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Epsilon Cores of Games with Limited Side Payments Nonemptiness and Equal Treatment[1]

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We introduce the concept of a parameterized collection of games with limited side payments, ruling out large transfers of utility. Under the assumption that the payoff set of the grand coalition is convex, we show that a game with limited side payments has a nonempty ε-core. Our main result is that, when some degree of side-paymentness within nearly-effective small groups is assumed and large transfers are prohibited, then all payoffs in the ε-core treat similar players similarly. A bound on the distance between ε-core payoffs of any two similar players is given in terms of the parameters describing the game. These results add to the literature, showing that games with many players and small effective groups have the properties of competitive markets. *Journal of Economic Literature* Classification Numbers: C71, C78, D71.

Key Words: cooperative games; equal treatment; core convergence; games without side payments (NTU games); large games; approximate cores; effective small groups; parameterized collections of games.

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1. INTRODUCTION

In many game theoretic and economic situations side payments may be freely made only within small groups of players. For example, marriage games may have transferable utility within marriages but may rule out side payments between marriages. Even if side payments were possible, large transfers of payoff may require cooperation and agreement among a large numbers of players and such agreements may be difficult to obtain.[4] In this paper we introduce the concept of games with limited side payments. Side payments are limited by the requirement that groups of players bounded in size, without making transfers between themselves, can achieve almost all feasible gains to collective activities. The ε-core consists of those feasible payoffs that cannot be improved upon by at least ε for each member of any coalition. We demonstrate conditions under which ε-cores of games with limited side payments are nonempty. Our main result is that any ε-core payoff has approximately the equal treatment property; that is, every payoff in the ε-core assigns similar payoffs to similar players.

Our results apply to games in parameterized collections. The parameters describing the games are (a) bounds on the number of approximate player types and the accuracy of the approximation and (b) a bound on the size of nearly effective groups of players and their distance from exact effectiveness. Two players are approximately the same types if they make similar contributions to coalitions; that is, if they are approximate substitutes. That nearly-effective groups are bounded in size means that all or almost all gains to collective activities can be realized by coalitions no larger than the bound. This can be viewed as a sort of "small group effectiveness" assumption.

Our interest in approximate cores stems from our interest in competition in diverse economic settings, such as those with local public goods, clubs, production, location, indivisibilities, nonmonotonicities, and other deviations from the classic Arrow–Debreu–McKenzie model. Both of our results contribute to a line of research investigating the "market-like" properties of large games under conditions limiting returns to group size; cf. Wooders (1983, 1994b). An important precursor is Shubik (1959), which introduced the idea of studying games as models of large economies. Another is Shapley and Shubik (1966), which shows that large exchange economies have nonempty approximate cores. A third is Shapley and Shubik (1969a), which demonstrates an equivalence between markets and totally balanced games. From an economic perspective, the remarkable

[4]This point is stressed in Kaneko and Wooders (1997). In more applied literature, this is a recognized problem in reaching trade and environmental agreements; see e.g. Abrego *et al.* (1997).

paper of Tiebout (1956), in which he conjectured that large economies with local public goods are "market-like," contains insights similar to those that lead to the current research.[5] Further motivation is provided for this research by Buchanan (1965), who stressed the need for a general theory, including as extreme cases both purely private and purely public goods economies and the need for "a theory of clubs, a theory of cooperative membership." Unlike detailed models of economies, models of large games can accommodate the entire spectrum from games derived from private goods economies to games derived from economies with purely public goods.

In large games the core is a stand-in for the competitive equilibrium.[6] Thus, a crucial result is that cores or approximate cores of large games are nonempty. Since the competitive equilibrium has the equal treatment property, for games to resemble competitive markets it is also crucial that approximate cores converge to equal treatment payoffs. Our result showing that ε-cores treat similar players similarly demonstrates that the equal treatment property of competitive equilibrium extends to approximate cores of large games with small effective groups and limited side payments.

To place our research in the context of the literature, recall that Shapley and Shubik (1966) initiated the study of nonemptiness of approximate cores of large exchange economies. Prior to the introduction of parameterized collections of games, other papers showing the nonemptiness of approximate cores of large games relied on the construct of a pregame. A pregame consists of a set of player types and a *single* worth function assigning a payoff to any group of players. The payoff to a group of players depends on the numbers of players of each type in the group. Given a total player set and a specification of the numbers of players of each type in the set, the worth function of the pregame induces a worth function on the player set and thus determines a game. When there are many players of each type and per capita payoffs are bounded (PCB) or, alternatively, when small groups of players are effective (SGE), then large games derived from pregames have nonempty approximate cores (Wooders 1983, 1994a,b).[7] The pregame

[5]See Wooders (1999) and the references therein.

[6]See, for example, discussions in Shapley and Shubik (1969a) and Wooders (1994a,b) for market games with side payments. For games without side payments, Billera (1974) and Mas-Colell (1975) are relevant. Hildenbrand and Kirman (1988, Chapt 4), among others, treat the core more explicitly as a standin for the competitive equilibrium in nonclassical exchange economies.

[7]In other words, PCB means that the supremum of average payoffs is finite while SGE means that relatively small groups are effective for the realization of all or almost all gains to collective activities. Unlike the earlier and subsequent papers on this topic, Wooders and Zame (1984) used a stronger condition, the boundedness of individual marginal contributions to coalitions.

structure, which requires that all games considered be subgames of some larger game, however, has hidden consequences. To illustrate, when there are sufficiently many players of each type which appears in the games, the assumptions of SGE and PCB are equivalent (Wooders 1994b, Section 5); this does not hold for parameterized collections of games. In addition, the pregame framework dictates that the payoff to a group is independent of the player set in which it's embedded; thus, widespread externalities are ruled out.[8] It is not, in fact, the construct of a pregame that's important to the prior nonemptiness results in the literature: it's the features of many close substitutes for each player and some form of small group effectiveness. The framework of parameterized collections of games incorporates games derived from pregames but is significantly broader. Our nonemptiness result states that, given the parameters describing a collection of games, there is a bound such that all games in the collection with more players than the bound have nonempty approximate cores. The bound depends on the parameters describing the games.

The equal treatment property of the core of a large exchange economy was initially demonstrated by Shubik (1959) for an example with quasilinear utilities and two types of players. As a step toward their celebrated result showing convergence of the core to the set of Walrasian equilibrium allocations, Debreu and Scarf (1963) showed that all outcomes in the core of a replicated, convex exchange economy have the equal treatment property. Equal treatment results in the game-theoretic literature have relied on the pregame structure. In the context of a pregame with transferable utility, when there are many close substitutes for each player and PCB is satisfied, ε-cores of large games are nonempty and receive approximately equal treatment in the sense that *most* players of the same type receive nearly the same payoffs (Wooders 1980,1994a).[9] In large games with *strictly* effective small groups (SSGE), that is, when *all* gains to cooperation can be realized by groups bounded in size, and with strongly comprehensive payoff sets, all payoffs in the core have the equal treatment property (Wooders, 1983, Theorem 3). In the current paper, the manner in which side payments are limited ensures that the core treats similar players similarly and simultaneously allows possibly ever-increasing returns to coalition size. Thus, our equal treatment result is stronger than prior results in three important ways: First, although it is not required that small groups are *strictly* effective,

[8]See for example Shapley and Shubik (1969b) or Hammond (1999) for economic examples and Kovalenkov and Wooders (1999a) for game-theoretic examples in the context of parameterized collections.

[9]See also Bennett and Wooders (1979), who apply earlier versions of these results to questions of firm formation and income distribution, and Shubik and Wooders (1982), who apply these results to economies with clubs, where individual agents may belong to multiple clubs.

we demonstrate that *all* players of the same approximate type are treated approximately equally. Second, when there are sufficiently many players of an approximate type, our result provides an explicit bound on the distance between the ε-core payoffs for any two players of that approximate type. Third, our results are not restricted to games with transferable utility nor to games derived from pregames.

In the next subsection we provide further motivation for our framework and some intuition into our main result.

1.1. *Intuition behind Our Model and the Main Result*

The form of our assumption that nearly-effective groups are bounded in size is crucial to our equal treatment result. Consider a simple example: Any two players can earn £2 and there are $2n + 1$ players in total; any division of players into two-player groups will leave one player left over. Now consider a payoff (not necessarily feasible) assigning one player £$2n$ and all other players £0. If utility were transferable (and the game is superadditive, which we assume throughout), then, for any n, this payoff would be feasible. When side payments are limited by the feature that all or almost all feasible payoffs can be realized by groups bounded in size without transfers between groups, for n larger than the bound any payoff assigning one player £$2n$ is infeasible. Suppose, for our example, we take the bound on group size as, say, 11 and allow transferability of utility only within groups containing no more than 11 members. Under these conditions a payoff giving 10 players zero, one player £10, and all other players £1.00 each would be feasible. Under the same conditions, however, no feasible payoff could assign any one player more than £10. The intuition behind this example holds generally; to define games with limited side payments we will require that any feasible payoff vector is nearly achievable by a partition of players into groups bounded in size. In our other papers (Kovalenkov and Wooders, 1999a,b), different notions of small group effectiveness are used and thus the sort of equal treatment property demonstrated in the current paper may not hold.

Turning to the question of equal treatment, consider a $2n$-player matching or marriage game with n identical males and n identical females. Suppose that the payoff to being unmatched is 0 and the payoff to any male–female pair is 1. It is well understood that for such a game the core has the equal treatment property. Moreover, for any payoff vector in the ε-core, the percentage of players of any gender whose payoff differs significantly from the average for their gender becomes small as n grows large. This follows from the observation that since every matched pair must receive as least $1 - 2\varepsilon$, the payoff that can be transferred to players receiving more than the average for their genders is bounded. For any $\varepsilon > 0$, however, payoffs in the ε-core may be far from equal treatment. If side payments are unlimited (that is, utility is transferable), there are ε-core payoffs that assign one player $2n\varepsilon$ more than any other player of the same gender.

Continuing the same example, suppose now that utility is transferable only within marriages but not between marriages. We might think of a matched pair of players as indulging in some activity that yields pleasure only to them. Alternatively, we may suppose that for some reason—political infeasibility, for example—transfers are not made between coalitions or that there is simply no reason for such transfers to be made. With this restriction, the behavior of ε-core payoffs changes dramatically. Now any payoff vector x in the ε-core for our example must have the property that for any two players i and j of the same gender, $|x_i - x_j| \leq 2\varepsilon$. Note that this result depends on some transferability of utility within marriages. If we rule out any side payments even within marriages, so that, for example, each member of a marriage could realize at most $\frac{1}{2}$, then we again lose the equal treatment property of the core and the result that ε-core payoff vectors treat similar players similarly. Thus, we require that there is some degree of side payments within nearly-effective small groups but must rule out, as the previous paragraph suggests, unlimited side payments.

The concept of parameterized collections of games is designed to treat games where most players have many approximate substitutes so the examples above all fall into this framework. The concept of limited side payments is designed to describe, as an extreme case, situations like the those discussed above where transfers can be made within nearly-effective coalitions, bounded in size, but large transfers are ruled out. As illustrated above, the property of limited side payments is built into the definition of nearly-effective groups. To give further intuition into how this will be done, note that in our marriage game ruling out transfers between marriages is equivalent to requiring that feasible payoff vectors can be realized by marriages. In a later section, where we demonstrate the equal treatment property of the core, we will also require some degree of transferability within nearly-effective coalitions.

1.2. *Organization of the paper*

In the next section we introduce the required definitions. Section 3 states our nonemptiness result and provides a number of examples. Section 4 introduces games with side-paymentness uniformly bounded away from zero only for nearly-effective small groups, states our result that the ε-core is approximately equal treatment, and provides additional examples. Section 5 provides some remarks on our model and results and relates our work to prior literature. Section 6 discusses related economies with limited side payments. Proofs are collected in Section 7 and two additional examples appear in an appendix.

2. DEFINITIONS

2.1. *Cooperative Games*: *Description and Notations*

Let $N = \{1, .., n\}$ denote a *set of players*. A nonempty subset of N is called a *coalition*. For any coalition S let $\mathbf{R}^S$ denote the $|S|$-dimensional Euclidean space with coordinates indexed by elements of S. For $x \in \mathbf{R}^N$, x_S will denote its restriction to $\mathbf{R}^S$. To order vectors in $\mathbf{R}^S$ we use the symbols $\gg$, $>$ and $\geq$ with their usual interpretations. The nonnegative orthant of $\mathbf{R}^S$ is denoted by $\mathbf{R}^S_+$ and the strictly positive orthant by $\mathbf{R}^S_{++}$. For $A \subset \mathbf{R}^S$, $\mathrm{co}(A)$ denotes the convex hull of A. We denote by $\vec{1}_S$ the vector of ones in $\mathbf{R}^S$, that is, $\vec{1}_S = (1, ..., 1) \in \mathbf{R}^S$. Each coalition S has a feasible set of payoffs or utilities denoted by $V_S \subset \mathbf{R}^S$. By agreement $V_\varnothing = \{0\}$ and $V_{\{i\}}$ is nonempty, closed and bounded from above for any i. Moreover we will assume that

$$\max\left\{x : x \in V_{\{i\}}\right\} = 0 \qquad \text{for any } i \in N. \tag{$*$}$$

It must be noted that $(*)$ is by no means restrictive since it can always be achieved by a normalization. It is convenient to describe the feasible utilities of a coalition as a subset of $\mathbf{R}^N$. For each coalition S let

$$V(S) := \left\{x \in \mathbf{R}^N : x_S \in V_S \text{ and } x_a = 0 \text{ for } a \notin S\right\}.$$

A *game without side payments* (called also an *NTU game* or simply a *game*) is a pair (N, V) where the correspondence $V : 2^N \longrightarrow \mathbf{R}^N$ is such that $V(S) \subset \{x \in \mathbf{R}^N : x_a = 0 \text{ for } a \notin S\}$ for any $S \subset N$ and satisfies the following properties:

(2.1) $V(S)$ is nonempty and closed for all $S \subset N$.

(2.2) $V(S) \cap \mathbf{R}^N_+$ is bounded for all $S \subset N$, in the sense that there is real number $K > 0$ such that if $x \in V(S) \cap \mathbf{R}^N_+$ then $x_i \leq K$ for all $i \in S$.

(2.3) $V(S_1) + V(S_2) \subset V(S_1 \bigcup S_2)$ for any disjoint $S_1, S_2 \subset N$ (superadditivity).

(2.4) $V(S) = V(S) - \{x \in \mathbf{R}^N_+ : x_a = 0 \text{ for } a \notin S\}$ for all $S \subset N$ (comprehensiveness).

2.2. *Games with Limited Side Payments*

To define parameterized collections of games we will need the concept of Hausdorff distance. For every two nonempty subsets E and F of a metric space (M, d) define the *Hausdorff distance between E and F*, $\text{dist}(E, F)$ (with respect to the metric d on M), by

$$\text{dist}(E, F) := \inf\{\varepsilon \in (0, \infty) : E \subset B_\varepsilon(F) \text{ and } F \subset B_\varepsilon(E)\},$$

where $B_\varepsilon(E) := \{x \in M : d(x, E) \leq \varepsilon\}$ denotes the ε-neighborhood of E.

Since payoff sets are unbounded below, we will use a modification of the concept of the Hausdorff distance so that the distance between two payoff sets is the distance between the intersection of the sets and a subset of Euclidean space. Let m^* be a fixed positive real number. Let M^* be a subset of Euclidean space $\mathbf{R}^N$ defined by $M^* := \{x \in \mathbf{R}^N : x_a \geq -m^* \text{ for any } a \in N\}$. For every two nonempty subsets E and F of Euclidean space $\mathbf{R}^N$ let $H_\infty[E, F]$ denote the Hausdorff distance between $E \cap M^*$ and $F \cap M^*$ with respect to the metric $\|x - y\|_\infty := \max_i |x_i - y_i|$ on Euclidean space $\mathbf{R}^N$.

The concepts defined below lead to the definition of parameterized collections of games. Given positive integers T and B, a parameterized collection of games with limited side payments is a collection of games each of which can be approximated by a game with T player types and with effective coalition sizes uniformly bounded by B. Note that other than having T approximate types and nearly-effective coalition sizes bounded by B, there is no necessary relationship between any games in the collection. To motivate the concepts required in the definition, each is related to analogous concepts in the pregame framework.

Observe that for a compact metric space of player types, given any real number $\delta > 0$, there is a partition (not necessarily unique) of the space of player types into a finite number of subsets, each consisting of player types who are "δ-similar" to each other. Parameterized collections of games do not restrict to a compact metric space of player types, but do employ the idea of a finite number of approximate types.

Let (N, V) be a game and let δ be a nonnegative real number. Given a set $W \subset \mathbf{R}^N$ and a permutation τ of N, let $\sigma_\tau(W)$ denote the set formed from W by permuting the values of the coordinates according to the associated permutation τ. Given a partition $\{N[t] : t = 1, \ldots, T\}$ of N, a permutation τ of N is *type-preserving* if, for any $i \in N$, $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A *δ-substitute partition* of N is a partition $\{N[t] : t = 1, \ldots, T\}$ of N with the property that, for any type-preserving permutation τ and any coalition S,

$$H_\infty[V(S), \sigma_\tau^{-1}(V(\tau(S)))] \leq \delta.$$

Note that in general a δ-substitute partitionof N is not uniquely determined. Moreover, in contrast to games derived from pregames, two games in a parameterized collection may have the same partitions but have no other relationship to each other.

A (δ, T)-type game is simply a game with a finite number T of approximate types. Let δ be a nonnegative real number and let T be a positive integer. A game (N, V) is a (δ, T)-*type game* if there is a T-member δ-substitute partition $\{N[t] : t = 1, \ldots, T\}$ of N. The set $N[t]$ is interpreted as an *approximate type*. Players in the same element of a δ-substitute partition are δ-*substitutes.* When $\delta = 0$, they are *exact substitutes*.

In all extant studies of approximate cores of large games, some conditions are required to limit gains to collective activities, such as boundedness of marginal contributions to coalitions, as in Wooders and Zame (1984), or the less restrictive conditions of per capita boundedness and/or small group effectiveness, as in Wooders (1980, 1983, 1994a,b), for example. Small groups are effective if all or *almost all* gains to collective activities can be realized by groups bounded in size of membership. The following notion formulates the idea of small effective groups in the context of parameterized collections of games. Informally, groups of players containing no more than B members are β-effective if, by restricting coalitions to having fewer than B members, the loss to each player is no more than β. Formally, let (N, V) be a game, let β be a nonnegative real number, and let B be a positive integer. For each group $S \subset N$, define a corresponding set $V(S; B) \subset \mathbf{R}^N$ in the manner.

$$V(S; B) := \bigcup_{\mathcal{P}_B(S)} \sum_{S' \in \mathcal{P}_B(S)} V(S'),$$

where the union is taken over all partitions $\mathcal{P}_B(S)$ of S in groups that are restricted to have no more than B members. Note that, by superadditivity, $V(S; B) \subset V(S)$ for any $S \subset N$ and, by construction, $V(S; B) = V(S)$ for $|S| \leq B$. The game (N, V) has β-*effective* B-*bounded groups* if for every group $S \subset N$

$$H_\infty[V(S), V(S; B)] \leq \beta.$$

When $\beta = 0$, 0-effective B-bounded groups are called *strictly effective* B-*bounded groups.*

Now we can define our collections of games. Given positive integers T and B and nonnegative real numbers δ and β, let $G((\delta, T), (\beta, B))$ be the collection of all (δ, T)-type games that have β-effective B-bounded groups. Then

$$G((\delta, T), (\beta, B))$$

is called a *parameterized collection of games with limited side payments*.

Our results hold for all parameters δ and β that are sufficiently small, specifically, for $2(\delta + \beta) < m^*$, where m^* is a positive real number used in the definition of the Hausdorff distance. In Section 4 we introduce one additional parameter to describe games with limited side payments.

3. NONEMPTINESS OF EQUAL TREATMENT ε-CORES OF GAMES WITH LIMITED SIDE PAYMENTS

Let (N, V) be a game. A payoff x is *ε–undominated* if for all $S \subset N$ and $y \in V(S)$ it is not the case that $y_S \gg x_S + \vec{1}_S \varepsilon$. The payoff x is *feasible* if $x \in V(N)$. The ε-core of a game (N, V) consists of all feasible and ε-undominated allocations. When $\varepsilon = 0$, the ε-core is the *core*.

Given nonnegative real numbers ε and δ, we will define the *equal treatment ε-core* of a game (N, V) relative to a partition $\{N[t]\}$ of the player set into δ-substitutes as the set of payoffs x in the ε-core with the property that, for each t and all i and j in $N[t]$, it holds that $x_i = x_j$. Thus, a payoff for a game is in the equal-treatment ε-core if all players of the same approximate type are assigned equal payoffs.

3.1. *The Theorem*

Some additional restrictions are required to obtain nonemptiness of ε-cores of large games with limited side payments. We will discuss the indispensability of these conditions in the next subsection.

Let C be a positive real number. A game (N, V) has a *per capita payoff bound* of C if, for all coalitions $S \subset N$,

$$\sum_{a \in S} x_a \leq C|S| \qquad \text{for any } x \in V(S).$$

Let ρ be a positive real number, $0 < \rho \leq 1$. A (δ, T)-type game (N, V) is *ρ-thick* if for each t it holds that $|N[t]| \geq \rho|N|$.

Our first theorem demonstrates that, with convexity of payoff sets of grand coalitions, all sufficiently large games in a parameterized collection have nonempty equal treatment ε-cores. The size of a game required to ensure nonemptiness of the equal treatment ε-core depends on the parameters describing the games, a bound on feasible per capita payoffs, and a bound on the thickness (the percentage of players of each approximate type) of the player set.

THEOREM 1. *Let T and B be positive integers. Let C and ρ be positive real numbers*, $0 < \rho \leq 1$. *Then for any* $\varepsilon > 0$ *exists an integer* $\eta(\varepsilon, T, B, C, \rho)$

such that if

(a) $(N, V) \in G((\delta, T), (\beta, B))$,
(b) $V(N)$ *is convex*,
(c) (N, V) *is* ρ-*thick*,
(d) (N, V) *has a per capita payoff bound of* C, *and*
(e) $|N| \geq \eta(\varepsilon, T, B, C, \rho)$,

then the equal treatment $(\varepsilon + \delta + \beta)$-*core of* (N, V) *is nonempty.*

Remark 1. As in Wooders (1983), our proofs do not require that $V(N)$ be convex but rather only that, for each approximate type, we have convexity over payoffs to players of the same approximate type. Similarly, rather than per capita boundedness, our proofs only require that the set of equal treatment payoffs, represented as a subset of $\mathbf{R}^T_+$, be bounded. Our assumptions are given in the slightly stronger forms for simplicity of statement.

The proof of Theorem 1 uses the following definition of the ε-remainder core and another nonemptiness result. Informally, the ε-remainder core allows some small proportion of players to be ignored; a payoff is in the ε-remainder core if it is feasible and if it is in the core of a subgame whose player set contains all the remaining players. The ε-remainder core is defined as follows. Let (N, V) be a game. A payoff x belongs to the ε-*remainder core* if, for some group $S \subset N$, $(|N| - |S|)/|N| \leq \varepsilon$ and x_S belongs to the core of the subgame (S, V). We have the following result (Kovalenkov and Wooders, 1999a, Theorem 1 (Subcase for $q = 0$)): Let T and B be positive integers. For any $\varepsilon > 0$, there exists an integer $\eta_1(\varepsilon, T, B)$ such that if

(a) $(N, V) \in G((0, T), (0, B))$ and
(b) $|N| \geq \eta_1(\varepsilon, T, B)$,

then the ε-remainder core of (N, V) is nonempty.

A Sketch of the Proof. In the proof of Theorem 1 we first approximate a game (N, V) with T approximate types and β-effective B-bounded groups by another game (N, V^0) with T exact player types and strictly effective B-bounded groups, that is, by a game $(N, V^0) \in G((0, T), (0, B))$. We then use the above result on nonemptiness of the ε-remainder core to obtain nonemptiness of the ε-remainder core of (N, V^0). This result implies that there is a core payoff in a large subgame (S, V) and that the proportion of "leftover" or "remainder" players is bounded by ε. Using per-capita boundedness and thickness of the player set, we can "smooth" payoffs in the ε-remainder core across players of the same approximate types and thus compensate the leftover players. The convexity assumption allows us to obtain equal treatment by averaging the payoffs across players of the same type while maintaining feasibility. We present the formal proof of Theorem 1 in Section 7. ■

In the next subsection we present examples clarifying Theorem 1.

3.2. *Examples*

Our first example illustrates the statement of Theorem 1.

EXAMPLE 1. Let (N, V_0) be a superadditive game where, for any two-person coalition $S = \{i, j\}$, $j \neq i$,

$$V_0(S) := \{x \in \mathbf{R}^N : x_i \leq 1, x_j \leq 1, \text{ and } x_k = 0 \text{ for } k \neq i, j\}$$

and, for each $i \in N$,

$$V_0(\{i\}) := \{x \in \mathbf{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

For an arbitrary coalition S the payoff set $V_0(S)$ is given as the *superadditive cover*, that is,

$$V_0(S) := \bigcup_{\mathcal{P}_2(S)} \sum_{S' \in \mathcal{P}_2(S)} V_0(S'),$$

where the union is taken over all partitions $\mathcal{P}_2(S)$ of S in groups that are restricted to have no more than two members.

We now define a game (N, V_{co}) in the following way. For each $S \subset N$, let $V_{\text{co}}(S)$ denote the convex cover of the payoff set $V_0(S)$, that is,

$$V_{\text{co}}(S) := \text{co}(V_0(S)).$$

Obviously the game (N, V_{co}) has convex payoff sets, one a player type, and a per capita bound of 1. We leave it to the reader to verify that for any positive integer $m \geq 3$ the game (N, V_{co}) has $(1/m)$-effective m-bounded groups. Thus the game (N, V_{co}) is a member of the class $G((0, 1), (1/m, m))$ and is 1-thick. Then, by Theorem 1, for $|N| \geq \eta(\varepsilon, 1, m, 1, 1)$ the equal treatment $(\varepsilon + 1/m)$-core of (N, V_{co}) is nonempty.

Therefore, for any $\varepsilon^0 > 0$ there is a positive integer $N(\varepsilon^0)$ such that for any $|N| \geq N(\varepsilon^0)$ the game (N, V_{co}) has a nonempty equal treatment ε^0-core. (For example, take an integer $m^0 \geq 2/\varepsilon^0$ and define $N(\varepsilon^0) \geq \eta(\varepsilon^0/2, 1, m^0, 1, 1)$.)

The next two examples illustrate that neither convexity of $V(N)$ nor thickness of the game can be omitted in the statement of Theorem 1.

EXAMPLE 2 (Convexity of $V(N)$). Consider a sequence of games without side payments $(N^m, V^m)_{m=1}^{\infty}$, where the mth game has $2m+1$ players. Suppose that any player alone can earn at most 0 units. Suppose that any two-player coalition can distribute a total payoff of 2 units in any agreed-upon way, while there is no transferability of payoff between coalitions. Suppose only one- and two-player coalitions are effective. Then the game is described by the following parameters: $T = 1, \rho = 1, B = 2, \delta = \beta = 0$, and per capita bound $C = 1$. Thus the game satisfies all the assumptions of Theorem 1 except convexity of $V(N)$. The $\frac{1}{3}$-core of the game, however, is empty for arbitrarily large values of m. (At any feasible payoff, at least one player gets 0 units and some other player no more than 1 unit. These two players can form a coalition and receive 2 units in total. This coalition can improve upon the given payoff for each of its members by $\frac{1}{2}$.)

EXAMPLE 3 (Thickness). Consider a sequence of games without side payments $(N^m, V^m)_{m=1}^{\infty}$, where the mth game has $m+3$ players. Suppose that there are two types of players. Let the mth game have 3 players of type A and m players of type B. Assume that only players of type A are essential in the following sense: Any coalition with less than two players of type A can get only 0 units or less for each of its players. Suppose that any coalition with two or three players of type A can get only two units which will be distributed between the players of type A in the coalition and 0 units or less for each of the players of type B. Note that there is no transferability of payoff from the players of type B to the players of type A. The game is then described by the following parameters: $T = 2, B = 3, \delta = \beta = 0$, and $C = 1$. Moreover, $V^m(N)$ is convex. The $\frac{1}{7}$-core of the game, however, is empty for arbitrarily large values of m. (For any feasible payoff there is a player of type A who receives no more than $\frac{2}{3}$ units. There is another player of type A that gets no more than one unit. These two players can form a coalition and receive two units in total. This coalition can improve upon the given payoff for each of its members by $\frac{1}{6}$, since $(2 - \frac{5}{3})\frac{1}{2} = \frac{1}{6}$.)

Remark 2 (Per Capita Boundedness and Small Group Effectiveness). Recall that for large games with side payments derived from pregames Wooders (1994b) established an equivalence between small group effectiveness and per capita boundedness.[10] Within the framework of parametrized collections of games, this equivalence no longer holds. This raises the question of whether either one or the other of the conditions can be dropped. Example A1, in the Appendix, shows that the condition of small-group effectiveness cannot be disregarded. The necessity of the assumption of per capita boundedness is not clear. Example A2 in the Appendix illustrates

[10] The games must be large in the sense that any player that appears in the game must have many close substitutes.

that per capita boundedness is not a consequence of the other conditions in Theorem 1 but we have no example showing that per capita boundedness is indispensable. Note, however, that per capita boundedness is a very mild assumption satisfied by most economic examples.

4. THE EQUAL TREATMENT PROPERTY OF ε-CORES OF GAMES WITH LIMITED SIDE PAYMENTS

In the previous section we presented results on nonemptiness of the ε-cores for large games with limited side payments. In fact, Theorem 1 shows nonemptiness of the equal-treatment ε-core as well. In the current section we focus on games where some degree of side payments is required for small groups. Specifically, we require q-comprehensiveness for nearly-effective groups. We then demonstrate that under this condition the ε-core of a game with limited side payments is equal treatment in a strong sense: any payoff in this ε-core treats *all* players that are approximate substitutes nearly equally.

4.1. *The Theorem*

Consider a set $W \subset \mathbf{R}^S$. Recall that W is *comprehensive* if $x \in W$ and $y \le x$ implies $y \in W$. The set W is *strongly comprehensive* if it is comprehensive, and whenever $x \in W, y \in W$, and $x < y$ there exists $z \in W$ such that $x \ll z$.[11]

Our second theorem requires a uniform version of strong comprehensiveness. Given $x \in \mathbf{R}^S$, $i, j \in S$, $0 \le q \le 1$, and $\phi \ge 0$, define a vector $x^q_{i,j}(\phi) \in \mathbf{R}^S$ by

$$(x^q_{i,j}(\phi))_i = x_i - \phi,$$
$$(x^q_{i,j}(\phi))_j = x_j + q\phi,$$

and

$$(x^q_{i,j}(\phi))_k = x_k \qquad \text{for } k \in S\backslash\{i, j\}.$$

The set W is *q-comprehensive* if W is comprehensive and if, for any $x \in W$, it holds that $(x^q_{i,j}(\phi)) \in W$ for any $i, j \in S$ and any $\phi \ge 0$.

The condition of q-comprehensiveness for $q > 0$ bounds the slope of the Pareto frontier of the set away from zero. For q equal to one, q-comprehensiveness is equivalent to the assumption of transferable utility.

[11] Informally, if one person can be made better off (while all the others remain at least as well off), then all persons can be made better off. This assumption, called "quasi-transferable utility" in Wooders (1983), has also been called "nonleveledness."

For q equal to zero, q-comprehensiveness means simply comprehensiveness. Note that if a set is q-comprehensive for some $q > 0$ then the set is q'-comprehensive for all q' with $q \geq q' \geq 0$.

Now let $G((\delta, T), (\beta, B))$ be a parameterized collection of games with limited side payments that was defined in Section 2. Let $0 \leq q \leq 1$ be given. Let $G((\delta, T), (\beta, B, q))$ denote the subcollection of games in $G((\delta, T), (\beta, B))$ with the property that if (N, V) is a member of the subcollection then for all $S \subset N$ with $|S| \leq B$, $V_S = \{x_S : x \in V(S)\}$ is q-comprehensive. Note that

$$\begin{aligned} G((\delta, T), (\beta, B, q)) &\subset G((\delta, T), (\beta, B, q')) \\ &\subset G((\delta, T), (\beta, B, 0) = G((\delta, T), (\beta, B)) \end{aligned}$$

for all q and q' such that $q \geq q' \geq 0$.

Our main theorem requires some degree of side-paymentness for small coalitions, those coalitions containing no more members than the given bound on nearly effective coalition sizes. Payoff sets for nearly-effective coalitions are required to satisfy q-comprehensiveness for some $q > 0$; that is, side payments can be made between players in nearly effective coalitions at a rate bounded below by q. For large coalitions, however, q can be arbitrarily small or zero. The theorem demonstrates that if a game with limited side payments has sufficiently many players of any particular approximate type, then all payoffs in the ε-core treat all members of that type nearly equally, that is, any two players of such an approximate type receive approximately the same ε-core payoffs. The distance between ε-core payoffs of any of the two players is bounded above by a function of the parameters describing the game. From inspection of the bound, given B and q, we can conclude that as approximate types become close to exact types, nearly effective groups become strictly effective and ε goes to zero, the ε-core "shrinks" to the equal-treatment ε-core.

THEOREM 2. *Let $(N, V) \in G((\delta, T), (\beta, B, q))$ where $q > 0$, and let $\{N[t]\}$ be a partition of N into δ-substitutes. If some payoff x belongs to the ε-core of (N, V) and for some t it holds that $|N[t]| > B$, then for any $t_1, t_2 \in N[t]$,*

$$|x_{t_1} - x_{t_2}| \leq \frac{2B}{q}(\varepsilon + \delta + \beta).$$

It is important to bear in mind in interpreting the theorem that for large groups q-comprehensiveness is not assumed. As noted previously, we have in mind situations such as economies with many small clubs or marriages, for example, where transfers may be made within the group, but large transfers, which would require the cooperation of large groups of players, are ruled out.

We present the proof of Theorem 2 in Section 7. In the next subsection we present examples clarifying Theorem 2.

4.2. *Examples*

The following simple example illustrates the statement of Theorem 2 and also shows that, in the central case of $\delta = \beta = 0$, the bound provided cannot be improved upon.

EXAMPLE 4. Let (N, V^*) be a superadditive game where for any two-person coalition $S = \{i, j\}$, $j \neq i$,

$$V^*(S) := \{x \in \mathbf{R}^N : x_i + x_j \leq 2, \text{ and } x_k = 0 \text{ for } k \neq i, j\}$$

and for each $i \in N$ let

$$V^*(\{i\}) := \{x \in \mathbf{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

For an arbitrary coalition S the payoff set $V^*(S)$ is given as the *superadditive cover*, that is,

$$V^*(S) := \bigcup_{\mathcal{P}_2(S)} \sum_{S' \in \mathcal{P}_2(S)} V^*(S'),$$

where the union is taken over all partitions $\mathcal{P}_2(S)$ of S in groups that are restricted to have no more than two members.

Observe that the game (N, V^*) has one exact player type and strictly effective two-bounded groups. Therefore $(N, V^*) \in G((0, 1), (0, 2))$. Moreover, V_S^* is 1-comprehensive for any $S \subset N$, $|S| \leq 2$. Let the game (N, V^*) have an even number of players, that is, $|N| = 2m$ for some positive integer m. Observe that the payoff which gives one unit for each player belongs to the core of (N, V^*). Thus, for any $\varepsilon \geq 0$, the ε-core of (N, V^*) is nonempty. Theorem 2 implies that for any $a, b \in N$ and any payoff x in the ε-core of (N, V^*), providing that $|N| > 2$, we must have

$$|x_a - x_b| \leq 4\varepsilon.$$

It is easy to check that this statement cannot be improved. Observe first that the restriction $|N| > 2$ is essential. If $|N| = 2$ then any division of two units between two players is a core payoff; thus, for $\varepsilon < \frac{1}{2}$ the bound of the theorem is not applicable. Now consider a payoff y that gives one unit to each of the players except two selected players (say, players a and b), $1 - 2\varepsilon$ to player a, and $1 + 2\varepsilon$ to player b. The reader can observe that even in the case $|N| > 2$ this payoff is in the ε-core of the game (N, V^*). (The payoff is obviously feasible, but it is also ε-undominated, since any pair of players get at y at least $2 - 2\varepsilon$.) But at this payoff it holds that $|y_a - y_b| = 4\varepsilon$. Thus the bound given in Theorem 2 cannot be improved.

It is well known that a theorem such as the above will not hold for games with transferable utility. The following example provides an illustration.

EXAMPLE 5a. Fix any $\varepsilon > 0$. Consider a game with transferable utility (N, v) that has one exact player type and a nonempty equal-treatment core. (There are many such games.) Take any payoff $x \in \mathbf{R}^N$ in the equal treatment core of (N, v). This determines a number $\bar{x}$ as the payoff for each player in N. Now select one player (say, player a). Consider the payoff y that assigns $\bar{x} - \varepsilon$ for all players except player a and $\bar{x} + (|N| - 1)\varepsilon$ to player a. Obviously the payoff y is feasible and ε-undominated. Thus y belongs to the ε-core of (N, v). However, for a very large $|N|$, y treats player a much better than it does the others.

The sharpest result that can be obtained for games with transferable utility induced by pregames satisfying per capita boundedness (or even strict small-group effectiveness) is that, for small values of ε, *most* players of the same type are treated approximately equally (Wooders, 1980, 1994a). For games without side payments that have q-comprehensive payoff sets for $q > 0$ (q sidepayments) it also does not hold that the ε-core treats all players of the same type approximately equally. The following example, quite similar to the previous one, demonstrates this point.

EXAMPLE 5b. Fix any $\varepsilon > 0$, $q > 0$. Consider a game (N, V) with q-comprehensive payoff sets, one exact player type, and a nonempty equal treatment core. (As above, there are many such games.) Take any payoff in the equal treatment core of (N, V) and define $\bar{x}$ as the payoff assigned to each player in N. Now select one player, say a. Consider the payoff y^q that assigns $\bar{x} - \varepsilon$ for all players except player a and $\bar{x} + q(|N| - 1)\varepsilon$ for a player a. Then the payoff y^q is feasible and ε-undominated. Thus y^q belongs to the ε-core of (N, V). For very large $|N|$, however, y^q treats one player much better than the others.

The above examples show that Theorem 2 depends crucially on the fact that side payments are limited. The other condition of Theorem 2, q-comprehensiveness ($q > 0$) for payoff sets of small coalitions, is also crucial for the equal-treatment property of the ε-core. Example 6 below, which continues Example 1, shows that without this restriction the ε-core may contain nonequal-treatment payoffs, even for games with limited side payments.

EXAMPLE 6. Recall the game (N, V_0) defined in Example 1. Observe that the game (N, V_0) has one exact player type, strictly effective 2-bounded groups, and a per capita bound of 1. Thus, $(N, V_0) \in G((0, 1), (0, 2))$. But, for any $q > 0$ and any coalition S of two players, the payoff sets $V(S)$ are not q-comprehensive. Let m be a positive integer. Let (N^m, V_0^m) be a game where the number of players in the set N^m is $2m+1$ and for any coalition $S \subset N^m$ $V_0^m(S) := V_0(S)$. Thus, each game (N^m, V_0^m) has an odd number of players.

It is easy to see that the payoff y giving 1 to each of $2m$ players and 0 to the last player (say player a) is in the core. (The "left-out" player, in a coalition by himself, cannot make both himself and a player in a two-person coalition better off—the player in the two-person coalition cannot be given more than 1.) However, for any player other than a, say, player b, we have $y_b = 1$. Thus $|y_a - y_b| = 1$: even though all players are exact substitutes and $\delta = \beta = 0$, the core does not converge to the equal-treatment core, even for an arbitrarily large m.

5. FURTHER REMARKS

Remark 3 (Relationships to Other Papers Using Parameterized Collections of Games).

(1) Our other results (Kovalenkov and Wooders, 1999a,b) for the ε-core depend on a notion of q-comprehensiveness. In fact, all our other results require that payoff sets be q-comprehensive with $q > 0$; the case $q = 0$ has not been treated. One of the achievements of the current paper is to treat the case where $q > 0$ only for small coalitions, thus allowing transfers within small effective coalitions but ruling out arbitrarily large transfers.

(2) The definition of β-effective B-bounded groups used in this paper is different than that in our previous papers. There, rather than requiring that $V(S;B)$ be close to $V(S)$ for every group S, we required that the "q-comprehensive cover" of $V(S;B)$ be close to $V(S)$. The notion of β-effective B-bounded groups used in this paper is important for our equal-treatment result.

Remark 4 (Small Group Effectiveness for Improvement). There are several possibilities for defining β-effective B-bounded groups. Our definition requires that partitions of the total player set into B-bounded groups must realize almost all payoffs in $V(N)$. This approach has the advantage of ease of exposition. Another possibility is to require that all improvements can be carried out by groups bounded in size. Let us say that a payoff x can be *β-improved upon* by some coalition S in a game (N, V) if there is a payoff $y \in V(S)$ such that $y_S \gg x_S - \beta\vec{1}_S$. Suppose that a game (N, V) satisfies the property that for each payoff $x \in V(N)$, if x can be *$(\beta/2)$-improved* by some coalition then there is a coalition S satisfying $|S| \leq B$ that can *β-improve* upon x. Then (N, V) has *β-improvement-effective B-bounded groups.* The condition that a game has bounded group sizes that are effective for improvement is obviously less restrictive than the condition defined earlier of β-effective B-bounded groups. The conditions are, however, interchangeable.[12] The only difference in our results, were we to use β-effectiveness for improvement, would be that the bounds would change. Our motivation in choosing the condition of β-effective B-bounded groups is that the condition is more easily verifiable since it only depends on what groups of individuals can do rather than on the entire game structure, and it seems more consistent with the sort of examples most frequently used in the literature.[13]

Remark 5 (Effectiveness of Relatively Small Groups). It may be possible to obtain analogues of the results of this paper for situations where bounds are placed on relative sizes of effective coalitions rather than on absolute sizes. We've chosen to bound the absolute sizes of near-effective coalitions since familiar examples of games and economies have natural bounds on absolute sizes of effective coalitions, for example, marriage models.

6. ECONOMIES WITH LIMITED SIDE PAYMENTS

In application, the conditions of our results may not be unduly restrictive. Indeed, in some game-theoretic and economic situations, limited side payments would appear to be natural, for example, in job-matching models (Kelso and Crawford, 1982; Roth and Sotomayor, 1990; and others). Even if side payments were possible, in large games it may require cooperation and agreement between a large number of players to make large transfers. In this case the framework of side payments, or even q-comprehensiveness for $q > 0$, may be quite problematic. On the other hand, the presumption of economics that there exist opportunities for gains to trade and exchange is at the heart of economics. Thus, the requirement of some degree of side payments within small coalitions may not be restrictive.

[12] See Wooders (1994a, Proposition 3.8) for the relationship between these two notions of effectiveness in the context of games derived from pregames. In other contexts, an example of the interchangeability of these conditions is given by Mas-Colell (1979) and Kaneko and Wooders (1989). Mas-Colell uses the fact that almost all improvement in exchange economies can be carried out by groups bounded in size while Kaneko and Wooders use the fact that almost all feasible outcomes can be achieved by partitions of the total player set into groups bounded in size.

[13] For example, marriages involve two persons; softball teams have nine members, and when average costs of production are downward sloping and bounded away from zero there is some size of plant that is "almost efficient."

There are also numerous specific examples to which our results apply. Consider a two-sided matching model where all gains to collective activities can be realized by groups consisting of one person from each side of the market and where utility is transferable within buyer–seller pairs. Our equal-treatment result applies. If there is some mechanism convexifying the total payoff set (for example, lotteries), then our nonemptiness result also holds. These remarks also apply to partitioning games.[14]

Another class of economic models whose derived games may fit well into our assumptions include economies with local public goods or clubs. (A survey is provided in Conley and Wooders, 1998). Kovalenkov and Wooders (1999a) introduce a model of an economy with clubs (and multiple memberships) and, under remarkably nonrestrictive conditions, demonstrate nonemptiness of approximate cores. With appropriate conditions on the economic models, the results of the current paper could be applied to models of economies with clubs.

7. PROOFS

Proof of Theorem 1. Recall that we required $2(\beta + \delta) < m^*$. Assume for now that $\varepsilon \leq \min\{m^*/2, BC\}$, implying that $(\varepsilon + \beta + \delta) < m^*$. We first consider the case where the game (N, V) has strongly comprehensive payoff sets and then use the result for this case to obtain the result for the general case.

Case 1. Suppose the game (N, V) has strongly comprehensive payoff sets. Define for each $S \subset N$

$$V'(S) := \bigcap \sigma_\tau^{-1}(V(\tau(S))),$$

where intersection is taken over all type-consistent permutations τ. Then $(N, V') \in G((0, T), (\beta, B))$ and $V'(N)$ is convex. Moreover, $V'(S) \subset V(S)$ and

$$H_\infty(V'(S), V(S)) \leq \delta$$

for any $S \subset N$. Now let us define for any $S \subset N$,

$$V^0(S) := V'(S; B).$$

Then $(N, V^0) \in G((0, T), (0, B))$. Moreover,

$$V^0(S) \subset V'(S) \subset V(S)$$

[14]See Garratt and Qin (1997) for examples of games with lotteries; Kaneko (1982) for NTU assignment markets; and Kaneko and Wooders (1982) and le Breton *et al.* (1992) for partitioning games.

and

$$H_\infty(V^0(S), V(S)) \le H_\infty(V^0(S), V'(S)) + H_\infty(V'(S), V(S)) \le \beta + \delta.$$

In addition, the game (N, V^0) has a per capita payoff bound of C and strongly comprehensive payoff sets.

Applying Kovalenkov and Wooders (1999a, Theorem 1) for the ε^0-remainder core, with

$$\varepsilon^0 := \frac{\rho}{BC}\varepsilon,$$

to the game (N, V^0), it follows that for $|N| \ge \eta_1(\varepsilon^0, T, B)$ there is some coalition S such that the subgame (S, V^0) has a nonempty core and $|S|/|N| \ge 1 - \varepsilon^0$. Let x be a payoff in the core of the subgame (S, V^0). Note that $\rho > \varepsilon^0$.

We now prove that for $|N| \ge (B+1)/(\rho - \varepsilon^0)$ the payoff x will have the equal-treatment property; that is, for every t and any $i, j \in N[t]$, $x_i = x_j$. Suppose there is a type t and two players $i_1, i_2 \in N[t] \cap S$ satisfying $x_{i_1} > x_{i_2}$. By feasibility of x, there is some partition $\{S^k\}$ of S, $|S^k| \le B$ for each k, such that $x \in \sum_k V^0(S^k)$. Then, since $|N| \ge (B+1)/(\rho - \varepsilon^0)$ we have that $|N[t] \cap S| \ge \rho|N| - \varepsilon^0|N| > B$ and there must exist $j_1, j_2 \in N[t] \cap S$ such that $x_{j_1} \in S^{k_1}, x_{j_2} \in S^{k_2}$ for $S^{k_1}, S^{k_2} \subset \{S^k\}$, $S^{k_1} \ne S^{k_2}$, and $x_{j_1} > x_{j_2}$. (If x_{i_1}, x_{i_2} are in the same element of $\{S^k\}$, there exists $j \in N[t] \bigcap S$ in the different element of $\{S^k\}$ than i_1, i_2 and it can't be $x_j = x_{i_1}$ and $x_j = x_{i_2}$ since $x_{i_1} \ne x_{i_2}$. Thus we can suppose without loss of generality that j_1 and j_2 are in different elements in $\{S^k\}$.) But then the player set $S^{k_1}\backslash\{j_1\}$ can form the coalition $(S^{k_1}\backslash\{j_1\}) \cup \{j_2\}$ with player j_2 rather then j_1. By strong comprehensiveness of payoff sets and since players j_1, j_2 are exact substitutes in the game (S, V^0), all players in the new coalition can be strictly better off than in x. This contradicts the supposition that x is a core payoff for the game (S, V^0). Thus x has the equal treatment property.

Now let $\eta'(\varepsilon, T, B, C, \rho) := \max\{\eta_1(\varepsilon^0, T, B), (B+1)/(\rho - \varepsilon^0)\}$, where, as above, $\varepsilon^0 = (\rho/BC)\varepsilon$. As we have shown, for $|N| \ge \eta'(\varepsilon, T, B, C, \rho)$ there is a coalition $S \subset N$ satisfying $|S|/|N| \ge 1 - \varepsilon^0$ and, for each t, $|S \bigcap N[t]| > B$, and there is a payoff $x \in \mathbf{R}^S$ in the equal-treatment core of (S, V^0).

Next, define a payoff $z \in \mathbf{R}^N$ that, for each t, assigns to each $a \in N[t]$ the same payoff assigned to players of type t by x. This payoff z is undominated in the game (N, V^0). (If payoff z were dominated by some coalition S_1 in (N, V^0) then z could be dominated by some coalition $S_2 \subset S_1$, $|S_2| \le B$. But there exists a coalition $S_3 \subset S$ with the same profile as S_2. This coalition S_3 dominates the payoff x in (S, V^0), contradicting the fact that x is in the core of (S, V^0).) Moreover, since $x \in \sum_k V^0(S^k)$ for some partition $\{S^k\}$

of S with $|S^k| \leq B$, by per capita boundedness $x_a \leq BC$ for any $a \in S$. Thus $z_a \leq BC$ for any $a \in S$.

Now construct a payoff $x' \in \mathbf{R}^N$ from the payoff for $x \in \mathbf{R}^S$ by

$$x'_S := x_S \text{ and } x'_a := 0 \qquad \text{for all } a \notin S.$$

Let $y \in \mathbf{R}^N$ be the average of all payoffs $\sigma_\tau(x')$ across all type-consistent permutations τ of N. Then y has an equal-treatment property. Moreover, for every t and any $a \in N[t]$,

$$z_a - y_a = z_a - \frac{|N[t] \cap S|}{|N[t]|} z_a = \frac{|N[t]| - |N[t] \cap S|}{|N[t]|} z_a$$
$$\leq \frac{|N| - |S|}{|N[t]|} z_a \leq \frac{\varepsilon^0}{\rho} z_a \leq \frac{\varepsilon^0}{\rho} BC = \varepsilon.$$

Since z is undominated in (N, V^0), y is ε-undominated in (N, V^0). Therefore for $|N| \geq \eta'(\varepsilon, T, B, C, \rho)$ the payoff y has the equal-treatment property and is $(\varepsilon + \beta + \delta)$-undominated in the original game (N, V). In addition, observe that $x' \in V^0(N) \subset V'(N)$. Therefore $\sigma_\tau(x') \in V'(N)$ for any type-consistent permutation τ of N. By the convexity of $V'(N)$ and by construction of y, it holds that $y \in V'(N) \subset V(N)$. Thus, for $|N| \geq \eta'(\varepsilon, T, B, C, \rho)$ the payoff y belongs to the equal-treatment $(\varepsilon + \beta + \delta)$-core of (N, V).

Case 2. Now we consider the general case, where the game (N, V) does not necessarily have strongly comprehensive payoff sets. We first approximate the game (N, V) by another game (N, V') with strongly comprehensive payoff sets. This approximation can be done sufficiently closely so that $V(S) \subset V'(S)$ and $H_\infty(V(S), V'(S)) \leq \varepsilon/2$ for any $S \subset N$. Note that the approximation is easy since, by our definition of the Hausdorff distance, we must ensure close approximation only on a compact set. (For details of such an approximation see Wooders (1983, Appendix).) Let $\eta(\varepsilon, T, B, C, \rho) := \eta'(\varepsilon/2, T, B, C, \rho)$. By Case 1 above, for $|N| \geq \eta(\varepsilon, T, B, C, \rho)$ the equal treatment $(\varepsilon/2 + \beta + \delta)$-core of the game (N, V') is nonempty. But then for $|N| \geq \eta(\varepsilon, T, B, C, \rho)$ the game (N, V) has a nonempty equal-treatment $(\varepsilon + \beta + \delta)$-core. This is the conclusion we need for $\varepsilon \leq \min\{m^*/2, BC\}$.

Finally for $\varepsilon > \min\{m^*/2, BC\}$, consider $\varepsilon' := \min\{m^*/2, BC\}$ and let $\eta(\varepsilon, T, B, C, \rho) := \eta(\varepsilon', T, B, C, \rho)$. Since $\varepsilon \geq \varepsilon'$, for $|N| \geq \eta(\varepsilon, T, B, C, \rho)$ again the equal-treatment $(\varepsilon + \beta + \delta)$-core of the game (N, V) is nonempty. ■

Proof of Theorem 2. Consider some payoff vector x in the ε-core of (N, V). Define a payoff vector y by $y(\{i\}) := x(\{i\}) - \beta$ for each $i \in N$. Since (N, V) has β-effective B-bounded groups, there exists a partition $\{S^k\}$ of N, with $|S^k| \leq B$ for each k, such that $y \in \sum_k V(S^k)$. Moreover, for the game (N, V), by construction the payoff y cannot be improved upon by any coalition $S \subset N$ for each of its members by more than $(\varepsilon + \beta)$.

Case 1. Consider the case where, according to the partition $\{S^k\}$, two players t_1 and t_2 of the same approximate type (that is, $t_1, t_2 \in N[t]$) are in different coalitions. Suppose $t_1 \in S^1$ and $t_2 \in S^2$. Suppose also that $y_{t_1} > y_{t_2}$. Then players in $S^1 \backslash \{t_1\}$ would prefer to form a new coalition with t_2 rather than t_1. Let $S = S^1 \backslash \{t_1\}$. Let τ denote a permutation of N that permutes only t_1 and t_2. Since players t_1 and t_2 are of the same approximate type, τ is a type-consistent permutation, and

$$H_\infty\left[V(S^1), \sigma_\tau(V(S \cup \{t_2\}))\right] \leq \delta.$$

Hence there exists a payoff z that is feasible for $S \bigcup \{t_2\}$ and close to the payoff y with the loss for any player in $S \bigcup \{t_2\}$ of not more then δ. Since transfers at rate q are possible, this new coalition can improve upon the payoff vector y by more than $(\varepsilon + \beta)$ for each player if

$$q(|y_{t_1} - y_{t_2}| - (\varepsilon + \beta) - \delta) > |S|\delta + |S|(\varepsilon + \beta).$$

Since y is an $(\varepsilon + \beta)$-core payoff we must have $|y_{t_1} - y_{t_2}| - (\varepsilon + \delta + \beta) \leq (1/q)|S|(\varepsilon + \delta + \beta) \leq (1/q)(B - 1)(\varepsilon + \delta + \beta)$. Hence, $|x_{t_1} - x_{t_2}| = |y_{t_1} - y_{t_2}| \leq (B/q)(\varepsilon + \delta + \beta)$.

Case 2. Now suppose that t_1 and t_2 are in the same element of the partition $\{S^k\}$. Then, since $|N[t]| > B$, there exists $t_3 \in N[t]$ that belongs to a different member of the partition $\{S^k\}$. Hence by Case 1 above; $|x_{t_1} - x_{t_2}| \leq |x_{t_1} - x_{t_3}| + |x_{t_3} - x_{t_2}| \leq (2B/q)(\varepsilon + \delta + \beta)$. ■

APPENDIX: ADDITIONAL EXAMPLES

In this section we present the two examples discussed in Remark 2. Both illustrate the importance of the conditions in Theorem 1. Example A1 appeared previously in Kovalenkov and Wooders (1999a, Example 3).

EXAMPLE A1 (The Indispensability of the Small Group Effectiveness Assumption). Consider a sequence of games $(N^m, v^m)_{m=1}^\infty$ with side payments and where the mth game has $3m$ players. Suppose that any coalition S consisting of at least $2m$ players can get up to $2m$ units of payoff to divide among its members, that is, $v^m(S) = 2m$. Assume that if $|S| < 2m$, then $v^m(S) = 0$. Observe that each game has one exact player type and a per capita Bound of 1. That is, $\rho = 1, T = 1, C = 1$, and $\delta = 0$. However, the $\frac{1}{7}$-core of the game is empty for arbitrarily large values of m.

For any feasible payoff there are m players that are assigned in total no more than $(2m/3m)m = \frac{2}{3}m$. There are another m players that are assigned in total no more than $(2m/2m)m = m$. These $2m$ players can form a coalition and receive $2m$ in total. This coalition can improve upon the given payoff for each of its members by $\frac{1}{6}$, since $(2m - \frac{5}{3}m)(1/2m) = \frac{1}{6}$.

Example A2 (The Independence of the per Capita Boundedness Assumption). Let m be an arbitrary positive integer. Let (N, V^m) be a superadditive game where, for any coalition S, with $|S| > 1$,

$$V^m(S) := \{x \in \mathbf{R}^N : \ x_i \leq m \text{ for } i \in S, \text{ and } x_k = 0 \text{ for } k \notin S\}$$

and, for each $i \in N$,

$$V^m(\{i\}) := \{x \in \mathbf{R}^N : x_i \leq 0 \text{ and } x_j = 0 \text{ for all } j \neq i\}.$$

Now let us define a collection of games

$$G^* := \left\{(N, V^m) : m \in \mathbf{Z}_+\right\}.$$

Obviously any game $(N, V) \in G^*$ has convex payoff sets and a nonempty core. Moreover, any game $(N, V) \in G^*$ has one (exact) player type and thus is 1-thick. We leave it to the reader to verify that any game $(N, V) \in G^*$ has strictly effective 3-bounded groups. Thus, $G^* \subset G((0, 1), (0, 3))$ and all games in G^* are 1-thick and have convex payoff sets and nonempty cores.

However, the class G^* is not restricted by any common per capita boundedness condition. For any positive integer C and any $m > C$, the game $(N, V^m) \in G^*$ does not satisfy the per capita bound of C. Hence the condition of the common per capita bound on the collection of games is independent of all other conditions (and the implication) in Theorem 1.

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