Equivalence of Games and Markets

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EQUIVALENCE OF GAMES AND MARKETS[1]

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The author proves an equivalence between large games with effective small groups of players and games generated by markets. Small groups are effective if all or almost all gains to collective activities can be achieved by groups bounded in size. A market is an exchange economy where all participants have concave, quasi-linear payoff functions. The market approximating a game is socially homogeneous—all participants have the same monotonic nondecreasing, and 1-homogeneous payoff function. Our results imply that any market (more generally, any economy with effective small groups) can be approximated by a socially homogeneous market.

KEYWORDS: Games, markets, market-game equivalence, small group effectiveness, social homogeneity.

1. SMALL GROUP EFFECTIVENESS AND SOCIALLY HOMOGENEOUS MARKETS

THIS PAPER ESTABLISHES AN EQUIVALENCE between socially homogeneous markets with payoff-constant returns and large games with effective small groups of players. Small groups are effective if all or almost all gains to collective activities can be realized by the activities of groups of players bounded in absolute size. A market is defined as a private-goods economy where all participants have concave payoff functions that are linear in money. The market is socially homogeneous if all participants have the same payoff function; the market satisfies payoff-constant returns if the payoff functions are 1-homogeneous. The market approximating a large game with effective small groups also has the property that the payoff function is continuous. When the continuity assumption is relaxed at the boundaries of the commodity space, an equivalence of large games and markets holds with only the apparently mild requirement of per capita boundedness of payoffs.[3] The economies to which the results apply include ones with nonmonotonicities, nonconvexities, and consumption sets unbounded from above and below. Also, the economies may have public goods,

[1] Previous versions of this paper include parts of *C.O.R.E* Discussion Paper No. 8842, "Large Games are Market Games" (1988) and University of Toronto Department of Economics Working Paper 1904 "Equivalence of Perfect Competition and Effective Small Groups" (1992).

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[3] The condition of per capita boundedness was introduced in the study of large games in Wooders (1979); see Kannai (1992) or Wooders (1992b) for a discussion of the model and the main result. Small group effectiveness was introduced in Wooders (1992a).

collectively consumed and produced goods, and other deviations from the Arrow-Debreu model of an exchange economy.

Our results are obtained in the framework of a model of games with a finite set of player types and side payments. A game consists of a set of players and a function, called the characteristic function, which assigns an amount of surplus to each group of players. The characteristic function is intended to provide an abstract summary description of economic data and is not intended to describe economic behavior; there is no presumption of cooperation.

Our first market-game equivalence result states that the class of large games with effective small groups and the class of continuous, socially homogeneous markets with payoff-constant returns are approximately equivalent. Our second market-game equivalence result establishes that per capita boundedness suffices to ensure the equivalence of large games and socially homogeneous markets with payoff-constant returns. Note that with only per-capita boundedness, the markets are not required to be continuous. Since games generated by markets satisfy per capita boundedness, our result establishes that large markets are approximated by socially homogeneous markets with payoff-constant returns. We also establish that with thickness of the player set, bounding the percentages of players of each type away from zero, per capita boundedness and small group effectiveness are equivalent, thus relating our two conditions and our two market-game equivalence results.

Since the conditions of continuity, social homogeneity, and payoff-constant returns appear restrictive, our first market-game equivalence result indicates the power of the condition of small group effectiveness. Since per capita boundedness simply bounds average payoffs away from infinity and thus clearly permits a variety of economic structures, our second market-game equivalence result indicates the broad applicability of small group effectiveness.

In the approach of this author the set of marketed commodities is viewed as endogenous. There may be many different definitions of commodities for a market approximating a game. For example, for a game derived from an economy, the commodities in an approximating market can be chosen to be the same as those specified for the economy.[4] In this paper it is established that a large game or a large economy, regardless of the number of types of commodities, is approximated by a market where the number of types of commodities is no larger than the number of types of participants.

The asymptotic equivalence of markets and economies with effective small groups is suggested by a number of other results. One such suggestion was made by Tiebout (1956), who argued that economies with local public goods are "market-like." More recently several papers have shown that cores of such economies converge to price taking equilibrium outcomes.[5] Moreover, core convergence holds in exchange economies even when coalitions are constrained

[4] An application using this approach appears in Wooders (1992c).

[5] See also Buchanan (1965). Results on convergence of cores in economies with public goods are reviewed in Wooders (1994).

to be small relative to the total economy.[6] Recall that Shapley and Shubik (1969) showed that large exchange economies have nonempty approximate cores. Wooders (1983) shows that large games satisfying per capita boundedness and with a finite number of types have nonempty approximate cores and Kaneko and Wooders (1986) show that continuum games with small (finite) coalitions have nonempty cores. Shapley and Shubik (1966) showed an equivalence between the class of totally balanced games, those with the property that the game and every subgame have nonempty cores, and the class of games derived from continuous markets. The nonemptiness of approximate cores of large games implies approximate balancedness and thus implies that large games are approximately market games. Shapley (1964) first showed that Shapley values of replicated exchange economies converge to core payoffs. With a strong form of small group effectiveness (boundedness of individual marginal contributions to coalitions), Wooders and Zame (1987) show that Shapley values of large games are in approximate cores.[7] Moreover, the following "market-like" properties of large games with effective small groups are shown in Wooders (1992b): approximate cores of large games converge if and only if small groups are effective; approximate cores of large games with small effective groups satisfy a game-theoretic analogue of the "Law of Demand" (an increase in the percentage of players of any type does not cause an increase and may cause a decrease in core payoffs to each player of that type); approximate cores of large games with effective small groups and many players of each type have an approximate equal-treatment property—a core payoff assigns most players of the same type approximately the same payoff.[8]

The research noted above raises the possibility not only that large games and economies with effective small groups *behave* like competitive markets but also that these games and economies *are* competitive markets or close to them. Such observations suggest our equivalence result, which explains the similarities between markets and games.

Some relaxation of the finite-type and side payment assumptions of this paper is possible. Using familiar techniques of approximating a compact metric space of types/attributes by finite sets of types, extensions of all our results can be obtained with a compact metric space of player types and with differentiated commodity markets, as in Mas-Colell (1975) or Jones (1984), for example.[9] Indeed, no topological structure on the set of player types is necessary to make apparent the closeness between a large game with effective small groups and a

[6] See Anderson (1992) for a survey including research on convergence of the core with restrictions on coalition size.

[7] We conjecture that this result holds under the milder condition of small group effectiveness. Wooders and Zame (1987) provide further references to related literature on the Shapley value of large games and economies.

[8] Further references to related literature are provided in Wooders (1992b).

[9] In fact, the first version of this paper, C.O.R.E Discussion Paper No. 8842 (1988), used a compact metric space of player types. The framework with a compact metric space of player types and the assumption of small group effectiveness is further studied in Wooders (1992, 1993b). The later paper contains some extensions of the results of this paper.

market game.[10] The formulation with a finite number of types of players, however, is fundamental and permits a number of other questions to be addressed. An advantage of the formulation with a metric space of attributes (either finite or infinite) is that it facilitates the study of the continuum limit. The restriction to games with side payments can also be relaxed and some analogues of our results still obtain.[11]

The results in this paper obviously apply to the special case of games and economies with bounded effective group sizes, those when all gains to improvement can be realized by strict subgroups of the population. Games with bounded effective group sizes and a finite number of types are basic to the research of this paper and other related works since they are especially tractable and since they approximate games with effective small groups. Some special properties of games with bounded effective group sizes include that for all sufficiently large games, all core payoffs have the equal-treatment property (Wooders (1983, Theorem 3)). For a discussion of the special properties of games with bounded effective group sizes and a unified presentation of a number of results, see Wooders (1992b).

It should be observed that we are not concerned here with game-theoretic justification for the price-taking hypothesis of perfect competition. Rather than showing any equivalence of outcomes of solution concepts, we establish a game-theoretic equivalence between large games and large markets themselves. We return to this in the concluding section.

2. GAMES AND PREGAMES

There is a given finite number T of types of players. Let Z_+^T denote the T-fold Cartesian product of the nonnegative integers. A *profile* $f = (f_1, \dots, f_T) \in Z_+^T$ is a group of players, described by the number f_t of players of each type t in the group. The profile describing a group consisting of only one player of type t and no other players is denoted by χ_t. Note that if f is a profile and r is a positive integer, then rf is a profile. Also, if f and g are profiles, then so is $f+g$. Given a profile f, define $\|f\| = \Sigma_t f_t$, called the *norm* of f; the norm of f is the total number of players in the group f. The *support* of a profile f is the set $\{t \in \{1, \dots, T\}\colon f_t \neq 0\}$. A *partition of a profile* f is a collection of profiles (f^k), not necessarily all distinct, satisfying $\Sigma_k f^k = f$. A partition of a profile is analogous to a partition of a set except that all members of a partition of a set are distinct.

Let Ψ be a function from the set Z_+^T of profiles to $\boldsymbol{R}_+$ with $\Psi(0) = 0$. The pair (T, Ψ) is called a *pregame* with *characteristic function* Ψ. The value $\Psi(f)$ is the total payoff a group of players f can achieve by collective activities of the

[10] See Wooders (1993a).

[11] See Wooders (1983) for the model of games without side payments, some relevant results, and Wooders (1991) for further results and references.

group membership. The pregame is *superadditive* if

$$(2.1)\qquad \Psi(f)=\max \sum_{g\in P}\Psi(g),$$

where P is a partition of f and the maximum is taken over all partitions of f.

A *game determined by a pregame* (T,Ψ), called simply a *game* or a *game in characteristic form*, is a pair $[n,\Psi]$ where n is a profile, interpreted as a description of the total player set in the game, and the characteristic function Ψ is restricted to subprofiles of n. Note that a pregame specifies payoffs for every profile whereas a game specifies a total population, and the only relevant profiles are those no larger than the population. A payoff for the game is a vector x in $\boldsymbol{R}^T$. A payoff x is *feasible* if $x\cdot n\leqslant\Psi(n)$, and it is *Pareto optimal* if $x\cdot n=\Psi(n)$.

A pregame (T,Ψ) satisfies *small group effectiveness* if it is superadditive and if, given any real number $\varepsilon>0$, there is an integer $\eta_0(\varepsilon)$ such that for each profile f, for some partition (f^k) of f:

$$(2.2)\qquad \|f^k\|\leqslant\eta_0(\varepsilon)\qquad \text{for each subprofile } f^k \text{ in the partition, and}$$

$$(2.3)\qquad \Psi(f)-\sum_k\Psi(f^k)\leqslant\varepsilon\|f\|;$$

for every profile f, almost all (within ε per capita) of the gains to collective activities can be realized by aggregating collective activities within group of participants bounded in absolute size.[12]

When a pregame satisfies small group effectiveness we say that it has *effective small groups*. Note that this condition does not rule out effective large groups, although it does imply that only smaller and smaller increases in per capita payoff can be achieved by the formation of larger and larger groups. Elsewhere the condition has been called "inessentiality of large groups;" collective activities of large groups are not essential for the realization of almost all gains to group formation. The term "group" is used instead of the more commonly used term "coalition" since "coalition" may suggest cooperative behavior. Small group effectiveness is intended to be a condition on economic and game-theoretic primitives rather than on behavior.[13]

3. MARKETS AND PREMARKETS

A premarket consists of a finite number of commodity types and of participant types. A participant type is described by two characteristics, a payoff function and an endowment. Formally, a *premarket* is a pair $(\boldsymbol{R}^M_+, A)$ where $\boldsymbol{R}^M_+$

[12] Note that the profile f in the definition may be arbitrarily large relative to the bound $\eta_0(\varepsilon)$. Thus, in "very large" coalitions, almost all gains to collective activities can be realized as the aggregate of activities of "negligible"' groups of participants. This suggests the f-core approach, introduced in Kaneko and Wooders (1986), where finite coalitions are effective in games with a continuum of players.

[13] The condition of small group effectiveness is less restrictive than boundedness of individual marginal contributions to coalitions, as in Wooders and Zame (1987) for example.

is the *commodity space* and A is the *set of types of participants*. A *commodity bundle* x is a vector $x \in \boldsymbol{R}_+^M$. The set $A = \{a^i\colon i = 1, \dots, I\}$ is a finite indexed collection of triples, $a^i = (\omega^i, u^i)$, where each ω^i is a commodity bundle, called the *endowment* of a participant of type i, and each u^i is a concave, monotonic nondecreasing function from the set of commodity bundles $\boldsymbol{R}_+^M$ to the reals, called the *payoff function* of a participant of type i.[14] Assume that payoff functions are normalized so that $u^i(\omega^i) \geqslant 0$ for all i. The premarket is *continuous* if the payoff functions of all participants are continuous and the premarket satisfies *payoff-constant returns* if the payoff functions of all participants are 1-homogeneous. The premarket $(\boldsymbol{R}_+^M, A)$ is *socially homogeneous* if there is a function u such that for all participants i, $u^i = u$.

A *market* determined by a premarket $(\boldsymbol{R}^M, A)$ is a pair $[n, A]$ where n is the *market population profile*, a vector in Z_+^I listing the number of participants with attributes a^i for each $i = 1, \dots, I$. We refer to participants with the same endowments and payoff functions as participants of the same "type."

REMARK 1: Socially homogeneous markets enjoy particularly pleasing properties. Since all participants have the same concave payoff function, competitive prices are determined by the subgradients of the payoff function. For almost all distributions of total endowments the competitive price system is uniquely determined (up to a normalization). Moreover, from these observations and from convergence of approximate cores to limiting competitive payoffs it follows that cores and in a certain sense, approximate cores, of large socially homogeneous economies are typically small. Since large games with effective small groups are approximately markets, they inherit these properties. See Wooders (1992b) for further discussion.

4. ASYMPTOTIC MARKET-GAME EQUIVALENCE

We establish an approximate equivalence between large economies with effective small groups, modeled as abstract games, and continuous, socially homogeneous markets with payoff-constant returns. Proofs are given in Section 7.

Let $(\boldsymbol{R}_+^M, A)$ be a premarket with $|A| = I$. For each profile $f = (f_1, \dots, f_I) \in Z_+^I$, define $\Lambda(f)$ by

$$\Lambda(f) = \max \sum_{i=1}^{I} \sum_{j=1}^{f_i} u^i(x^{ij}), \tag{4.1}$$

where the maximum is taken over the set of commodity bundles $\{x^{ij}\}$ satisfying

$$\sum_{i=1}^{I} \sum_{j=1}^{f_i} x^{ij} = \sum_{i=1}^{I} f_i \omega^i. \tag{4.2}$$

[14] For interpretation, we may think of the payoff function of trader i as defined by $U^i(x, \xi) = u^i(x) + \xi$, where ξ is a real number, to be thought of as "final money balance." The money plays no formal role in this paper.

The pair (I, Λ) is *the pregame induced by the premarket* $(\boldsymbol{R}_+^M, A)$. Any pregame which can be induced by some premarket is a *market pregame*. We say that *a premarket satisfies small group effectiveness* if its induced pregame satisfies small group effectiveness.

Let (T, Ψ) and $(\hat{T}, \hat{\Psi})$ be two pregames. The pregames (T, Ψ) and $(T, \hat{\Psi})$ are *asymptotically equivalent* if $\hat{T} = T$ and if, given any real number $\varepsilon > 0$, there is an integer $\eta_1(\varepsilon)$ such that for all profiles f with $\|f\| \geqslant \eta_1(\varepsilon)$

$$(4.3) \qquad |\hat{\Psi}(f) - \Psi(f)| \leqslant \varepsilon\|f\|.$$

Asymptotic equivalence of two pregames implies that for any large profile the per capita payoffs assignable to that profile by the two characteristic functions are approximately equal. A pregame (T, Ψ) and a premarket $(\boldsymbol{R}_+^M, A)$ are said to be *asymptotically equivalent* if the pregame is asymptotically equivalent to the pregame determined by the premarket. This requires, of course, that the number $|A|$ of types of participants in the premarket equals the number of types T of players in the pregame. The following two Theorems lead to our first market-game equivalence result.

THEOREM 1: *A continuous, socially homogeneous premarket with payoff-constant returns satisfies small group effectiveness.*

THEOREM 2: *A pregame (T, Ψ) has effective small groups if and only if there is an asymptotically equivalent, continuous, and socially homogeneous premarket $(\boldsymbol{R}_+^M, A)$ satisfying payoff-constant returns.*

Theorem 1 states that continuous, socially homogeneous market pregames satisfy small group effectiveness. Theorem 2 states that the class of pregames with effective small groups is asymptotically equivalent to the class of continuous, socially homogeneous market pregames satisfying payoff-constant returns. The following conclusion is the main result of this paper.

THEOREM 3 (Market-Game Equivalence): *The class of continuous, socially homogeneous premarkets with payoff-constant returns and the class of pregames with effective small groups are asymptotically equivalent.*

A market with many participants is the basic model of a perfectly competitive market. Among other properties, large markets satisfy nonemptiness of the core and convergence of the core to the Walrasian equilibrium payoffs. Our result shows that large games with effective small groups are asymptotically equivalent to market games. This implies that for any large game with effective small groups, a Walrasian outcome of an approximating market is an approximately feasible payoff for the game. Therefore, for the game, there are Pareto optimal payoffs that are "close" to Walrasian payoffs of an approximating market. Since these payoffs are close to Walrasian payoffs we can regard them as "approximate competitive equilibrium" payoffs. When the number of participants is large, economies with effective small groups have approximate competitive outcomes; there is some set of commodities and a price system for these commodities that constitutes an approximate competitive equilibrium. This remark holds even for economies with public goods, as in Wooders (1980) for example.

REMARK 2: The condition of payoff-constant returns in Theorem 1 is included for symmetry with the conditions of Theorem 2; it is not required for the conclusion of the Theorem. In the next section, we show that provided the percentages of players of each type are bounded away from zero, without any additional assumptions market pregames satisfy small group effectiveness. The condition of social homogeneity in Theorem 1 is important in limiting the effects of "scarce" player types, those appearing in the total player set in vanishingly small proportions.

REMARK 3: Small group effectiveness is a natural and powerful condition, and applies to diverse economies. As we show in the next section, if we ignore boundaries of the commodity space, small group effectiveness is equivalent to per capita boundedness. Thus, the more restrictive the conditions on the premarkets approximating pregames with effective small groups, the more striking the result. In the above theorems the conditions on the premarkets are chosen to be restrictive. It can be demonstrated that if either continuity of social homogeneity is relaxed, Theorem 2 no longer holds. Our results in this paper imply that any premarket with effective small groups is asymptotically equivalent to a socially homogeneous premarket. Other conditions ensuring asymptotic social homogeneity of markets and economies more generally are discussed in Wooders (1993b).

5. PER CAPITA BOUNDEDNESS AND THICK MARKETS

Collections of profiles with the property that the percentages of players of each type are bounded away from zero are called "thick." Formally, let $\rho > 0$ be a real number and define

$$P(\rho) = \left\{ f \in Z_+^T \text{: for each } t = 1, \ldots, T \text{ either } \left(\frac{1}{\|f\|}\right) f_t > \rho \right.$$
$$\left. \text{or } \left(\frac{1}{\|f\|}\right) f_t = 0 \right\}.$$

The set $P(\rho)$ is called the set of *ρ-thick* profiles.

A pregame (T, Ψ) satisfies *per capita boundedness* if there is a constant A such that for all profiles f it holds that

$$\frac{\Psi(f)}{\|f\|} < A. \tag{5.1}$$

The per capita boundedness condition was introduced in Wooders (1979, 1983).

When profiles are required to be thick, per capita boundedness is equivalent to small group effectiveness.

THEOREM 4: *Let* (T, Ψ) *be a pregame and let* $\varepsilon > 0$ *and* $\rho > 0$ *be given positive real numbers. Then*

(5.2) *there is an integer* $\eta_1(\varepsilon, \rho)$ *such that*:

for each profile f in $P(\rho)$, *there is a partition* (f^k) *of f with*

$\|f^k\| \leqslant \eta_1(\varepsilon, \rho)$ *for each subprofile* f^k *in the partition and*

$$\Psi(f) - \sum_k \Psi(f^k) \leqslant \varepsilon\|f\|$$

if and only if there is a constant A such that (5.1) *holds for all f in* $P(\rho)$.

Let (T, Ψ) and $(\hat{T}, \hat{\Psi})$ be two pregames. The two pregames are *asymptotically quasi-equivalent* if $T = \hat{T}$ and if for each $\rho > 0$ (4.3) holds for all ρ-thick profiles, that is,

given any positive real numbers $\varepsilon > 0$ and $\rho > 0$ there is an integer $\eta_2(\varepsilon, \rho)$ such that for all profiles f in $P(\rho)$ with $\|f\| \geqslant \eta_2(\varepsilon, \rho)$,

$$|\Psi(f) - \hat{\Psi}(f)| \leqslant \varepsilon\|f\|.$$

We say *that a premarket is asymptotically quasi-equivalent to a pregame* if the pregame derived from the premarket is asymptotically quasi-equivalent to the pregame. Also, *two premarkets are asymptotically quasi-equivalent* if their induced pregames are asymptotically quasi-equivalent.

The next two Theorems are analogues of Theorems 1 and 2.

THEOREM 5: *A market pregame satisfies per capita boundedness.*

THEOREM 6: *A pregame* (T, Ψ) *satisfies per capita boundedness if and only if there is a socially homogeneous premarket* $(\boldsymbol{R}_+^M, A)$ *with payoff-constant returns such that* (T, Ψ) *and* $(\boldsymbol{R}_+^M, A)$ *are asymptotically quasi-equivalent.*

Theorems 5 and 6 illustrate that besides ensuring boundedness of per capita payoffs, the role of small group effectiveness in our results is to ensure that arbitrarily small percentages of players of "scarce types" cannot have significant effects on per capita payoffs of large groups. Our second market-game equivalence result follows.

THEOREM 7 (Quasi-Equivalence of Games Satisfying Per Capita Boundedness and Socially Homogeneous Markets): *The class of socially homogeneous premarkets with payoff-constant returns and the class of pregames satisfying per capita boundedness are asymptotically quasi-equivalent.*

REMARK 4: With a finite number of types of participants (or of commodities), the condition of per capita boundedness appears extremely mild and easy to apply. Moreover, similar conditions have a long history in economy theory. Small group effectiveness has the advantage that besides ensuring that there are continuous approximating markets, it is applicable to situations where all players/commodities may differ, for example, situations with a compact metric space of attributes (Wooders (1992a, 1993b), for example) and ones with no topology on the space of player types (Wooders (1993c)).

6. THE NUMBER OF COMMODITIES

To show that a pregame with effective small groups is asymptotically equivalent to a premarket a particular premarket is constructed. This premarket is itself of interest as it has a natural interpretation and permits the statement of another result. In the premarket constructed the commodities are the players and the derived markets can be interpreted as ones where players hire the participation of other players in groups. This interpretation is especially suited to economies with production, economies with clubs, labor markets, and attribute games in general, where, as in the cost allocation literature, the players of a game are interpreted as commodities or as attributes of economic participants. The payoff function is constructed as follows. Let (T, Ψ) be a pregame with effective small groups. Define a function u by

$$(6.1) \qquad u(x) = \|x\| \lim_{\nu \to \infty} \frac{\Psi(f^\nu)}{\|f^\nu\|},$$

where $\{f^\nu\}$ is any sequence of profiles with the properties that $\|x\| \lim_{\nu \to \infty} (1/\|f^\nu\|) f^\nu = x$. Provided that the pregame (T, Ψ) satisfies small group effectiveness the function u is well-defined. Our final Theorem establishes that *whatever* the number of types of commodities in a premarket, the premarket can be approximated by another premarket with the number of commodities no larger than the number of types of participants and the payoff function of all the participants can be taken as that given by (6.1).

THEOREM 8: *Let $(\boldsymbol{R}_+^M, A)$ be a premarket satisfying small group effectiveness and let $|A| = I$. Let (I, Λ) denote the derived pregame. Then there is an asymptotically equivalent socially homogeneous premarket $(\boldsymbol{R}_+^{M'}, A')$ satisfying payoff-constant returns and with the number of commodities M' equal to the number I of types of players in the premarket $(\boldsymbol{R}_+^M, A)$.*

An analogous result holds for premarkets and asymptotically quasi-equivalent premarkets.

7. PROOFS OF THE THEOREMS

Small group effectiveness dictates that all or almost all gains to collective activities can be realized by groups bounded in absolute size. This implies that, in a certain sense, the games can be approximated by ones with bounded effective group sizes. Most results on large games with effective small groups referenced in this paper were first proven for games satisfying the stronger requirement. We proceed here also by first proving a result for games with bounded effective group sizes.

Let (T, Ψ) be a pregame. The pregame satisfies *bounded effective group sizes* if there is an integer B such that for each profile f there is a partition (f^k) of f satisfying

$$\|f^k\| \leqslant B \text{ for each } k \quad \text{and} \tag{7.1}$$

$$\Psi(f) - \sum_k \Psi(f^k) = 0; \tag{7.2}$$

all gains to collective activities can be realized by groups bounded in size by B.

We will use the observation that from concavity, the value of the maximum in (4.1) is unchanged when the allocation $\{x^{ij}\}$ is required to have the equal-treatment property, that is, for each i and for all j and j', $x^{ij} = x^{ij'}$.

LEMMA 1: *Let (T, Ψ) be a superadditive pregame with bounded effective group sizes. Let $\{f^\nu\}$ and $\{g^\nu\}$ be sequences of profiles with the properties that*

$$\|f^\nu\| \to \infty \quad \textit{and} \quad \|g^\nu\| \to \infty \textit{ as } \nu \textit{ becomes large} \quad \textit{and} \tag{7.3}$$

$$\lim_{\nu \to \infty} \left(\frac{1}{\|f^\nu\|} \right) f^\nu = \lim_{\nu \to \infty} \left(\frac{1}{\|g^\nu\|} \right) g^\nu. \tag{7.4}$$

Then

$$\lim_{\nu \to \infty} \left(\frac{1}{\|f^\nu\|} \right) \Psi(f^\nu) = \lim_{\nu \to \infty} \left(\frac{1}{\|g^\nu\|} \right) \Psi(g^\nu). \tag{7.5}$$

PROOF OF LEMMA 1: We prove the result by contradiction. Suppose that (T, Ψ), $\{f^\nu\}$, and $\{g^\nu\}$ satisfy the conditions of the Lemma. Let B denote a bound on effective group sizes. Define $A = \max \Psi(f)$, where the maximum is taken over all profiles f with $\|f\| \leqslant B$. Since $(\Psi(h)/\|h\|) \leqslant A$ for all profiles h it holds that the sequence $\{\Psi(f^\nu)/\|f^\nu\}$ has a converging subsequence. Suppose the sequences $\{\Psi(f^\nu)/\|f^\nu\|\}$ and $\{\Psi(g^\nu)/\|g^\nu\|\}$ both converge and there is a positive real number $\delta > 0$ such that

$$\lim_{\nu \to \infty} \frac{\Psi(f^\nu)}{\|f^\nu\|} > \lim_{\nu \to \infty} \frac{\Psi(g^\nu)}{\|g^\nu\|} + 2\delta. \tag{7.6}$$

Define $f := \lim_{\nu \to \infty}(1/\|f^\nu\|)f^\nu = \lim_{\nu \to \infty}(1/\|g^\nu\|)g^\nu$. Define the sequences of profiles $\{h^\nu\}$ and $\{l^\nu\}$ by

$$h_t^\nu = f_t^\nu \quad \text{if} \quad f_t \neq 0, \tag{7.7}$$

$$h_t^\nu = 0 \quad \text{if} \quad f_t = 0, \quad \text{and} \tag{7.8}$$

$$l_t^\nu = f_t^\nu - h_t^\nu. \tag{7.9}$$

From boundedness of effective group sizes it follows that

$$\Psi(f^\nu) - \Psi(h^\nu) \leqslant AB\|l^\nu\|$$

and therefore, since $(\|l^\nu\|/\|f^\nu\|) \to 0$ as $\nu \to \infty$, $\lim_{\nu\to\infty}(\Psi(f^\nu)/\|f^\nu\|) = \lim_{\nu\to\infty}(\Psi(h^\nu)/\|h^\nu\|)$.

Define $L := \lim_{\nu\to\infty}(\Psi(f^\nu)/\|f^\nu\|)$. Let ν_0 be sufficiently large so that for all $\nu \geqslant \nu_0$ it holds that

$$\frac{\Psi(h^\nu)}{\|h^\nu\|} \geqslant L - \delta. \tag{7.10}$$

From (7.4) we can suppose without loss of generality that the support of h^ν is contained in the support of $g^{\nu'}$ for all ν and ν' (that is, $h_t^\nu \neq 0$ implies $g_t^\nu \neq 0$). It follows that there is an integer $\nu_1 \geqslant \nu_0$ such that for all sufficiently large terms ν, for some integer r_ν,

$$g^\nu \geqslant r_\nu h^{\nu_1} \quad \text{and}$$

$$\frac{r_\nu\|h^{\nu_1}\|}{\|g^\nu\|} \geqslant \frac{L-2\delta}{L-\delta}, \tag{7.11}$$

that is, g^ν permits r_ν copies of h^{ν_1}, with only a small percentage of "leftovers." From superadditivity, (7.10) and (7.11),

$$\frac{\Psi(g^\nu)}{\|g^\nu\|} \geqslant \frac{r_\nu\Psi(h^{\nu_1})}{r_\nu\|h^{\nu_1}\|}\cdot\frac{r_\nu\|h^{\nu_1}\|}{\|g^\nu\|} \geqslant (L-2\delta) = \lim_{\nu\to\infty}\frac{\Psi(f^\nu)}{\|f^\nu\|} - 2\delta,$$

for all $\nu \geqslant \nu_1$, a contradiction to (7.6).

The result now follows from the fact per capita payoffs are bounded. Thus, in general, the sequences $\{\Psi(f^\nu)/\|f^\nu\|\}$ and $\{\Psi(g^\nu)/\|g^\nu\|\}$ have converging subsequences. Since any pair of converging subsequences converge to the same limit both sequences must converge to the same limit. *Q.E.D.*

Lemma 2: *Let (T,Ψ) be a pregame satisfying small groups effectiveness. Let $\{f^\nu\}$ and $\{g^\nu\}$ be sequences of profiles satisfying* (7.3) *and* (7.4). *Then* (7.5) *holds.*

Proof of Lemma 2: We prove the result by contradiction. Suppose that (T,Ψ), $\{f^\nu\}$, and $\{g^\nu\}$ satisfy the conditions of the Lemma but not (7.5). Since small group effectiveness implies per capita boundedness (which is easy to show), we can suppose without loss of generality that the sequences $\{(1/\|f^\nu\|)\Psi(f^\nu)\}$ and $\{(1/\|g^\nu\|)\Psi(g^\nu)\}$ both converge and that for some positive real number $\varepsilon_0 > 0$, for all terms f^ν and g^ν in the sequences,

$$\left|\left(\frac{1}{\|f^\nu\|}\right)\Psi(f^\nu) - \left(\frac{1}{\|g^\nu\|}\right)\Psi(g^\nu)\right| > 3\varepsilon_0. \tag{7.12}$$

From small group effectiveness we may choose an integer B with the property that for each profile h there is a partition (h^k) of h satisfying

$$\|h^k\| \leqslant B \quad \text{for each } k \text{ and}$$

$$\Psi(h) - \sum_k \Psi(h^k) \leqslant \varepsilon_0\|h\|.$$

We next construct a pregame with effective group sizes bounded by B. For each profile f define $\Gamma(f)$ by

$$\Gamma(f) = \max \sum_k \Psi(f^k),$$

where the maximum is taken over all partitions (f^k) of f with $\|f^k\| \leqslant B$ for each k. Note that for the choice of B it follows that $\Psi(h) - \Gamma(h) \leqslant \varepsilon_0 \|h\|$ for all profiles h. Then, by Lemma 1, there is an integer ν_0 sufficiently large so that, for all $\nu \geqslant \nu_0$,

$$\begin{aligned} &\left| \frac{\Psi(f^\nu)}{\|f^\nu\|} - \frac{\Psi(g^\nu)}{\|g^\nu\|} \right| \\ &\quad \leqslant \left| \frac{\Psi(f^\nu)}{\|f^\nu\|} - \frac{\Gamma(f^\nu)}{\|f^\nu\|} \right| + \left| \frac{\Gamma(f^\nu)}{\|f^\nu\|} - \frac{\Gamma(g^\nu)}{\|g^\nu\|} \right| + \left| \frac{\Gamma(g^\nu)}{\|g^\nu\|} - \frac{\Psi(g^\nu)}{\|g^\nu\|} \right| \\ &\quad \leqslant 3\varepsilon_0, \end{aligned}$$

a contradiction to (7.12). *Q.E.D.*

It is convenient to prove Theorem 4 next.

Proof of Theorem 4: Let (T, Ψ) be a pregame with bounded per capita payoffs and suppose that the conclusion of the Theorem is false. Then there are real numbers ρ_0 and ε_0 and a sequence of profiles $\{f^\nu\}$ such that

$$\|f^\nu\| \to \infty \quad \text{as} \quad \nu \to \infty,$$

$$f^\nu \in P(\rho) \quad \text{for each } \nu, \text{ and}$$

for each f^ν, for every partition $(f^{\nu k})$ of f^ν with $\|f^{\nu k}\| < \nu$ for each k, it holds that

$$\Psi(f^\nu) - \sum_k \Psi(f^{\nu k}) > 2\varepsilon_0 \|f^\nu\|. \tag{7.13}$$

Without loss of generality we can assume that for some $Q \leqslant T$, $f^\nu \in R^Q_{++} \times \{0\}^{T \setminus Q}$ for each ν. By passing to a subsequence if necessary, we can suppose that the sequence $\{(1/\|f^\nu\|) f^\nu\}$ converges, say to $f \in R^Q_{++} \times \{0\}^{T\setminus Q}$. Again passing to a subsequence if necessary, from per capita boundedness we can suppose that the sequence $\{\Psi(f^\nu)/\|f^\nu\|\}$ converges, say to the real number L. Since the sequence $\{\Psi(f^\nu)/\|f^\nu\|\}$ converges, there is an integer ν_0 sufficiently large so that

$$\frac{\Psi(f^{\nu_0})}{\|f^{\nu_0}\|} \leqslant L + \varepsilon_0$$

and for all $\nu \geqslant \nu_0$ it holds that

$$\left| \frac{\Psi(f^\nu)}{\|f^\nu\|} - \frac{\Psi(f^{\nu_0})}{\|f^{\nu_0}\|} \right| < \varepsilon_0.$$

Let ν_1 be sufficiently large so that for each $\nu \geqslant \nu_1$, for some integer r_ν and profile m^ν it holds that

$$f^\nu = r_\nu f^{\nu_0} + m^\nu, \quad \text{and}$$

$$\frac{\|m^\nu\|}{\|f^\nu\|}(L+\varepsilon_0) \leqslant \varepsilon_0;$$

this is possible since each term f^ν is in $P(\rho)$. (A proof of this fact is provided in Wooders and Zame (1987).) From the above inequalities, for all ν sufficiently large we obtain the estimate:

$$\begin{aligned}
&\left|\frac{\Psi(f^\nu)}{\|f^\nu\|} - r_\nu \frac{\Psi(f^{\nu_0})}{\|f^\nu\|}\right| \\
&\quad \leqslant \left|\frac{\Psi(f^\nu)}{\|f^\nu\|} - \frac{\Psi(f^{\nu_0})}{\|f^{\nu_0}\|}\right| + \left|\frac{\Psi(f^{\nu_0})}{\|f^{\nu_0}\|} - r_\nu \frac{\Psi(f^{\nu_0})}{\|f^\nu\|}\right| \\
&\quad \leqslant \varepsilon_0 + \frac{\Psi(f^{\nu_0})}{\|f^{\nu_0}\|}\frac{\|m^\nu\|}{\|f^\nu\|} \\
&\quad \leqslant \varepsilon_0 + (L+\varepsilon_0)\frac{\|m^\nu\|}{\|f^\nu\|} \leqslant 2\varepsilon_0.
\end{aligned}$$

This yields a contradiction to (7.13) since for all ν sufficiently large it follows that

$$\Psi(f^\nu) - r_\nu \sum_k \Psi(f^{\nu_0}) - \sum_t m_t^\nu \Psi(\chi^t) \leqslant 2\varepsilon_0 \|f^\nu\|.$$

We leave the other direction to the reader. *Q.E.D.*

PROOF OF THEOREM 1: Let (I, Λ) be a market pregame derived from a premarket $(\mathbf{R}_+^M, A)$ with I types of participants and where all participants have the same 1-homogeneous and continuous payoff function, denoted by u. Suppose (I, Λ) does not satisfy small group effectiveness. Then there is a positive real number ε_0 and a sequence of profiles $\{f^\nu\}$ with the property that for any partition $(f^{\nu k})$ of f^ν where, for each k, $\|f^{\nu k}\| \leqslant \nu$ it holds that

$$(7.14) \quad \Lambda(f^\nu) - \sum_k \Lambda(f^{\nu k}) > 3\varepsilon_0 \|f^\nu\|.$$

Suppose, without loss of generality, that there is a vector f in $\mathbf{R}_+^I$ such that $(1/\|f^\nu\|)f^\nu$ converges to f. Define the sequences $\{h^\nu\}$ and $\{l^\nu\}$ by equations (7.7), (7.8), and (7.9). Since the sequence $\{h^\nu\}$ has the property that the percentage of players of each type (that appears in the profiles) is bounded away from zero, from Theorem 4 there is an integer B such that for each ν, for some partition $(h^{\nu k})$ of h^ν with $\|h^{\nu k}\| \leqslant B$ for each k,

$$(7.15) \quad \Lambda(h^\nu) - \sum_k \Lambda(h^{\nu k}) \leqslant \varepsilon_0 \|h^\nu\|.$$

Note that since u is concave, for any profile g it holds that $\Lambda(f) = u(\Sigma_i g_i \omega^i)$. From Rockafellar (1972, Theorem 10.4), u is Lipschitzian on the simplex. Since both u and Λ are 1-homogeneous, there is a constant A such that for each integer ν

$$(7.16)\qquad \left| \frac{\lambda(f^{\nu})}{\|f^{\nu}\|} - \frac{\Lambda(h^{\nu})}{\|h^{\nu}\|} \right| \leqslant A \left\| \frac{1}{\|f^{\nu}\|} f^{\nu} - \frac{1}{\|h^{\nu}\|} h^{\nu} \right\|.$$

Select an integer ν_0 sufficiently large so that

$$A \left\| \frac{1}{\|f^{\nu}\|} f^{\nu} - \frac{1}{\|h^{\nu}\|} h^{\nu} \right\| < \varepsilon_0.$$

From superadditivity, (7.15) and (7.16), and the fact that $(\|h^{\nu}\|/\|f^{\nu}\|) \to 1$ as η becomes large, it follows that for all sufficiently large ν

$$\left| \frac{\Lambda(f^{\nu})}{\|f^{\nu}\|} - \frac{\Lambda(h^{\nu})}{\|f^{\nu}\|} \right| \leqslant 2\varepsilon_0.$$

It now follows that for all sufficiently large ν,

$$\begin{aligned} &\Lambda(f^{\nu}) - \sum_k \Lambda(h^{\nu k}) - \sum_i l_i^{\nu} \Lambda(\chi_i) \\ &\quad \leqslant |\Lambda(f^{\nu}) - \Lambda(h^{\nu})| + \left| \Lambda(h^{\nu}) - \sum_k \Lambda(h^{\nu k}) \right| \\ &\quad \leqslant 3\varepsilon_0 \|f^{\nu}\|, \end{aligned}$$

which is a contradiction. *Q.E.D.*

Proof of Theorem 2: Let (T, Ψ) be a pregame. From Lemma 2 the function u, given by (6.1), is well-defined and continuous. Since it is 1-homogeneous and superadditive, u is concave.

Now consider the premarket $(\mathbf{R}_+^T, A)$ where $|A| = T$ and

$$a^i = (\chi_t, u)$$

for each $t = 1, \ldots, T$. The pair $(\mathbf{R}_+^T, A)$ is a continuous, socially homogeneous premarket with payoff-constant returns. From the constraints of u it follows that the pregame (T, Ψ) is asymptotically equivalent to the pregame derived from $(\mathbf{R}_+^T, A)$; we leave the details to the reader. This proves one part of the Theorem.

Next, let $(\mathbf{R}_+^M, A)$ be a socially homogeneous premarket with derived game (T, Λ) and with the property that (T, Λ) and (T, Ψ) are asymptotically equivalent. Let u denote the payoff function of the participants in the premarket. From Theorem 1 (T, Λ) has effective small groups. Thus, given $\varepsilon > 0$ there is an integer $\eta_0(\varepsilon)$ such that for each profile f there is a partition of f, say $(f^k$: $k = 1, \ldots, K)$, satisfying

$$(7.17)\qquad \|f^k k\| \leqslant \eta_0(\varepsilon) \quad \text{and}$$

$$(7.18)\qquad \Lambda(f) - \sum_k \Lambda(f^k) < \varepsilon \|f\|.$$

From asymptotic equivalence there is an integer $\eta_1(\varepsilon) \geqslant \eta_0(\varepsilon)$ such that for each profile f with

(7.19) $\quad \|f\| \geqslant \eta_1(\varepsilon)$

it holds that

(7.20) $\quad |\Psi(f) - \Lambda(f)| \leqslant \varepsilon \|f\|.$

The following Lemma leads to the conclusion of the proof.

LEMMA 3: *Let f be a profile and let $(f^k\colon k = 1, \ldots, K)$ be a partition of f satisfying* (7.17) *and* (7.18). *Then there are integers $0 = m_0 < m_1 < \ldots < m_q < \ldots < m_Q = K$ such that, for each $q = 0, \ldots, Q-1$,*

$$\eta_1(\varepsilon) \leqslant \left\| \sum_{k=m_q+1}^{m_{q+1}} f^k \right\| \leqslant 4\eta_1(\varepsilon).$$

PROOF OF LEMMA 3: Suppose $\|f\| \leqslant 4\eta_1(\varepsilon)$. In this case, the partition (f) satisfies the required properties. Therefore we suppose that $\|f\| > 4\eta_1(\varepsilon)$. Let (f^k) be a partition of f satisfying (7.17) and (7.18). There exists an integer m_1 such that

$$\left\| \sum_{k=1}^{m_1-1} f^k \right\| \leqslant \eta_1(\varepsilon) \quad \text{and}$$

$$\left\| \sum_{k=1}^{m_1} f^k \right\| > \eta_1(\varepsilon).$$

It follows that

$$\left\| \sum_{k=1}^{m_1} f^k \right\| = \left\| \sum_{k=1}^{m_1-1} f^k \right\| + \|f^{m_1}\| \leqslant \eta_1(\varepsilon) + \eta_0(\varepsilon) \leqslant 2\eta_1(\varepsilon) \quad \text{and}$$

$$\left\| \sum_{k=m_1+1}^{K} f^k \right\| = \|f\| - \left\| \sum_{k=1}^{m_1} f^k \right\| \geqslant 4\eta_1(\varepsilon) - 2\eta_1(\varepsilon) = 2\eta_1(\varepsilon).$$

Case 1: If $2\eta_1(\varepsilon) \leqslant \|\sum_{k=m_1+1}^{K} f^k\| \leqslant 4\eta_1(\varepsilon)$ let $g^1 = \sum_{k=1}^{m_1}$ and let $g^2 = \sum_{k=m_1+1}^{K} f^k$.

Case 2: If $\|\sum_{k=m_1+1}^{K} f^k\| > 4\eta_1(\varepsilon)$, then there is an integer m_2 such that

$$\nu_1(\varepsilon) \leqslant \left\| \sum_{k=m_2+1}^{m_2} f^k \right\| \leqslant 2\eta_1(\varepsilon) \quad \text{and}$$

$$\left\| \sum_{k=m_2+1}^{K} f^k \right\| > 2\eta_1(\varepsilon).$$

In this case, define $g^1 = \sum_{k=1}^{m_1} f^k$, and $g^2 = \sum_{k=m_1+1}^{m_2} f^k$.

Next consider $\Sigma_{k=m_2+1}^{K} f^k$. Depending on whether

$$\left\| \sum_{k=m_2+1}^{K} f^k \right\| \leqslant 4\eta_1(\varepsilon) \quad \text{or} \quad \left\| \sum_{k=m_2+1}^{K} f^k \right\| > 4\eta_1(\varepsilon)$$

proceed as in Case 1 or 2 to determine m_3 and define $g^3 = \Sigma_{k=m_2+1}^{m_3} f^k$, satisfying the required conditions.

We can argue repeatedly as above to determine integers $m_1, \ldots, m_Q$ and profiles $g^1, \ldots, g^Q$ such that

$$g^1 = \sum_{k=1}^{m_1} f^k,$$

$$g^2 = \sum_{k=m_1+1}^{m_2} f^k, \ldots, \quad \text{and}$$

$$g^Q = \sum_{k=m_{Q-1}+1}^{K} f^k,$$

where $\eta_1(\varepsilon) \leqslant \|g^q\| \leqslant 4\eta_1(\varepsilon)$ for all $q = 1, \ldots, m_Q$. This completes the proof of the Lemma. *Q.E.D.*

To complete the proof of the Theorem, from superadditivity, (7.17) and (7.18), and Lemma 3 it now follows that

$$\begin{aligned} &\Psi(f) - \sum_{q}^{Q} \Psi(g^q) \\ &\quad \leqslant |\Psi(f) - \Lambda(f)| + \left| \Lambda(f) - \sum_{q=1}^{Q} \Lambda(g^q) \right| \\ &\quad\quad + \left| \sum_{q=1}^{Q} \Lambda(g^q) - \sum_{q=1}^{Q} \Psi(g^q) \right| \\ &\quad \leqslant 3\varepsilon \|f\|, \end{aligned}$$

where $\{g^q\}$ is defined as in the proof of Lemma 3. Thus, given any $\varepsilon > 0$ the integer $4\eta_1(\varepsilon)$ satisfies the properties required in the definition of small group effectiveness applied to the pregame (T, Ψ). This completes the proof of the Theorem. *Q.E.D.*

Theorem 5 is a standard result. With the appropriate construction of a premarket from a pregame satisfying per capita boundedness, the proof of Theorem 6 is essentially the same as the proof of Theorem 2. We note only that for any x in $\mathbf{R}^T_{++}$ the definition of the payoff function for the premarket approximating the pregame can be taken as that given by (6.1) with the approximating profiles taken to have the same support as the commodity bundle.

Theorem 8 was proved during the course of the proof of Theorem 2 so we have reached the conclusion of this section.

8. DISCUSSION

Economic models with effective small groups may include ones with multiple marketplaces, firms, communities, or clubs. Small group effectiveness ensures that when there is in total a large number of participants, then there are, at least potentially, "many" groups. The intuition underlying our results is that in economies with effective small groups, competition within groups for shares of the surplus generated by the group, and competition between groups for participants, lead to a competitive outcome. This intuition emerges especially from the study of economies with collectively produced and/or consumed goods, such as ones with local public goods (see, for example, Tiebout (1956) and Wooders (1980)). It follows from the results of the current paper that one set of commodities for which an economy is approximately a market is the set of participants themselves. Intuitively, participants sell their participation in groups in return for a share of the surplus generated by the group.[15]

Our concepts and results can be applied in a variety of contexts, for example, in investigations of the "Coase Theorem" (Coase (1960)). One such investigation of the effects of property rights assignments is carried out in Wooders (1992c). Provided that assignments of property rights are bounded, small group effectiveness of pregames (called "technologies" in this application, as in Wooders and Zame (1987)) ensures convergence of approximate cores to competitive prices for attributes of players and to approximate attribute core payoffs. Other applications include ones to economies with collective production and to economies with local public goods or shared goods more generally; some such applications are reviewed in Wooders (1992b). Another application may be to financial markets with unbounded short sales.

The approach of this paper is quite distinct from established theories of competitive markets. Three major theories are Cournot's noncooperative equilibrium theory, Edgeworth's contract theory, and Clark's marginal productivity theory, revived in Ostroy's no-surplus theory.[16] The equivalence result of this paper suggests that small group effectiveness may be a powerful condition for the study of these theories; the theories have been applied to markets, and asymptotically economies with effective small groups are markets. Some results

[15] These ideas have been a reoccurring theme in the research leading to this paper. Ostroy (1984) also stresses prices for participants.

[16] See Cournot (1938), and more recently, Shubik (1973), Hart (1979), and Novshek and Sonnenschein (1978) for the Cournotian theory, Anderson (1992) for a survey of research on Edgeworth's contract theory and the equivalence of cooperation and equilibrium outcomes, and Ostroy (1980, 1984) for the marginal productivity theory.

in these directions include Hammond, Kaneko, and Wooders (1989) on the equivalence of the core and the competitive outcomes in large economies with widespread externalities and effective small groups (finite coalitions in the continuum).

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