooders (1980b) shows that if a sequence of games satisfies per capita boundedness then the sequence is asymptotically totally balanced – given any epsilon greater than zero and any subgame of any game in the sequence, when we replicate the set of players in that subgame, for all sufficiently large replications the epsilon-core of the replicated subgame is non-empty.[4] Moreover, epsilon-cores of large games converge to competitive payoffs of limiting market games and thus have an equal-treatment property, that is, most players who are identical receive nearly the same payoffs. Under the stronger assumption of minimum efficient scale for coalitions, for all sufficiently large games, all payoffs in the core have the equal treatment property and approximate cores converge to the (equal-treatment) cores of the associated balanced cover games.

In this paper, a model of a sequence of replica economies with coalition production and local public goods, where agents are allowed to be members of possibly more than one club, is developed. There may be multiple public and private goods. No convexity assumptions are required, either in consumption or production. For simplicity, the model allows crowding only in production, that is, the production possibilities open to a club depend on the membership of the club but preferences are unaffected by club memberships.[5] Few restrictions are placed on the model; the major ones are that the asymptotic growth of utility functions is no more than linear and the production correspondences are such that positive outputs do not become virtually free in per-capita terms as the economies become large. We show that the sequence of games derived from the economies is a sequence of near-market games. From the results of Wooders (1980b) it follows that for all sufficiently large economies, approximate cores are nonempty and converge to the equal-treatment core – that is, in large games approximate cores treat almost all identical players nearly equally.

When we further assume that *all* increasing returns to coalition size are realized by some finite economy – there is a minimum efficient scale of coalitions rather than a near minimum efficient scale – we obtain a stronger core convergence result. The ε-cores of the derived sequence of games converge to the cores of the associated balanced cover games. These results depend on Wooders (1979) results showing that when there is a minimum efficient scale, the limit of ε-cores equals the limit of the cores of the associated balanced cover games.

An investigation of competitive equilibria (or competitive-like equilibria since we

[4]Results in the same spirit are contained in Wooders (1980a). These, however, are for economies with a local public good and anonymous crowding – individuals are crowded only by the *numbers* of individuals in their jurisdiction/club. This implies that equilibrium jurisdictions/clubs/coalitions consist of individuals who are identical or who have the same equilibrium "demands" for crowding and public goods. While the anonymous crowding model is economically interesting, from the perspective of game theory it is a significantly more restrictive than general games with a finite number of types, as in Wooders (1980b). We thank Yakar Kannai for stressing that the relationship between Wooders (1980a) and 1980b) should be discussed.

[5]This significantly simplifies the model. If we allowed preferences of an agent to depend on the sizes and compositions of the clubs containing that agent, as in Wooders (1981b), for example, the analysis would be more complicated but, with appropriate modifications of assumptions, the same results would still obtain For some recent related results, see Conley and Wooders (1998b) and Kovalenkov and Wooders (1997a,b).

are dealing with a class of economies to which the classic definition of the competitive equilibrium does not apply[6]) and convergence of the core to the equilibrium payoffs is beyond the scope of this paper. We note however that since the economies under consideration generate games with side payments, from our results and those of Shapley and Shubik (1969) it follows that large economies can be represented by markets where each player owns one unit of a commodity identified with his "type." Since, under our assumptions, approximate cores treat similar players similarily it follows that approximate cores converge to equilibrium outcomes of the representing markets.[7]

Except under some very special conditions (satiation or "asymptotic" satiation, i.e., marginal utilities go to zero as the amount of the public good increases) replica economies with pure public goods do not generate near-market games. Since games derived from economies with pure public goods may well be market games (i.e., totally balanced),[8] this suggests that, in line with the incentives literature, it is when we consider economies with many players that game-theoretic properties of private-goods or "market-like" economies and those of pure public goods economies differ.

Another question to be addressed is when is an economic situation "adequately" represented by the characteristic function of the derived game? What is "adequate" is, to some extent, a matter of opinion and depends on the ultimate purpose of the model. For our purposes, we say an economic situation has the "c-game" property[9] (i.e., in our view is adequately represented by its derived game), if what a coalition can achieve once it has formed is independent of the actions of the complimentary coalition. Exchange economies obviously have the c-game property; those with pure public goods might not. Those with coalition production and local public goods raise modelling and interpretation problems concerning their "c-gameness"; this will be discussed later.

As indicated above, in this paper we analyze replication sequences of economies – ones with a fixed number of types of agents and increasing numbers of agents of each type. To extend the analysis to economies with a continuum of agents or with simply a "large" number of agents appears to pose different problems than extensions of this nature for private goods exchange economies.[10] The technical results in this paper are based on results concerning non-emptiness and convergence of approximate cores of large replica games (games with a fixed distribution of a finite number of player types). The extension of the analysis to economies with a large number of agents (but not necessarily replica economies) could, we believe, be carried out if results

[6] Some illustrations of such convergence are demonstrated in Boehm (1974) for economies with coalition production and in Wooders (1980a,1981b) for ones with local public goods. See also Bennett and Wooders (1979) where the results of Wooders (1979) are applied to questions of firm formation. For more recent results see, for example, Conley and Wooders (1998a) and references therein.

[7] For the market explicitly constructed by Shapley and Shubik where there is only one player of each type, the equivalence of the core and the equilibrium outcomes holds in finite economies. This is no longer true in the setting of our model. But in the limit, core-equilibrium equivalence obtains. See Wooders (1994a,b) for related results.

[8] See Rosenthal (1976).

[9] See Shubik (1982, pp. 130-131, 354).

[10] Since this paper was written, some such results have been obtained. See especially Kaneko and Wooders (1986) and Conley and Wooders (1998b) and further references in these papers.

were available for large (not-necessarily replica) games analogous to those for large replica games.[11] Our purpose in this paper is to introduce concepts of near-market economies and near-market games; it is not to obtain the most general results that might be possible.

The models are also restricted to ones with a freely transferable medium of exchange. As has been shown, such a restriction is not necessary.[12]

In Section 2 a mathematical structure and analysis for near-market games is presented. Section 3 contains our model of a sequence of replica economies whose derived games are near-market games. Section 4 deals with the application and interpretation of near-market games together with a closer examination of the modelling problems involved. Section 5 discusses some of the subsequent literature on market games and on economies with clubs. All proofs are contained in the appendices.

1.3. Near Markets

In the remainder of this paper we argue that for large economies, with appropriate conditions essentially limiting the effects of externalities and indivisibilities, near-market games provide a valuable tool for analysis of the first six cases described in Table 1. For nonlocal public goods we suggest that the physical conditions require a direct modelling of the voting or other socio-political mechanism. This is noted in discussion of several special cases in Section 4.

Even though our analysis is carried out in terms of games with sidepayments we conjecture that if there exists a "quasi-transferable" utility (i.e., a good which is divisible, always desirable and a substitute for any other good) our results hold substantially.[13]

2. Near-Market Games

2.1. Games

We first review some game-theoretic concepts.

A **game** (with sidepayments) is an ordered pair (N, v) where $N = \{1, \ldots, n\}$ is a finite set, called the set of **players** and v is a real-valued function mapping subsets of N into $\mathbf{R}_+$ with $v(\phi) = 0$. Two players i and j are **substitutes** if given any subset S of N where $i \notin S$ and $j \notin S$, we have

$$v(S \cup \{i\}) = v(S \cup \{j\}).$$

[11] Some such results have been obtained. See, for example, Wooders and Zame (1984) and Kovalenkov and Wooders (1997a,b).

[12] In this regard, see Wooders (1981a) and Shubik and Wooders (1981a,b). The assumption of a freely transferable medium of exchange simplifies the analysis while allowing quite complex economic sutations. Recent relevant papers include Kovalenkov and Wooders (1997a,b) and Conley and Wooders (1998b).

[13] This conjecture, when made, was based on the fact that analogous development of approximate core theory has been carried out for games without side payments (see Wooders 1981a and Shubik and Wooders 1982b) and some results in the spirit of those herein had been obtained (see Wooders 1980a,1981a). Recent relevant results include Conley and Wooders (1998b).

A **subgame** (S, v) of (N, v) is an ordered pair consisting of a non-empty subset S of N and the function v restricted to subsets of S. The game (N, v) is **superadditive** if for all disjoint subsets S and S' of N, we have

$$v(S) + v(S') \leq v(S \cup S').$$

A **payoff** for the game is a vector $B = (B^1, \ldots, B^n) \in \mathbf{R}^n_+$, the non-negative orthant of the n-fold Cartesian product of the reals. A **payoff** B is feasible if

$$\sum_{i \in N} B^i \leq v(N).$$

Given $\varepsilon \geq 0$, a payoff is in the (weak) ε-**core**[14] if it is feasible and if, for all non-empty subsets S of N,

$$\sum_{i \in S} B^i \geq v(S) - \varepsilon \mid S \mid$$

where $\mid S \mid$ denotes the cardinal number of the set S. When $\varepsilon = 0$, the ε-core is called simply the **core**.

Given a game (N, v), let $(N, \widetilde{v})$ denote the **totally balanced cover** of (N, v); the function $\widetilde{v}$ is the smallest real-valued function such that, for all non-empty subsets S of N, the subgame $(S, \widetilde{v})$ has a non-empty core and $v(S) \leq \widetilde{v}\,(S)$.

2.2. Sequences of Games

A sequence of games $(N_r, v_r)_{r=1}^{\infty}$ is **superadditive** if each game (N_r, v_r) is superadditive. The sequence is **per-capita bounded** if there is a constant K, independent of r, such that

$$v_r(N_r)/ \mid N_r \mid \leq K$$

for all r.

Let $(N_r, v_r)_{r=1}^{\infty}$ be a sequence of games where, for some positive integer T, for each r the set of players N_r contains rT players denoted by $N_r = \{(t, q) : t = 1, \ldots, T,\ q = 1, \ldots, r\}$. For each r and each t, let $[t]_r = \{(t, q) : q = 1, \ldots, r\}$. The sequence is a **sequence of replica games** if

(1) $N_r \subset N_{r+1}$ for all r;

(2) for each r and all subsets S of $N_r, v_r(S) \leq v_{r'}(S)$ whenever $r' \geq r$;

(3) for each r and each t all players in $[t]_r$ are substitutes for each other.

Note that positive externalities are allowed in the sense that what a coalition can achieve may increase when the total player set increases in size: Negative externalities could also be allowed. For the purposes of this paper, what is crucial is that the sequences $\frac{v_r(N_r)}{|N_r|}$ and $\frac{v_r(s)}{|s|}$ converge to limiting values. This is discussed further in Section 5.

[14] This concept was introduced by Shapley (1966).

Throughout the following, given a sequence of replica games $(N_r, v_r)_{r=1}^{\infty}$ we define $[t]_r$ as above and call the members of $[t]_r$ **players of type** t. We also assume there are T types of players and denote the set of players N_r as above.

Give a sequence of replica games $(N_r, v_r)_{r=1}^{\infty}$ and a subset S of N_r for some r, define the vector $\rho(S) = (s_1, \ldots, s_T)$ by its coordinates $s_t = \mid S \cap [t]_r \mid$; the vector $\rho(S)$ is called the **profile of** S and is simply a list of the numbers of players of each type contained in S. Let I denote the T-fold Cartesian product of the non-negative integers. Observe that for any r and any subset S of N_r, we have $\rho(S) \in I$. Also, since players of the same type are substitutes, if S and S' are two subsets with the same profiles then for any r such that $S \subseteq N_r$ and $S' \subseteq N_r$, we have $v_r(S) = v_r(S')$. Consequently the function v_r can be completely determined by a mapping from a subset of I to the reals. In the following, given r and a profile s of a subset of N_r, we define $v_r(s)$ as $v_r(S)$ for any $S \subseteq N_r$ with $\rho(S) = s$.[15] Given $s \in I$ we write $\mid s \mid = \sum_{t=1}^{T} s_t$ since when $\rho(S) = s$, we have $\mid S \mid = \sum_{t=1}^{T} s_t$.

We say a sequence of games satisfies the property of "**near minimum efficient scale**" (for coalitions), NMES, if it is per-capita bounded. If the sequence is also superadditive, we say the sequence is a sequence of **near market games**.

We remark that when a sequence of replica games is per-capita bounded and superadditive, both $\widetilde{v}(N_r)/\mid N_r \mid$ and $v(N_r)/\mid N_r \mid$ converge and to the same limit. In this case, for r sufficiently large, the per-capita gains to forming a coalition larger than N_r are "small." This motivates our term "near-minimum efficient scale."

A sequence of replica games $(N_r, v_r)_{r=1}^{\infty}$ is **asymptotically totally balanced** if, given any r, any subset S of N_r, and any $\varepsilon > 0$, there is an n^* such that for all $n \geq n^*$ we have

$$\frac{\widetilde{v}_{nr}(S_n)}{\mid S_n \mid} - \frac{v_{nr}(S_n)}{\mid S_n \mid} < \varepsilon,$$

where v_{nr} denotes the function v_r, with $r' = nr$ and S_n is any subset of N_r, with $\rho(S_n) = n\rho(S)$. It can easily be verified (and follows from well-known results, cf. Shapley (1967)), that given any game (N, v), we have

$$\frac{\widetilde{v}(N)}{\mid N \mid} - \frac{v(N)}{\mid N \mid} < \varepsilon$$

if and only if the ε-core of the game is non-empty. Consequently, given any subset S of N_r for some r and any sequence of subsets (S_n) satisfying the properties required above, for all n sufficiently large the subgames (S_n, v_{nr}) have non-empty ε-cores.

The following theorem provides sufficient conditions for asymptotic total balancedness of sequences of replica games.

Theorem 2.1. *(Wooders 1980b).*[16] *Let $(N_r, v_r)_{r=1}^{\infty}$ be a sequence of near market games. Then the sequence is asymptotically totally balanced.*

[15] This abuse of notation should create no confusion. We note that we typically denote subsets by upper case letters and profiles by lower case ones.

[16] This result also follows from Wooders (1981a) since games with side payments are a special case of games without side payments.

For a proof, we refer the reader to Wooders (1980b, Theorem 3) or Wooders (1994b, Theorem 3.2).

2.3. Convergence of Cores

In the remainder of this section, we consider convergence properties of ε-cores of sequences of replica games. These results are from Wooders (1979,1980b).[17]

The next theorem shows that as r goes to infinity and ε goes to zero, in "the limit", only equal-treatment payoffs are in the ε-cores where an **equal treatment payoff** x for N_r has the property that, for each t, $x^{tq} = x^{tq'}$ for all q and q'.

Theorem 2.2. *Wooders (1980b). Let $(N_r, v_r)_{r=1}^{\infty}$ be a sequence of near-market games. Then, given any $\delta > 0$ and any $\lambda > 0$, there is an ε^* and an r^* such that for all $\varepsilon \in [0, \varepsilon^*]$ and for all $r \geq r^*$, if B is in the ε-core of (N_r, v_r) then*

$$| \{(t,q) \in N_r : \| B^{tq} - \overline{B}_t \| > \delta\} | /r < \lambda$$

where $\overline{B}_t = \sum_{q=1}^{r} B^{tq}/r$ and $\| \cdot \|$ denote the absolute value.

Less formally, Theorem 2.2 states that for small ε and large r, any payoff in the ε-core has the property that the percentage of players whose payoffs differ significantly (by more than δ) from the average payoff for their type can be made arbitrarily small (less than λ). We remark that since the theorem holds for $\varepsilon \geq 0$, it follows that for all sufficiently large r any payoff in the core of the r^{th} balanced cover game has "nearly" the equal-treatment property. For a proof of the result, we refer the reader to Wooders (1980b, Theorem 5) or to Wooders (1994b, Proposition A.1.2).

Given a sequence of replica games $(N_r, v_r)_{r=1}^{\infty}$, let B be an equal-treatment payoff in the ε-core of (N_r, v_r) for some r and some $\varepsilon \geq 0$. Since, for each t, $B^{tq} = B^{tq'}$ for all q and q', B can be completely described by a vector $\overline{B} \in \mathbf{R}^T$ where, for each t, $\overline{B}_t = B^{tq}$ for (any) q. We say that $\overline{B}$ **represents** an equal-treatment payoff in the ε-core of (N_r, v_r). Given $\varepsilon > 0$, for each r let

$$\begin{aligned} \mathcal{B}_r(\varepsilon) &= \{\overline{B} \in^T : \overline{B} \text{ represents an equal treatment} \\ &\qquad \text{payoff in the } \varepsilon - \text{core of } (N_r, v_r)\} \end{aligned}$$

.

Given $\varepsilon > 0$, let $L(\mathcal{B}(\varepsilon))$ denote the closed limit[18] of the sequence of sets $(\mathcal{B}_r(\varepsilon))_{r=1}^{\infty}$. In the appendix it is shown that this limit exists. Define

$$A^* = \cap_{\varepsilon > 0} L(\mathcal{B}(\varepsilon)); \tag{2.1}$$

[17] In Wooders (1992) the results of this section are shown to hold for private goods exchange economies with quasi-linear utilities,

[18] See Hildenbrand (1974) , pp. 15-17 for a definition and some properties of the closed limit of a sequence of sets.

then A^* is the **asymptotic core** of (N_r, v_r). Informally, A^* is the limit of the ε-cores and $B^* \in \mathbf{R}^T$ is in the asymptotic core if, given any $\varepsilon > 0$, there is a sequence $(\overline{B}_r^{\varepsilon})_{r=1}^{\infty}$ where $\overline{B}_r^{\varepsilon} \in \mathcal{B}_r(\varepsilon)$ for each r and $\overline{B}_r^{\varepsilon} \to B^*$ as r goes to infinity.

Given a sequence of replica games, $(N_r, v_r)_{r=1}^{\infty}$, let $\mathcal{C}(r)$ denote the subset of $\mathbf{R}^T$ where each member C of $\mathcal{C}(r)$ represents an equal-treatment payoff in the core of the balanced cover game $(N_r, \widetilde{v}_r)$. Let $L(\mathcal{C})$ denote the closed limit of the sequence of sets $(\mathcal{C}(r))_{r=1}^{\infty}$.

An example in Wooders (1980b) illustrates that approximate cores do not necessarily converge to cores of the corresponding balanced cover games (or the cores of the games themselves, if nonempty). The example is simple: Let $T = 2$. For $s \in I$ define

$$v(s) = \left\{ \begin{array}{ll} \min\{s_1 - 1, s_2\} & \text{when } s_1 \geq 1 \\ 0 & \text{otherwise.} \end{array} \right\}$$

Let the profile of N_1 be $m = (1, 1)$. The only equal-treatment payoff in the core of the rth replica game is represented by $(\frac{r-1}{r}, 0)$. It follows that $L(\mathcal{C}) = (1, 0)$. It can be verified, however, that A^* is the set $\{(p_1, p_2) : p_1 \geq 0,\ p_2 \geq 0 \text{ and } p_1 + p_2 = 1\}$. Informally, for the ε-core, the fact that there is an "excess" player of type 1 is virtually irrelevant, but the cores of the games depend crutially on the exact proportions of players of each type. As shown in Wooders (1980b), under the assumption of per capita boundedness, the asymptotic core is equivalent to the core of a certain continuum limit game, where a set of measure zero can be ignored. For the purposes of the current paper, since we do not consider continuum economies, we will only appeal to the following convergence result.

We now state conditions under which $L(\mathcal{C})$ exist and equal A^*. We say a sequence of superadditive replica games $(N_r, v_r)_{r=1}^{\infty}$ has a **minimum efficient scale** for coalitions, MES, if there is an r^* such that for all $r \geq r^*$ and $r' \geq r^*$ we have $\widetilde{v}_r\ (N_r)/r = \widetilde{v}_{r'}\ (N_{r'})/r$.[19] We call the replication number r^* an **MES bound** (a minimum efficient scale bound). We remark that when r^* is an MES bound for the sequence, given $r > r^*$ if B is a payoff in the core of (N_r, v_r) then B has the equal-treatment property; this is easily verified (and shown in Wooders (1979, Theorem 6) and Wooders (1983, Theorem 3) for games without side payments).

Theorem 2.3. *(Wooders 1979b). Let $(N_r, v_r)_{r=1}^{\infty}$ be a sequence of near-market games. If the sequence has the MES property, then $L(\mathcal{C})$ exists and equals A^*.*

See Wooders (1979) or (1980b, Theorem 7) for a proof.

3. Near-market economies

3.1. Introduction to Near-Market Economies

In this section we develop a model of a sequence of replica economies with private goods, local public goods, and coalition production. Minimal restrictions are imposed

[19] Alternative formulations of MES are contained in Wooders (1979,1981a) and Shubik and Wooders (1982a).

on the model yet we are able to show that the sequence of derived games is superadditive and per-capita bounded and thus the sequence is asymptotically totally balanced. Therefore, the class of replication economies we consider are near-market economies — for large replications the derived games are near-market games.

3.2. The Model

Let us begin with an informal discussion of the model. There are a finite number T of player types and, in the r^{th} game, there are r players of type t.[20] There are ℓ private commodities and m public goods. The utility functions of agents are defined over $\mathbf{R}_+^{\ell+m}$. (This implies that preferences of agents are unaffected by the memberships of clubs.) A club structure of a set S of agents is a collection of subsets of S, called *clubs*. For each subset of agents, there is a given admissible set of club structures. It may be, for example, that the only admissible club structure of S is a partition of S. Alternatively, a club structure may allow an agent to belong to many clubs. Admissible club structures are required to satisfy certain natural properties. These properties imply that the game derived from the economy is superadditive. Given an admissible club structure, each club in the structure may produce public goods for the benefit of its members. The production possibilities available to a club depend on the club structure. Any two clubs with the same number of players of each type have the same production technology available to them. Similarily, private goods are produced by firms. A firm structure of a set S of agents is a collection of subsets of S. Admissible firm structures are required to satisfy the same properties as admissible club structures.

A **sequence of replica economies** $(\mathsf{E}_r)_{r=1}^{\infty}$ is defined as a sequence of septuples

$$\mathsf{E}_r = (N_r, \mathbf{R}^\ell, \mathbf{R}_+^m, U_r, W_r, (\mathcal{J}_r, Z_r), (\mathcal{F}_r, Y_r))$$

where:

$N_r = \{(t,q) \ : \ t = 1, \ldots, T, \ q = 1, \ldots, r\}$ is the set of agents;

$\mathbf{R}_+^\ell$ is the private commodity space;

$\mathbf{R}_+^m$ is the public commodity space;

$U_r = \{u^{tq} \ : \ (t,q) \in N_r\}$ is an indexed collection of utility functions mapping $\mathbf{R}_+^{m+\ell}$ into $\mathbf{R}_+^1$ with the property that for some linear function L and some real number c we have $u^{tq}(x,y) \leq L(x,y) + c$ for all (x,y) in $\mathbf{R}_+^{m+\ell}$ and for all (t,q) in N_r and $u^{tq} = u^{tq'}$ for all $q, q' \in 1, ..., r$.

$W_r = \{w^{tq} \in \mathbf{R}^\ell \ : \ (t,q) \in N_r\}$ is an indexed collection of endowment vectors, each in $\mathbf{R}_+^\ell$ (no public goods are initially endowed) and $w^{tq} = w^{tq'}$ for all $q, q' \in 1, ..., r$;

$(\mathcal{J}_r, Z_r)$ is a pair of correspondences, where

[20] This is not as restrictive as it might seem, since players of the different types may be identical.

$\mathcal{J}_r$, called the **allowable club structure correspondence**, maps non-empty subsets S of N_r into non-empty subsets of S and

Z_r, called the **public goods production correspondence**, maps subsets S of N_r into subsets of $\mathbf{R}^m_+ \times \mathbf{R}^\ell$; and

$(\mathcal{F}_r, Y_r)$ is another pair of correspondences where

$\mathcal{F}_r$, called the **allowable firm structures correspondence,** maps non-empty subsets S of N_r into collections of subsets of S, and

Y_r, the **private goods production correspondence**, maps non-empty subsets S of N_r into subsets of $\mathbf{R}^\ell$.

A number of further specifications are made on the components of a sequence of replica economies:

(1) $N_r \subset N_{r+1}$ for each r;

(2) For each t, each q and q' in $\{q'' \ : \ q'' = 1, \ldots r\}$, and all r, $u^{tq} = u^{tq'}$; i.e., all agents of the same type have the same utility functions and the same initial endowments. Also, $u^{tq}(w^{tq}, 0) > 0$ for all $(t,q) \in N_r$ and for all r (this assumption is for technical convenience);

(3) Given r and $S \subseteq N_r$, and $J(S) \in \mathcal{J}_r(S)$, $J(S)$ is **an allowable club structure of** S. Allowable club structures $J(S)$ are required to satisfy the properties that

(a) $S \subseteq \cup_{S' \in J(S)} S'$ (allowable club structures of S cover S)

(b) if $J(S) \in \mathcal{J}_r(S)$ and $J(S') \in \mathcal{J}_r(S')$, where S and S' are non-empty, disjoint subsets of agents, then $\{S'' \subseteq N_r \ : \ S'' \in J(S) \cup J(S')\} \in \mathcal{J}_r(S \cup S')$;

(c) given $S \subseteq N_r$ and $r' \geq r$, if $J(S)$ is in $\mathcal{J}_r(S)$; then $J(S)$ is in $\mathcal{J}_{r'}(S)$;

(d) if S and S' are non-empty subsets of N_r with the same profiles, then there is a one-to-one mapping, say ψ, of $\mathcal{J}_r(S)$ onto $\mathcal{J}_r(S')$ such that if $\psi(J(S)) = J(S')$, then the collection of profiles of members of $J(S)$ (not all necessarily distinct), equals those of $J(S')$.

Condition (a) simply ensures that allowable club structures of a subset S of agents cover S. Condition (b) ensures that the game derived from the economy is superadditive. Condition (c) ensures that the possible club structures of a subset of agents does not decrease when the size of the economy increases, and finally, (d) implies that admissible club structures depend only on profiles and not names of individual agents.

We remark that in Wooders (1980a) the allowable club structures of a set of agents is the set of all partitions of agents. The present formulation of allowable club structures is sufficiently general to include the possibility that $\mathcal{J}_r(S)$ is the set of all partitions of S and also to include the possibility of "over-lapping" clubs (an agent could possibly belong to several clubs simultaneously).

(4) The public goods production correspondence Z_r is required to satisfy the properties that

(a) given $S \subseteq N_r$ and $r' > r$, $Z_r(S) \subseteq Z_{r'}(S)$;

(b) if, for any r, S and S' are non-empty subsets of N_r with the same profiles then $Z_r(S) = Z_r(S')$, i.e., the production possibility set for a subset of N_r depends only on the profile of that coalition;

(c) $0 \in Z_r(S)$ for all non-empty subsets S of N_r and for all r.

(5) Given a subset S of N_r, **allowable firm structures** $F(S) \in \mathcal{F}_r(S)$ are required to satisfy the same properties as allowable club structures, i.e., (a)-(d) of (3) above. Also, the private goods production correspondence is assumed to satisfy the same properties as the public goods production correspondence, (a)- (c) of (4) above.

For private goods, we also define the **aggregate production correspondence**. Given r and $S \subseteq N_r$, define

$$\overline{Y}_r(S) = \bigcup_{F(S) \in \mathcal{F}_r(S)} \sum_{S' \in F(S)} Y_r(S') \ :$$

then $\overline{Y}_r(\cdot)$ is the r^{th} **aggregate production correspondence**. Note that $\overline{Y}_r(\cdot)$ is **superadditive**; given any two disjoint, non-empty subsets S and S' of N_r, we have $\overline{Y}_r(S) + \overline{Y}_r(S') \subseteq \overline{Y}_r(S \cup S')$.

The above consists of a description of the components of a sequence of replica economies and members of the sequence. In the following, we introduce additional definitions which enable us to relate production decisions to consumption decisions and to define feasible states of an economy.

An N_r-**allocation** is a vector

$$(x, y) = (x^{11}, \ldots, x^{Tr}, y^{11}, \ldots, y^{Tr}) \in \mathbf{R}_+^{Tr(m+\ell)}$$

where (x^{tq}, y^{tq}) is a commodity bundle for the $(t, q)^{th}$ agent. Given any r and any non-empty subset S of N_r, an S-**allocation** is an N_r-allocation where $x^{tq} = 0$ and $y^{tq} = 0$ if $(t, q) \notin S$.

Given r and a non-empty subset S of N_r, an S-**private goods production plan** is a vector $y \in \overline{Y}_r(S)$.

It is possible, and obvious, that we could define an aggregate public goods production correspondence analogously to the definition of the aggregate private goods production correspondence. However, because of the "non-transferability" of public goods produced in one club to the members of another club, we proceed differently in defining public goods production plans. In particular, we keep track of the club structure associated with a public goods production plan. Given a non-empty subset S of N_r, an S-**public goods production plan** is an ordered pair,

$$\varphi(S) = (J(S), \{(x_{S'}, z_{S'}) \in Z_r(S') \ : \ S' \in J(S)\})$$

where $J(S) \in \mathcal{J}_r(S)$.

Given r and a non-empty subset S of N_r, and S-**state of the economy**, $e(S)$, is an ordered triple,

$$e(S) = (\overline{y}, \varphi(S), (x, y))$$

where $\overline{y}$ is a private goods production plan for S, $\varphi(S)$ is an S-public goods production plan, and (x, y) is an S-allocation where

$$x^{tq} = \sum_{\substack{tq \in S' \\ S' \in J(S)}} x'_S, \text{ for all } (t, q) \in S$$

(the $(t, q)^{th}$ agent consumes the total outputs of public goods produced in all clubs of which he is a member). The state is S-**feasible** if

$$\sum_{(t,q) \in S} (y^{tq} - w^{tq}) + \sum_{s' \in J(S)} z_{S'} = \overline{y}.$$

An N_r-state of the economy is called simply **a state of the** r^{th} **economy**, or, when no confusion is likely to arise, simply a **state of the economy**.

Given $S \subseteq N_r$, let

$$\begin{aligned} A_r(S) \quad = \quad & \{(x, y) \text{ : there is an } S\text{-feasible state of the economy} \\ & \text{with associated } S\text{-allocation } (x, y)\}. \end{aligned}$$

The set $A_r(S)$ is called the set of S-**attainable allocations**.

$$\text{Proj}_S A_r(S) \subseteq \text{Proj}_S A_{r'}(S)$$

where $\text{Proj}_S A_{r''}(S)$ denotes the projection of the set $A_{r''}(S)$ onto the subset of $\mathbf{R}^{|S|(m+\ell)}$ associated with the members of S for any replication number r''.

3.3. The Derived Games

We now define the sequence of games derived from the sequence of economies.

Given r and $S \subseteq N_r$, define

$$v_r(S) = \sup_{(x,y) \in A_r(S)} \sum_{(t,q) \in S} u^{tq}(x^{tq}, y^{tq}) \text{ when } S \neq \phi \text{ and}$$

$$v_r(S) = 0 \text{ otherwise.}$$

Observe that the pair (N_r, v_r) is a game with sidepayments. The generation of a game with sidepayments presumes, as usual, the existence of a freely transferable medium of exchange. There is no need to introduce this explicitly.[21] It is straightforward to verify that the sequence of derived games is a sequence of replica games.

[21] To explicitly introduce the medium of exchange, add another private good to the consumption set, but allow this to be achieved in negative as well as positive amounts, so that the consumption set is $\mathbf{R}_+^\ell \times \mathbf{R}_+^m \times \mathbf{R}$ and take the utility functions as $\widehat{u}^{tq}(x, y, \xi) = u^{tq}(x, y) + \xi$.

3.4. Near-Market Economies

Without further restrictions on the economies, in particular on production, there is no assurance that the derived sequence of games is a sequence of near-market games. The restrictions required are, informally, that positive production does not become virtually free as the economies become large. Formally, we assume

A1. There is a closed convex cone $Y^* \subset \mathbf{R}^\ell$, with $-\mathbf{R}^\ell_+ \subseteq Y^*$ and $Y^* \cap \mathbf{R}^\ell_+ = \{0\}$, such that $Y_r(S) \subseteq Y^*$ for all subsets S of N_r and for all r.

A2. There is a closed convex cone $Z^* \subseteq \mathbf{R}^m_+ \times \mathbf{R}^\ell$, with $\{0\} \times -\mathbf{R}^\ell \subseteq Z^*$ and $Z^* \cap \mathbf{R}^m_+ \times \mathbf{R}^\ell_+ = \{0\}$, such that for any r, any non-empty subset $S \subseteq N_r$, and any allowable club structure $J(S) \in \mathcal{J}_r(S)$, we have

$$\sum_{S' \in J(S)} (x_{S'}, z_{S'}) \in \{(x,z) \in R^{m+\ell} \ : \ (\mid S \mid x, z) \in Z^*\}.$$

The first assumption is clear. The second is that there is some set Z^* satisfying the properties of a standard private goods production set and public goods are never cheaper per capita than private goods would be if they were produced with the public set Z^*.

An example of production correspondences which satisfy A2 in the one-private-good, one-public-good case is given by the production functions $x + z / \mid S \mid = 0$ for each subset S of N_r and for all r with $\mathcal{J}_r(S)$ equal to the set of partitions of S. Here the per capita costs of the public good, in terms of the inputs, is constant and independent of the size and composition of the club structure.

To see what A2 rules out, suppose all coalitions have the production set Z determined by the production function $x + z = 0$, there is only one private good and one public good, and again $\mathcal{J}_r(S)$ is the set of all partitions of S. To show that A2 is not satisfied, let $x_r = 2/r$ and $z_r = -(2/r)$ for each positive integer r. Observe that $(x_r, z_r) \in Z(S)$ for all $S \subseteq N_r$ and for all r. Choose a sequence of subsets $S_r \subseteq N_r$ for each r such that $\mid S_r \mid = r$. We then have $\lim_{r\to\infty} \mid S_r \mid x_r = 2$ and $\lim_{r\to\infty} z_r = 0$. This contradicts A2 since Z^* is closed and $Z^* \cap \mathbf{R}^2_+ = \{0\}$.

Together, the assumptions on Z_r and Y_r and the boundedness above of the utility functions (by a linear function plus a constant) imply that the sequence of derived games is per-capita bounded. Since the sequence of derived games is also superadditive, it is a sequence of near-market games and asymptotically totally balanced.

Theorem 3.1. *Let $(\mathsf{E}_r)_{r=1}^\infty$ be a sequence of replica economies satisfying A1 and A2. Then the derived sequence of replica games is a sequence of near-market games and is asymptotically totally balanced.*

In Appendix 2, we show that the derived sequence of games is per capita bounded and superadditive. Theorem 3.1 then follows as a Corollary to Theorem 2.1. (All theorems in this section are proven in Appendix 1.)

3.5. Convergence of ε-Cores to Cores

We now impose a further restriction on production which will have the result that the derived sequence of games has the MES property. Informally, the assumption ensures that any state of the economy that can be improved upon can be improved upon by a coalition bounded in size bounded in size by $\rho(N_{r^*})$ for some r^*.

We require the following definition: given r and $(x,y) \in A_r(N_r)$ a **permutation by types** of (x,y) is an N_r-allocation, say (x',y') where for each type t there is a one-to-one mapping of $\{(x^{tq}, y^{tq}) \ : \ q = 1 \ldots, r\}$ onto itself. Note that since (x,y) is an N_r-attainable allocation, (x',y') is an N_r-attainable allocation.

A3. There is an r^* such that for all positive integers n, given $r' = nr^*$ for some n and $(x,y) \in A_{r'}(N_{r'})$, there is a permutation by types of (x,y), say (x',y'), and an $(x^*,y^*) \in A_{r^*}(N_{r^*})$ where, for each $(t,q) \in N_{r^*}$, we have $u^{tq}(x^{*tq}, y^{*tq}) \geq u^{t'q'}(x'^{t'q'})$ for all $(t',q') \in N_r$, with $t' = t$ and $q' = q, 2q \ldots, nq$.

Informally, A3 is the assumption that agents can do as well for themselves in coalitions with profiles less than or equal to $\rho(N_{r^*})$ as they can in coalitions with profiles less than or equal to $n\rho(N_{r^*})$–increasing returns to coalition size are exhausted by economies with profiles less than or equal to that of N_{r^*}.

Theorem 3.2. *Let $(\mathsf{E}_r)_{r=1}^{\infty}$ be a sequence of replica economies satisfying assumptions A1 to A3. Then the derived sequence of replica games $(N_r, v_r)_{r=1}^{\infty}$ has the MES property.*

It is an immediate consequence of Theorems 2.2, 2.3, and 3.2, that the ε-cores of the derived games $(N_r, v_r)_{r=1}^{\infty}$ converge to the cores of the balanced cover games as r becomes large, i.e., for the derived sequence of games, $A^* = L(\mathcal{C})$. (Since we have made an MES assumption, only equal-treatment payoffs are in the cores of the balanced cover games for sufficiently large replications. Theorem 2.2 ensures the convergence of the ε-cores to equal-treatment payoffs.)

4. The Framework of the Models

In essence, when the core of a market game associated with an exchange economy is studied the technical and institutional assumptions which are made are that individual ownership of all economic goods is recognized and that all goods can be transferred, without costs or technical difficulty, among all individuals.[22] In particular, implicit in these assumptions is that the economic reality can be well represented by the characteristic function. Shapley and Shubik in Shapley (1974) have suggested the term "c-game" for a model that is adequately represented as a game by its characteristic function and we use that term here. An exchange economy has the c-game property

[22] When we discuss the core here, we refer to the core of the market game as constrasted with the set of undominated distributions of resources in the distrbution space of the economy which map into the core of the market game. This distinction may not appear to be of much importance when considering exchange economies, but it is helpful when we consider more general economies.

since what a coalition S can achieve via exchange of goods among the members of S is independent of exchanges carried out among the members of the complementary coalition $\overline{S}$.

The introduction of indivisibilities or other nonconvexities does nothing to influence the c-game property. It remains reasonable to assume that a set S can exchange among its members in any way they desire, even if the items are indivisible.

If we assume that production processes manifest constant returns, then no distinction need be made between individual or joint ownership of the technology. The c-game property is preserved. A group S can produce and exchange as it pleases regardless of the actions of $\overline{S}$.

Production and exchange without constant returns poses new problems. If the technology is commonly owned who gets to use it first now matters. This can be avoided by making the reasonable assumption that it is owned by groups, cf. Boehm (1974), Hildenbrand (1974).

In the models used previously and in this paper, the c-game property is preserved between clubs but not necessarily within given clubs. The feasible consumptions, productions, and exchanges of the members of a collection of clubs, acting as a coalition, is independent of the actions of the complementary collection of clubs; this is because there are no "spill-overs" of the public goods between clubs. Within a club, what a subset of agents can consume can be affected by the actions of the other members of the club; the public goods are "pure" public goods within the club. However, from the viewpoint of the study of the core, it is the c-gameness between clubs that is relevant.

As we discussed in the introduction, the modelling of private goods exchange economies as games in characteristic function form is "natural" – the economies have the *c*-game property. For club economies, because production technologies are associated with groups and because exclusion is possible (and also may well be desirable from the point of view of optimality) in consumption of local public goods, we have the set-up of a *c*-game; from the point of view of the study of the core, the appropriate construction of the characteristic function is obvious.

For pure public goods, as we have argued, additional specifications must be made to construct the characteristic function. It needs to be specified who has access to what technology (one could use a coalition production model, for example) and what a coalition can consume of the public goods produced by the complementary coalition.

4.1. The Firm and Replication

Modelling difficulties that arise in treating production (and also public goods) can be illustrated by examining the alternative ways of replicating an extremely simple example.

We begin by utilizing a simple example to illustrate modelling difficulties finessed by the assumption of group production and how to construct a sequence of replicated economies given group production.

Suppose that an individual i has an initial endowment of $(1, 0, s_i)$ where the first item is an individually owned good which can be used as an input to produce the second good. There is one firm which produces the second good with a production function given by $y = \sqrt{x}$ where y is the output and $x = \sum_{i=1}^{n} x_i$ is the total input. The utility function of an individual i is given by $U_i(x_i, y_i) = y_i$ where $y = \sum_{i=1}^{n} y_i$. The ownership share of the firm by individual i is s_i and $\sum_{i=1}^{n} s_i = 1$. Let p be the price of the output and fix the price of the input as one.

The manager of the firm will attempt to maximize

$$\Pi = p\sqrt{x} - x \text{ where } 0 \leq x_i \leq 1$$

and each individual i will attempt to maximize

$$y_i \text{ subject to } x_i + s_i\Pi = py_i.$$

Thus $x_i = 1$ and $p = 2\sqrt{n}$; hence $\Pi = n$ and $y = \sqrt{n}$.

Now $0 \leq s_i \leq 1$; thus, depending upon the distribution of shares an individual i could obtain y_i where

$$\frac{1}{2\sqrt{n}} \leq y_i \leq \frac{n+1}{2\sqrt{n}}.$$

Suppose that the original economy was of size $n = 9$ with only one type of agent. If a type has both utility functions and endowments the same for all members we require that $s_i = 1/9$. At this point we have to decide how to describe the characteristic function for this game and furthermore we need to describe how to calculate the characteristic functions for the games arising from the replicated economy.

Alternative 1. Shares are ignored, any subgroup may use the technology. Here the game is immediately inessential, the characteristic function[23] is given by $f(s) = s, s = 0, 1, \ldots, 9$. This is equivalent to creating n independent technologies.

Alternative 2. A simple majority at least is required to operate the firm. The game becomes

$$\begin{aligned} f(s) &= 0 \quad \text{for} \quad s \leq n/2 \\ &= \sqrt{s} \quad \text{for} \quad n/2 < s \leq n. \end{aligned}$$

This game has no core as can be seen from Figure 1.

Alternative 3. Unanimity is required to operate the firm. The game becomes

$$\begin{aligned} f(s) &= 0 \quad \text{for} \quad 0 \leq s \leq n-1 \\ &= \sqrt{n} \quad \text{for} \quad s = n. \end{aligned}$$

Here all imputations are in the core.

[23]For a type symmetric game instead of using the notation $v(S)$ to stand for the amount obained by the sets of agents we can use $f(s)$ where $s = |S|$ as all coalitions of the same size obtain the same amount.

Figure 4.1:

Alternative 4. Any simple majority can use the production function, but it can only take the product in proportion to its shares

$$\begin{aligned} f(s) &= 0 \quad \text{for} \quad s \le n/2 \\ &= \frac{s}{n}\sqrt{s} \quad \text{for} \quad \frac{n}{2} < s \le n \end{aligned}$$

It is easy to check that this has a core.

We now turn to replication. What do we mean by a replication of the 9 agents, 1 firm economy? We may consider at least two types of replication. In the first there is still one firm but we replicate the number of agents and shares. Thus the endowments in the r^{th} replication become $(1, 0, 1/9r)$ where although all individuals may have the same number of shares as previously, the percentage ownership of the firm has gone down by l/r. In the second form of replication the firms are increased as well. At this point we may wish to consider the distinction between giving all agents a portfolio in each firm, or enlarging the number of types. For example, for one replication do we consider one type with $(1, 0, 1/2n, 1/2n)$ where the last two entries indicate shares in the two firms or do we consider two types with $(1, 0, 1/n, 0)$ and $(1, 0, 0, 1/n)$?

Replication 1. There is only one firm, regardless of the replication number. The characteristic function becomes

$$\begin{aligned} f(s) &= 0 \quad \text{for} \quad s \le nr/2 \\ &= \sqrt{s} \quad \text{for} \quad nr/2 < s \le nr \end{aligned}$$

for the simple majority case. There is no core. Moreover, given any $\varepsilon > 0$ there is an n_0 such that for all n greater than n_0, the game has an empty ε-core. The sequence of games with $n = 1, 2, \ldots$ is not a sequence of replica games since the payoff to a given coalition eventually decreases as the set of players containing that coalition becomes large.

Replication 2a. ("balanced portfolio")

$$\begin{aligned} f(s) &= 0 \quad \text{for} \quad s \leq nr/2 \\ &= r\sqrt{s/r} \quad \text{for} \quad nr/2 < s \leq nr \end{aligned}$$

This sequence of games has cores or ε-cores and is a sequence of replica games.

Replication 2b. (different types)

$$f(s_1, s_2, \ldots, s_r) = p\sqrt{\frac{\sum_{j=1}^{r} \delta_j s_j}{p}} \qquad \text{where} \quad \delta_j \quad = 1 \quad \text{if} \quad s_j < n/2$$
$$= 0 \quad \text{if} \quad s_j \leq n/2$$
$$\text{and} \quad p = \sum_{j=1}^{r} \delta_j .$$

This sequence of games has a core or ε-core.

4.2. The Firm as a Local Public Good

We could interpret the example in 4.1 in terms of a local public goods and endogenous clubs. Consider an economy of size n with a minimal size club of 9 where each individual has an initial endowment of $(1, 0)$, the first item is a private good which can be converted to a public good according to $y = \sqrt{\sum x_i}$. The utility function for each individual i is $U_i(x_i, y, s) = y/s$ where s is the number of individuals in a club.

It is easy to see that this formulation leads to the formation of as many 9 person clubs as possible. If $n = 09$ there is a core, otherwise an ε-core. Furthermore implicit in this formulation is the idea that all individuals of the same type have an equal share of the club to which they belong. Thus the analogue with the corporate economy can be completed by considering shares in clubs.

4.3. The Pure Public Good

The example noted in 5.2 can be extended to an example illustrating a pure public good by replacing $U_i(x_i, y_i) = y_i$, by $U(x_i, y) = y$ where $y = \sqrt{\sum_i x_i}$.

Suppose that no subgroup smaller than 9 can produce. Then

$$\begin{aligned} f(s) &= 0 \quad \text{for} \quad s < 9 \\ &= s\sqrt{s} \quad \text{for} \quad s \geq 9. \end{aligned}$$

This game has a large nonconverging core. Note also that the sequence of derived games does not satisfy the near-minimum efficient scale property.

4.4. The framework revisited

The above discussion may indicate that when we consider sequences of economies, the c-game property within a fixed economy does not necessarily imply *self sufficiency*, that what a coalition can achieve is independent of the economy in which it is embedded. In fact, the model of this paper does not necessarily satisfy self-sufficiency – what a coalition can achieve may increase as the size of the economy in which it is embedded increased. What is crucial for our results is that, for each coalition S, the limit $\lim v_r(N_r)/r$ exists and, for each r and each $S \subseteq N_r$ the limit $\lim_{r'} v_{r'}(S)$ exists. Thus, widespread negative externalities could be allowed. For example, consider a game where all players are identical but for any given coalition S and for r sufficiently large so that $S \subseteq N_r$, $\frac{v_r(S)}{|S|}$decreases to $|S|$ as r grows large. Such situations could be accommodated at the cost of more complexity of the model.

5. Recent literature

While the convergence results in this paper for economies with clubs are still novel, since the paper was written in 1982 a number of related results have appeared in the literature and the reader may wish some some updated discussion and references.

The nonemptiness of ε-cores of large games with side payments and with a fixed distribution of player types reported in this paper has been extended to hold uniformly for all games derived from pregames with a compact metric space of player types (Wooders and Zame 1984, Wooders 1992a).[24] A *pregame* is a pair consisting of a compact metric space of player types and a worth function, ascribing a payoff to any coalition of players, described by their types. The worth of a coalition depends continuously on the types of the coalition members. The framework of Wooders and Zame (1984), as that of Wooders (1979), however, imposes *self-sufficiency* – what a coalition can achieve must be independent of the total player set in which it is embedded. This is essential for the uniformity[25] of the result of Wooders and Zame (1984) and Wooders (1992a).

The current paper and the prior game-theoretic results used suggest that there is a relationship between minimum efficient scale – that *all* gains to collective activities can be realized by groups of players bounded in size – and per capita boundedness.[26] In fact, Wooders (1994a) formulates an "approximate" version of MES called small group effectiveness – that all or amost all gains to collective activites can be realized by groups of players bounded in size – and demonstrates an equivalence between small group effectiveness and per capita boundedness. Wooders (1994a,b) shows that under the condition of small group effectiveness, the per capita payoff function $\frac{v(N)}{|N|}$

[24] Wooders and Zame (1984) requires boundedness of marginal contributions to coalitions while Wooders (1992a) requires only the milder conditions of small group effectiveness and per capita boundedness.

[25] That is, the result is for all sufficently large games derived from a pregame rather than for sequences of games with a fixed distribution of player types.

[26] This relationship is very clear in the proof technique of Wooders (1980b). Essentially, sequences of games satisfying per capita boundedness are approximated by sequences satisfying minimum efficient scale.

converges uniformly to the concave "utility function" of a certain market game, and for small ε and large player sets N, the core of the market game approximates the ε-core of the game.

There have been further advances in the club-theoretic literature, especially in the area of equilibrium and the equivalence of the core and the set of competitive outcomes. We refer the reader to Conley and Wooders (1998a) and references therein for a more up-to-date survey. As the work in the current paper suggests, when small groups of participants can realize all or almost all gains to collective activities, then large economies are competitive.

The convergence of approximate cores reported in this paper is extended in Wooders (1992b) to demonstrate that in economies with side-payments (quasi-linear utilities, in economics terminology) core payoffs converge to competitive prices.[27] Another sort of core convergence result was demonstrated by Engl and Scotchmer (1991).. Roughly, these writers consider a situation where players are characterized by their attributes, represented by points in a finite-dimensional Euclidean space. Attributes may be, for example, endowments of commodities and in fact, their model is equivalent to that of an exchange economy with quasi-linear utilities. They show that under an assumption of uniform convergence of the per capita payoff function, in per capita terms epsilon core payoffs to large groups of players are approximated by a linear function on attribute space. To illustrate the differences in approaches between Engl and Scotchmer (1991) and the convergence results reported in the current paper, consider an n-person game where all people are identical and as n becomes large, $\frac{v(n)}{n}$ converges to $\overline{x}$.[28] Engl and Scotchmer's result implies that for all sufficiently large coalitions S if x is in the ε-core, then $\left|\frac{x(S)}{|S|}-\overline{x}\right|$ is "small". More generally, if there are T types and players of type and x is in the ε-core, then for some $p \in \mathbf{R}^T$ representing an equal-treatment payoff in the ε-core,

$$\text{Engl-Scotchmer: } \left|\frac{x(S)}{|S|}-\overline{x}\cdot p\right| \text{ is "small".}$$

In contrast, the results of the current paper and Wooders' convergence results all show that for each t, for "most" players i of type t,

$$\text{Wooders (1980b): } |x_i - p_t\,| \text{ is "small".}^{29}$$

Examples illustrating further distinctions between the convergence results of Engl and Scotchmer and those reported in this paper are provided in Wooders (1992b). We note that there is a large literature on convergence of cores to equilibrium outcomes in economies with private goods only which is outside the scope of the current paper; see Anderson (1992) for a recent survey.

[27] In the same context, Shapley and Shubik (1966) show nonemptiness of approximate cores. They proceed, however, by convexifying preferences rather than by "balancifying" over small groups, as in the research of this author. In addition, Shapley and Shubik do not show convergence to approximate cores, as in Wooders (1980b,1992).

[28] In spirit, the Engl-Scotchmer result shows convergence in mean while our results show convergence in probability.

The result of Engl and Scotchmer (1991) may suggest that it is possible to hedonically price attributes of players, that is, to infer from core payoffs values of attributes. However, hedonic pricing requires *hedonic independence* – that the worth of a bundle of attributes equals the sum of the worths of the attributes in the bundle. Conley and Wooders (1995) provide examples showing that in Tiebout economies (or club economies) in general hedonic independence is not satisfied.

An aspect of economies satisfying minimum efficient scale that is not addressed in the current paper is "monotonicity," that is, when the abundance of players of a given type is increased then the core payoff to players of that type does not increase and may decrease. This was suggested in Wooders (1979) and is shown in Scotchmer and Wooders (1988). Wooders (1992b,1994a) also provides related results, allowing the abundances of all player types to change simultaneously, and demonstrates that a stronger condition of "cyclic monotonicity" is satisfied.

A difficulty with the pregame framework, used in the above papers, is that self-sufficiency rules out economy-wide externalities. Moreover, the pregame structure has hidden consequences. For example, the equivalence of small group effectiveness and per capita boundedness does not necessarily hold outside the pregame framework. To overcome these restrictions, recently, Kovalenkov and Wooders (1997a) introduced the framework of parameterized collections of games with and without side payments. This framework treats classes of games that can be described by certain parameters: (a) the number of approximate types of players and a bound on the distance between players of the same approximate type, and (b) the size of near-effective groups of players and a bound on how close such groups must be to being completely effective. In this framework, the payoff to a group of players need not depend on the size of the economy in which the group is embedded. Thus, widespread positive or negative externalities are allowed. Moreover, uniform results (that is, results for all large games, rather than for sequences) are obtained. Kovalenkov and Wooders (1997b) applies their game-theoretic results to a model with clubs. Unlike the current paper, Kovalenkov and Wooders (1997b) allows agents to be affected by the characteristics of the memberships of the clubs to which they belong. Kovalenkov and Wooders (1998) establishes monotonicity of core payoffs for large games where the results are stated in terms of the parameters describing the parameterized collection of games.

Finally, another issue discussed in the current paper is convergence of the core to competitive outcomes in economies with public goods. A sufficient condition for this convergence is demonstrated in Conley (1994).

6. Appendix

In this appendix, Theorems 3.1 and 3.2 are proven.

Proof of Theorem 3.1

To prove the theorem, we need only prove that the sequence of games derived from the economies is superadditive and per-capita bounded. It is straightforward to verify that the sequence is superadditive; thus we omit the proof.

To prove per-capita boundedness of the sequence of games, we construct another sequence of economies, say the *-economies, so that the sequence of games derived from the *-economies, denoted by $(N_r, v_r^*)_{r=1}^{\infty}$, is per-capita bounded and has the property that for all r and for all non-empty subsets S of N_r, we have $v_r(S) \leq v_r^*(S)$.

For the *-economies, we let Y^* be the private goods production possibility set available to all coalitions $S \subseteq N_r$ for all r. Observe that for any firm structures, say $F_r(S)$ and $F_r^{'}(S)$ we have

$$\sum_{S' \in F_r(S)} Y^* = \sum_{S' \in F_r^{'}(S)} Y^*$$

so the firm structure will be irrelevant. Of course, $Y_r(S) \subseteq Y^*$ and for any firm structure $F(S)$, $\sum_{S' \in F(S)} Y^* \subseteq Y^*$.

Given r and a subset S of N_r, define $Z^*(S) = \{(x, z) \in \mathbf{R}^{m+\ell} \ : \ (\mid S \mid x, z) \in Z^*\}$. Note that $Z^*(S)$ is a closed convex cone with vertex $\{0\}$ and $\{0\} \times -\mathbf{R}_+^{\ell} \subseteq Z^*(S)$; this follows from the assumptions on Z^*. Let $\varphi(S) = (J(S), \{(x'_S, z'_S) \in Z_r(S') \ : S' \in J(S)\})$ be an S-public goods production plan. From assumption A.2, there is an $(x, z) \in Z^*(S)$ such that

(a) for each $(t, q) \in S$, we have $\displaystyle\sum_{\{S' \in J(S) : (t,q) \in S'\}} x_{S'} \leq x$,

and

(b) $\displaystyle\sum_{S' \in J(S)} z'_S \leq z.$

Informally, there is an (x, z) in $Z^*(S)$ "at least as good as" any S-public goods production plan in the sense that with the production possibility set $Z^*(S)$, the agents can consume as much of the local public goods while using no more of the inputs. (Note, however, because agents do not necessarily have monotonic increasing preferences for the local public goods, they might not prefer to have more of them.)

For the purposes of this theorem, we can restrict our attention to states of the *-economies with associated club structures $\{\{(t, q)\} : (t, q) \in N_r\}$ since with this club structure all agents can be made "at least as well-off" as with any other club structure. To see this, given any r and any non-empty subset S of N_r, let $(x, z) \in Z^*(S)$ so $(\mid S \mid x, z) \in (Z)^*$ and $(x, z/ \mid S \mid) \in Z^*(\{(t, q)\})$ for each $(t, q) \in S$; thus with the club structure $\{\{(t, q)\} : (t, q) \in N_r\}$ each agent in S can consume x and total inputs are unchanged.

In the *-economies, the agents will all have the same utility functions. For each r and for all $(t, q) \in N_r$, define $u^{*tq}(x, y) = L(x, y)$ where L is a linear function such that $u^{tq}(x, y) \leq L(x, y) + c$ for some constant c for all $(x, y) \in \mathbf{R}_+^m + \mathbf{R}_+^{\ell}$ and for all $(t, q) \in N_1$.

For each r and each non-empty subset S of N_r, let $A_r^*(S)$ denote the set of S-attainable allocations for the *-economies.

For each r and all non-empty subsets S of N_r, define

$$v_r^*(S) = \sup_{(x,y) \in A_r^*(S)} \sum_{(t,q) \in S} L^*(x^{tq}, y^{tq}).$$

The finiteness of this "sup" is ensured by the linearity of the function L^* and boundedness of $A_r^*(S)$. Also,

$$v_r^*(S) = \sup_{(x,y)\in A_r^*(S)} L\left(\sum_{tq\in S} x^{tq}, \sum_{tq\in S} y^{tq}\right).$$

Let K be a real number such that $K > v_1^*(N_1)$. We will show that K is a per-capita bound for the sequence of games (N_r, v_r^*). Suppose not. Then there is an r' and a feasible state of the r'^{th} $*$-economy, say $e^*(N_{r'}) = (\overline{y}, J(N_{r'}), \{x_{S'}, z_{S'}) \in Z^*(S') : S \in J(N_{r'})\})), (x,y))$, such that $L(\sum_{tq\in N_{r'}} x^{tq}, \sum_{tq\in N_{r'}} y^{tq}) > Kr'$. We can assume without any loss that $F(N_{r'}) = \{N_{r'}\}$ and $J(N_{r'}) = \{\{(t,q) : (t,q) \in N_{r'}\}$. Also we can assume that $x^{tq} = x^{t'q'}$ and $y^{tq} = y^{t'q'}$ for all (t,q) and (t',q') in $N_{r'}$. We claim that there is an $(x',y') \in A_1^*(N_1)$. with $x'^{tq} = x^{tq}$ and $y'^{tq} = y^{tq}$ for all $(t,q) \in N_1$, which will yield a contradiction. Since $e^*(N_{r'})$ is feasible (for the r'^{th} economy), we have $\overline{y} \in Y^*$. Observe that $\overline{y}/r' \in Y^*$. Also, for some z^{tq} for each $(t,q) \in N_{r'}$, we have $(x^{tq}, y^{tq}) \in Z^*(\{(t,q)\})$, and

$$\sum_{tq\in N_{r'}} (y^{tq} - w^{tq}) + \sum_{tq\in N_{r'}} z^{tq} = \overline{y} \tag{6.1}$$

so

$$\sum_{tq\in N_1} (y^{tq} - w^{tq}) + \sum_{tq\in N_1} z^{tq} = \overline{y}/r. \tag{6.2}$$

This proves our assertion that $(x',y') \in A_1^*(N_1)$. But then

$$\begin{aligned} &L(\textstyle\sum_{tq\in N_1} x'^{tq}, \sum_{tq\in N_1} y'^{tq}) = \\ &\tfrac{1}{r'}L(\textstyle\sum_{tq\in N_{r'}} x^{tq}, \sum_{tq\in N_{r'}} y^{tq}) > K \end{aligned} \tag{6.3}$$

which is a contradiction. Therefore the sequence $(N_r, v_r)_{r=1}^{\infty}$ is per capita bounded.

Since $u^{tq}(x,y) \le L(x,Y)+c$ for all $(x,y) \in \mathbf{R}_+^m \times \mathbf{R}_+^\ell$, and from the construction of the production possibilities set for the *-economies, we have $v_r(N_r) \le v_r^*(N_r) + c \mid N_r \mid$ for all r. Therefore the sequence $(N_r, v_r)_{r=1}^{\infty}$ is per-capita bounded. ∎

Proof of Theorem 3.2

Given $\delta > 0$, r^* satisfying the conditions of A3, and $r' = nr^*$ for some positive integer n, there is an $(x,y) \in A_{r'}(N_{r'})$ such that

$$v_{r'}(N_{r'}) \le \sum_{tq\in N_{r'}} u^{tq}(x^{tq}, y^{tq}) + \delta.$$

From A3, there is an $(x^*, y^*) \in A_{r^*}(N_{r^*})$ such that

$$\sum_{tq\in N_{r'}} u^{tq}(x^{tq}, y^{tq}) \le n \sum_{tq\in N_{r^*}} u^{tq}(x^{tq}, y^{tq}).$$

Therefore, it follows that

$$\frac{v_{r'}(N_{r'})}{r'} \leq \frac{v_{r^*}(N_{r^*})}{r^*}$$

and, from superadditivity, that

$$\frac{v_{r'}(N_{r'})}{r'} \geq \frac{v_{r^*}(N_{r^*})}{r^*}$$

so

$$\frac{v_{r'}(N_{r'})}{r'} = \frac{v_{r^*}(N_{r^*})}{r^*}$$

Since this holds for all positive integers n, we have

$$\lim_{r\to\infty} \frac{v_r(N_r)}{r} = \frac{v_{r^*}(N_{r^*})}{r^*}.$$

Since

$$\lim_{r\to\infty} \frac{\widetilde{v}_r\ (N_r)}{r} = \lim_{r\to\infty} \frac{v_r(N_r)}{r}$$

we have

$$\lim_{r\to\infty} \frac{\widetilde{v}_r\ (N_r)}{r} = \frac{v_{r^*}(N_{r^*})}{r^*}.$$

Since $\frac{v_r(N_r)}{r} \leq \frac{\widetilde{v}_r(N_r)}{r}$ for all r and since $\frac{\widetilde{v}_r(N_r)}{r}$ is non-decreasing, we have

$$\frac{\widetilde{v}_r\ (N_r)}{r} = \frac{\widetilde{v}_{r^*}\ (N_{r^*})}{r} \quad \text{for} \quad r \geq r^*.$$

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