Journal of Mathematical Economics 34 (2000) 439–469
JOURNAL OF Mathematical ECONOMICS
www.elsevier.com/locate/jmateco

Inconsequential arbitrage ☆

Frank H. Page Jr. [a,*], Myrna H. Wooders [b,1], Paulo K. Monteiro [c]

[a] *Department of Finance, University of Alabama, Tuscaloosa, AL 35487, USA*
[b] *Department of Economics, University of Warwick, Coventry CV4 7AL, UK*
[c] *EPGE/FGV, Praia de Botafogo 190 sala 1125, 22253-900 Rio de Janeiro, RJ, Brazil*

Received 4 April 1999; received in revised form 27 June 2000; accepted 7 July 2000

Abstract

We introduce the concept of inconsequential arbitrage and, in the context of a model allowing short-sales and half-lines in indifference surfaces, prove that inconsequential arbitrage is sufficient for existence of equilibrium. Moreover, with a slightly stronger condition of nonsatiation than that required for existence of equilibrium and with a mild uniformity condition on arbitrage opportunities, we show that inconsequential arbitrage, the existence of a Pareto optimal allocation, and compactness of the set of utility possibilities are equivalent. Thus, when all equilibria are Pareto optimal — for example, when local nonsatiation holds — inconsequential arbitrage is necessary and sufficient for existence of an equilibrium. By further strengthening our nonsatiation condition, we obtain a second welfare theorem for exchange economies allowing short sales.

Finally, we compare inconsequential arbitrage to the conditions limiting arbitrage of Hart [Hart, O.D., 1974. J. Econ. Theory 9, 293–311], Werner [Werner, J., 1987. Econometrica 55, abs1403–1418], Dana et al. [Dana, R.A., Le Van, C., Magnien, F., 1999. J. Econ. Theory 87, 169–193] and Allouch [Allouch, N., 1999. Equilibrium and no market arbitrage. CERMSEM, Universite de Paris I]. For example, we show that the condition of Hart (translated to a general equilibrium setting) and the condition of werner are equivalent. We then show that the Hart/Werner conditions imply inconsequential arbitrage. To highlight the extent to which we extend Hart and Werner, we construct an example of an exchange economy in which inconsequential arbitrage holds (and is necessary and sufficient for existence), while the Hart/Werner conditions do not hold. © 2000 Elsevier Science S.A. All rights reserved.

Keywords: Arbitrage; Pareto optimality; Recession cones

☆ This paper was presented at the Winter Meeting of the Econometric Society in Chicago, January 1998.

* Corresponding author. Tel.: +1-205-348-0590; fax: +1-205-348-6713
E-mail addresses: fpage@cba.ua.edu (F.H. Page Jr.), mwooders@chass.utoronto.ca (M.H. Wooders), pklm@fgv.br (P.K. Monteiro).
[1] Also, Department of Economics, University of Toronto, Ont., Canada, M5S lAl.

PII: S0304-4068(00)00052-5

1. Introduction

We introduce the condition of *inconsequential arbitrage*. This condition ensures that arbitrarily large arbitrage opportunities are inconsequential from the viewpoint of existence of equilibrium. In particular, we show that in an exchange economy requiring local nonsatiation at rational allocations but allowing short sales and half-lines in indifference surfaces, inconsequential arbitrage is sufficient for existence of a quasi-equilibrium — and with the addition of two standard assumptions, we show that it is sufficient for existence of an equilibrium. With a slightly stronger condition of nonsatiation than that required for existence of a quasi-equilibrium and with a mild uniformity condition on arbitrage opportunities, we establish the equivalence of inconsequential arbitrage, existence of a Pareto optimal allocation, and compactness of the set of utility possibilities. Thus, under any condition ensuring that all equilibrium allocations are Pareto optimal (such as, local nonsatiation at rational allocations), inconsequential arbitrage is necessary and sufficient for existence of an equilibrium — as well as necessary and sufficient for existence of a Pareto optimal allocation and compactness of the set of utility possibilities. By further strengthening our nonsatiation condition, we obtain a second welfare theorem for exchange economies allowing short sales. This result generalizes the second welfare theorem in Page and Wooders (1996a). [2]

Prior papers in the literature on existence of equilibrium in models with consumption sets unbounded below — models with short sales — have focused first on establishing sufficient conditions for existence of equilibrium and second, on necessary conditions. In the context of these models, conditions that are necessary and sufficient for existence of equilibrium are also necessary and sufficient for the existence of a Pareto optimal allocation (Page and Wooders, 1993, 1996a). Typically, to show that a condition limiting arbitrage is necessary for existence of equilibrium, the prior literature has required that there be no half-lines in indifference surfaces (Werner, 1987; Page and Wooders, 1993) [3] or that at most one agent have half-lines in indifference surfaces (Page and Wooders, 1996a). In an asset market setting, for example, this means that (at most) one agent can be risk-neutral at extreme levels of wealth; thus the limitations of the prior models are of economic significance. In addition, many papers in the literature establishing sufficient conditions for existence of equilibrium have required that arbitrage opportunities be uniform with respect to endowment. Such uniformity is required, for example, by Hart (1974) and Werner (1987) for their sufficiency results. In the current paper, none of these restrictions are required.

[2] Within the context of a model allowing local satiation, Hurwicz (1996) proves a second welfare theorem assuming that consumption sets are bounded from below. Here, within the context of a model *not* allowing local satiation, we carry out our proof of a second welfare theorem allowing short sales, and hence allowing consumption sets to be *unbounded* from below. In a more recent paper, Hurwicz and Richter (2000) have established a second welfare theorem, again without the assumption of local nonsatiation, within the context of a general model in which the boundedness from below assumption has been relaxed.

[3] According to Werner, a relationship between his condition limiting arbitrage opportunities and compactness was first remarked by J.-F. Mertens in private conversation. The equivalence of the (closely related) Page–Wooders condition of 'no unbounded arbitrage' opportunities compactness of utility possibilities, and the existence of a Pareto-optimal point was established in Page and Wooders (1996a,b).

Conditions limiting arbitrage found in the literature generally fall into three broad categories:

1. *conditions on net trades*, for example, Hart (1974), Page (1987), Nielsen (1989), Page and Wooders (1993, 1996a,b), Allouch (1999) — including the condition of inconsequential arbitrage introduced here;
2. *conditions on prices*, for example, Green (1973), Grandmont (1977, 1982), Hammond (1983), and Werner (1987).
3. *conditions on the set of utility possibilities* (*namely, compactness*), for example, Brown and Werner (1995), Dana et al. (1999).

We compare inconsequential arbitrage to the conditions limiting arbitrage introduced by Hart (1974) and Werner (1987), as well as to conditions recently introduced by Dana et al. (1999) and Allouch (1999). For example, we show that the condition of Hart (translated to a general equilibrium setting) and the condition of Werner are equivalent, and more importantly, we show that the Hart/Werner conditions imply inconsequential arbitrage. We also show that under the assumption of no half-lines in indifference surfaces, the Hart/Werner conditions and inconsequential arbitrage are equivalent. In order to highlight the extent to which we extend Hart (1974) and Werner (1987), we construct an example (Example 3 below) of an exchange economy in which inconsequential arbitrage holds (and is necessary and sufficient for existence), while the Hart/Werner conditions do not hold. Inconsequential arbitrage, however, has some limitations. In order to make these more clearly understood, we also construct an example (Example 4 below) of an exchange economy in which inconsequential arbitrage does not hold, nor do the Hart/Werner conditions. But an equilibrium does exist. Finally, under additional conditions on the model, we show that if agents' indifference surfaces contain no half-lines, then inconsequential arbitrage, the Hart/Werner conditions, the Dana, Le Van, and Magnien condition, and Allouch's condition are all equivalent — and in turn, equivalent to the existence of a Pareto optimal allocation, compactness of the set of utility possibilities, and the existence of equilibrium.

In a recent paper, Allouch (1999) has shown that inconsequential arbitrage implies compactness of the set of utility possibilities. Moreover, Dana et al. (1999) have shown that if local satiation is ruled out, then compactness of utility possibilities implies existence of a quasi-equilibrium. Allouch (1999) also introduces a new condition, bounded arbitrage, and in a model more general than the one developed here, shows that bounded arbitrage implies the existence of a quasi-equilibrium. Allouch (1999) also shows that if local satiation is ruled out, then his condition of bounded arbitrage is equivalent to compactness of utility possibilities.

Inconsequential arbitrage builds on the condition of no unbounded arbitrage in Page and Wooders (1993, 1996a,b) (see also Page (1987)). That condition ensures that no group of agents can make mutually compatible, unbounded and utility-nondecreasing trades. The result of Page and Wooders (1993, 1996a), that existence of equilibrium implies no unbounded arbitrage, requires that no group of agents can make mutually compatible, unbounded and utility *increasing* trades. In general, no unbounded arbitrage implies inconsequential arbitrage. While the condition of no unbounded arbitrage focuses on expanding utility nondecreasing or increasing trades, inconsequential arbitrage focuses on contracting net trades without decreasing utility. This change, from the focus on expanding trades to the focus on decreasing trades, enables us to treat economies with half-lines in indifference surfaces.

It is this ability to contract net trades in directions determined by arbitrage without decreasing utility that allows for existence of an equilibrium in the presence of half-lines in indifference surfaces. The approach taken in this paper can also be viewed as an outgrowth of Hart (1974), showing existence of equilibrium in an asset-market model with unbounded short sales. Studying his paper suggested the ‘back up’ argument used in our proofs and the modification of no unbounded arbitrage leading to inconsequential arbitrage. In essence, we are extending the Hart (1974) model of an asset market and his condition limiting arbitrage opportunities to a general equilibrium model. Hart’s condition is stated explicitly in terms of the structure of asset returns; this is not required in our work. Our framework and results include Hart’s as a special case.

For an excellent exposition of the different notions of arbitrage in the prior literature, including a discussion of Chichilnisky (1997) and prior papers, we refer the reader to Dana et al. (1999).[4] Another survey, including asset market models in addition to general equilibrium models, is provided in Page and Wooders (1999). For excellent treatments of arbitrage and existence of equilibrium in infinite dimensional settings see Brown and Werner (1995) and Dana et al. (1997).

2. An economy with short sales

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ denote an *exchange economy*. Each agent j has *consumption set* $X_j \subset \mathbb{R}^L$ and *endowment* ω_j. The jth agent’s preferences $P_j(\cdot)$ over X_j are specified via a *utility function* $u_j(\cdot)$: $X_j \to \mathbb{R}$ as follows

$$P_j(x_j) := \{x \in X_j : u_j(x) > u_j(x_j)\}. \tag{1}$$

The *weak preferred set* is given by

$$\hat{P}_j(x_j) := \{x \in X_j : u_j(x) \geq u_j(x_j)\}. \tag{2}$$

The set of *individually rational allocations* at endowments $\omega = (\omega_1, \dots, \omega_n)$ is given by

$$A(\omega) = \left\{ (x_1, \dots, x_n) \in X_1 \times \cdots \times X_n : \sum_{j=1}^n x_j = \sum_{j=1}^n \omega_j \text{ and } x_j \in \hat{P}_j(\omega_j) \quad \text{for all } j \right\}. \tag{3}$$

We shall denote by $A_j(\omega)$ the projection of $A(\omega)$ onto X_j.

We shall maintain the following assumptions throughout: for each agent $j = 1, \dots, n$,

[A-1] $u_j(\cdot)$ is upper semicontinuous and quasi-concave;

[A-2] $\omega_j \in X_j$ and X_j is closed and convex.

We shall also maintain the following nonsatiation assumption:

[4] For further discussion see Monteiro et al. (1998, 1999, 2000).

[A-3] *Local nonsatiation at rational allocations*. For any rational allocation $(x_1, \dots, x_n) \in A(\omega)$, $\forall j$ $P_j(x_j) \neq \emptyset$ and $\mathrm{cl}\, P_j(x_j) = \hat{P}_j(x_j)$.

In **[A-3]**, 'cl' denotes closure. Thus, at a rational allocation $(x_1, \dots, x_n) \in A(\omega)$, $P_j(x_j)$ is nonempty and x_j is in the boundary of the preferred set $P_j(x_j)$ for each agent *j*.

Given prices $p \in \mathcal{B} := \{p' \in \mathbb{R}^L : \|p'\| \leq 1\}$ the *budget* set for the *j*th agent is given by

$$B(\omega_j, p) = \{x \in X_j : \langle x, p\rangle \leq \langle \omega_j, p\rangle\},$$

and the interior of the budget set relative to X_j is given by

$$F(\omega_j, p) = \{x_j \in X_j : \langle x_j, p\rangle < \langle \omega_j, p\rangle\},$$

An *equilibrium* for the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ is an $(n+1)$-tuple of vectors $(\bar{x}_1, \dots, \bar{x}_n, \bar{p})$ such that

1. $(\bar{x}_1, \dots, \bar{x}_n) \in A(\omega)$;
2. $\bar{p} \in \mathcal{B}\backslash\{0\}$;
3. for each *j*, $\langle \bar{x}_j, \bar{p}\rangle = \langle \omega_j, \bar{p}\rangle$ and $P_j(\bar{x}_j) \cap B(\omega_j, \bar{p}) = \emptyset$.

   Thus, $(\bar{x}_1, \dots, \bar{x}_n, \bar{p})$ is an equilibrium if and only if, for each *j*, $x_j \in P_j(\bar{x}_j)$ implies that $\langle x_j, \bar{p}\rangle > \langle \bar{x}_j, \bar{p}\rangle = \langle \omega_j, \bar{p}\rangle$.

We say that $(\bar{x}_1, \dots, \bar{x}_n, \bar{p})$ is a *quasi*-equilibrium if

1. $(\bar{x}_1, \dots, \bar{x}_n) \in A(\omega)$;
2. $\bar{p} \in \mathcal{B}\backslash\{0\}$;
3. for each *j*, $\bar{x}_j \in B(\omega_j, \bar{p})$ and $P_j(\bar{x}_j) \cap F(\omega_j, \bar{p}) = \emptyset$.

   Thus, $(\bar{x}_1, \dots, \bar{x}_n, \bar{p})$ is a quasi-equilibrium if and only if for each *j*, $x_j \in P_j(\bar{x}_j)$ implies that $\langle x_j, \bar{p}\rangle \geq \langle \omega_j, \bar{p}\rangle$.

Note that every equilibrium is a quasi-equilibrium.

3. Inconsequential arbitrage

We define the *j*th *agent's arbitrage cone at endowments* $\omega_j \in X_j$ as the closed convex cone containing the origin given by

$$R(\hat{P}_j(\omega_j)) = \{y_j \in \mathbb{R}^L : \text{for } x'_j \in \hat{P}_j(\omega_j) \text{ and } \lambda \geq 0,\ x'_j + \lambda y_j \in \hat{P}_j(\omega_j)\}.$$

Thus, if $y_j \in R(\hat{P}_j(\omega_j))$, then for all $\lambda \geq 0$ and all $x_j \in \hat{P}_j(\omega_j)$, $x_j + \lambda y_j \in X_j$ and $u_j(x_j + \lambda y_j) \geq u_j(\omega_j)$. The agent's arbitrage cone at ω_j, then, is the recession cone corresponding the weak preferred set $\hat{P}_j(\omega_j)$ (see Rockafellar (1970), Section 8). Equivalently, $y_j \in R(\hat{P}_j(\omega_j))$ if and only if y_j is a cluster point of some sequence $\{\lambda^k x_j^k\}_k$ where the sequence of positive numbers $\{\lambda^k\}_k$ is such that $\lambda^k \downarrow 0$, and where for all *k*, $x_j^k \in \hat{P}_j(\omega_j)$; see Rockafellar (1970), Theorem 8.2.

An *arbitrage* $\omega = (\omega_1, \dots, \omega_n)$ is an *n*-tuple of net trades $y = (y_1, \dots, y_n)$ such that *y* is the limit of some sequence $\{(\lambda^k x_1^k, \dots, \lambda^k x_n^k\}_k$ with $\lambda^k \downarrow 0$ and $x^k = (x_1^k, \dots, x_n^k) \in A(\omega)$ for all *k*.[5] We shall denote by arb(ω) the set of all arbitrages at ω. Also, we shall denote by

[5] Note that $(y_1, \dots, y_n) \in \mathrm{arb}(\omega)$ implies that $\sum_j y_j = 0$.

arbseq$^{\omega}$(y) the set of all sequences $\{x^k\}_k$ of rational allocations (i.e. $x^k \in A(\omega)$ for all k) such that $\lambda^k x^k \to y$ for some sequence $\{\lambda^k\}_k$ of positive real numbers with $\lambda^k \downarrow 0$. Note that the set of all arbitrages, *arb*(ω), is the recession cone corresponding to the set of all individually rational allocations, $A(\omega)$ (see Rockafellar (1970), Section 9).

We say that an arbitrage $y = (y_1, \ldots, y_n) \in \mathrm{arb}(\omega)$ is in the *back-up set at* ω, denoted by bus(ω), if for each sequence $\{x^k\}_k \in \mathrm{arbseq}^{\omega}(y)$, there exists an $\varepsilon > 0$ such that for all k sufficiently large and all agents j,

$$x_j^k - \varepsilon y_j \in X_j \quad \text{and} \quad u_j(x_j^k - \varepsilon y_j) \geq u_j(x_j^k). \tag{4}$$

An economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies *inconsequential arbitrage at* ω if

$$\mathrm{arb}(\omega) \subseteq \mathrm{bus}(\omega). \tag{5}$$

In words, an arbitrage $y \in \mathrm{arb}(\omega)$ is inconsequential (i.e. is contained in the backup set at endowments *bus*(ω)) if for sufficiently large allocations $x \in A(\omega)$ in the $y = (y_1, \ldots, y_n)$ 'directions' from the endowment ω, each agent j can reduce his consumption by a small amount in the $-y_j$ direction without reducing his utility.

4. Implications of inconsequential arbitrage

4.1. Increasing cones and exhaustible arbitrages

A set closely related to the jth agent's arbitrage cone at x_j is the increasing cone at x_j given by

$$\boldsymbol{I}_j(x_j) := \{y_j \in R(\hat{P}_j(x_j)) : \forall\lambda \geq 0, \exists\lambda' > \lambda \text{ such that } u_j(x_j + \lambda' y_j) > u_j(x_j + \lambda y_j)\}.$$

We say that an arbitrage $y = (y_1, \ldots, y_n) \in \mathrm{arb}(\omega)$ is *exhaustible* at ω, if for all rational allocations $x = (x_1, \ldots, x_n) \in A(\omega)$ and all agents j,

$$y_j \in R(\hat{P}_j(\omega_j)) \backslash \boldsymbol{I}_j(x_j).$$

Thus, an arbitrage y fails to be exhaustible at ω if for some rational allocation $x \in A(\omega)$ and some agent j, $y_j \in \boldsymbol{I}_j(x_j)$. We shall denote by ea($\omega$) the set of all arbitrages that are exhaustible at ω. Note that if $y_j \in R(\hat{P}_j(\omega_j)) \backslash \boldsymbol{I}_j(x_j)$, then there exists a $\lambda_{x_j} \geq 0$ such that for $\lambda \geq \lambda_{x_j}$, $u_j(x_j + \lambda y_j)$ is nonincreasing in λ. Thus, there does not exist a $\lambda > \lambda_{x_j}$ such that $u_j(x_j + \lambda y_j) > u_j(x_j + \lambda_{x_j} y_j)$, and thus at $x_j + \lambda_{x_j} y_j$ all arbitrages in the y_j direction are exhausted.

Theorem 1 (Inconsequentiality implies exhaustibility). *Let* $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *be an economy satisfying* **[A-1]–[A-2]**. *The following statements are true.*

1. *If y is an inconsequential arbitrage at* ω (*i.e.* $y \in \mathrm{bus}(\omega)$), *then y is exhaustible at* ω (*i.e.* $y \in \mathrm{ea}(\omega)$). *Thus,*

$$\mathrm{bus}(\omega) \subseteq \mathrm{ea}(\omega).$$

2. *If the economy satisfies inconsequential arbitrage, then*

$$\mathrm{bus}(\omega) = \mathrm{ea}(\omega) = \mathrm{arb}(\omega).$$

Conversely, if $\mathrm{bus}(\omega)$ *is a proper subset of* $\mathrm{ea}(\omega)$, *then the economy fails to satisfy inconsequential arbitrage.* [6]

4.2. Compactness of the set of utility possibilities

Define the set of individually rational utility possibilities as follows:

$$U(\omega) := \{(u_1, \ldots, u_n) \in \mathbb{R}^n : \exists x \in A(\omega) \text{ such that } u_j(\omega_j) \leq u_j \leq u_j(x_j) \forall j\}. \quad (6)$$

Also for $k = 1, 2, \ldots$ define the set of k bounded individually rational allocations as follows:

$$C^k(\omega) := \left\{ (x_1, \ldots, x_n) \in A(\omega) : \sum_j \|x_j\| \leq k \right\}. \quad (7)$$

As follows easily from our next Theorem, inconsequential arbitrage implies that the set of individually rational utility possibilities is compact. [7]

Theorem 2 (Inconsequentiality and boundedness of utility). *Let* $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *be an economy satisfying* [**A-1**]–[**A-2**], *and inconsequential arbitrage. The following statements are true.*
1. *There exists* k_0 *such that* $\forall u \in U(\omega) \exists x \in C^{k_0}(\omega)$ *such that* $u_j \leq u_j(x_j) \forall j$.
2. *For each agent j there exists* $\bar{z}_j \in X_j$ *such that for all* $x_j \in A_j(w)$, $u_j(x_j) \leq u_j(\bar{z}_j)$.

5. Main existence result

Our main result states that in an economy allowing short-sales, and half-lines in indifference surfaces, inconsequential arbitrage at endowments is sufficient for existence of a quasi-equilibrium.

Theorem 3 (Inconsequential arbitrage implies the existence of a quasi-equilibrium). *Let* $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *be an economy satisfying* [**A-1**]–[**A-3**]. *If the economy satisfies inconsequential arbitrage at endowments* $\omega = (\omega_1, \ldots, \omega_n)$, *then there exists a quasi-equilibrium* $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$. *Moreover, if the quasi-equilibrium* $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ *is such that for each agent j,*
1. $\inf_{x \in X_j} \langle x, \bar{p} \rangle < \langle \omega_j, \bar{p} \rangle$,

[6] A sufficient condition for inconsequential arbitrage in terms of proximal trading volume appears in the working paper version of this paper (see Page et al. (2000)).
[7] This result, as well as its proof, suggested to us by Cuong Le Van, are in the spirit of Allouch's Proposition 5.2 (see Allouch (1999)).

2. *$P_j(\bar{x}_j)$ is open relative to X_j,*
then $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is an equilibrium.

Condition (1) in Theorem 1 will hold automatically if for each agent j, $\omega_j \in \mathrm{int} X_j$.[8] Condition (2) will hold automatically if for each agent j the utility function, $u_j(\cdot)$, is continuous.

6. Pareto optimality, inconsequential arbitrage, and existence of equilibrium

6.1. Two equivalence results

Under a stronger condition of nonsatiation and under an assumption that arbitrage cones are uniform across rational allocations, the existence of a Pareto optimal rational allocation, inconsequential arbitrage, and compactness of the set of utility possibilities are equivalent.[9] The additional assumptions required are:

[**A-4**] *Nonsatiation off the bus*.

If $y \in \mathrm{arb}(\omega)\backslash \mathrm{bus}(\omega)$, then for each rational allocation $x = (x_1, \ldots, x_n) \in A(\omega)$ there is at least one agent j such that for some $\lambda_j > 0$

$$x_j + \lambda_j y_j \in P_j(x_j).$$

[**A-5**] *Uniformity of arbitrage cones across rational allocations*.

For all rational allocations $x = (x_1, \ldots, x_n) \in A(\omega)$

$$R(\hat{P}_j(x_j)) = R(\hat{P}_j(\omega_j)) \quad \text{for all agents } j.$$

Theorem 4 (The equivalence of existence of a Pareto optimal allocation, inconsequential arbitrage, and compactness of utility possibilities). *Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying* [**A-1**]–[**A-2**] *and* [**A-4**]–[**A-5**]. *Then the following statements are equivalent:*
1. *$(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ has a rational Pareto optimal allocation.*
2. *$(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies inconsequential arbitrage.*
3. *The set of individually rational utility possibilities $U(\omega)$ is compact.*

By assumption [**A-3**] thick indifference surfaces at rational allocations are ruled out. Thus [**A-3**] guarantees that all equilibria are Pareto optimal. Our next two assumptions guarantee that every quasi-equilibrium is an equilibrium.

[**A-2**]* for each agent $j = 1, \ldots, n$, $\omega_j \in \mathrm{int}\, X_j$ and X_j is closed and convex.

[**A-6**] For all rational allocations $x = (x_1, \ldots, x_n) \in A(\omega)$, $\forall j$, $P_j(x_j)$ is open relative to X_j.

We have the following Corollary

---

[8] Here int denotes interior.

[9] A rational allocation $(x_1, \ldots, x_n) \in A(\omega)$ is Pareto-optimal if there does not exist another rational allocation $(x'_1, \ldots, x'_n) \in A(\omega)$ such that $u_j(x'_j) \geq u_j(x_j)$ for all agents j and for at least one agent j', $u_{j'}(x'_{j'}) > u_{j'}(x_{j'})$.

Corollary 5 (The equivalence of existence of a Pareto optimal allocation, inconsequential arbitrage, compactness of utility possibilities, and existence of equilibrium). *Let* $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *be an economy satisfying* [**A-1**], [**A-2**]*, [**A-3**], [**A-4**]–[**A-5**] *and* [**A-6**]. *Then the following statements are equivalent:*

1. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *has a rational Pareto optimal allocation*.
2. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *satisfies inconsequential arbitrage*.
3. *The set of individually rational utility possibilities* $U(\omega)$ *is compact*.
4. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *has an equilibrium*.

Remarks 1. (a) It can also be shown that nonemptiness of the core implies inconsequential arbitrage, as shown in Page and Wooders (1993, 1996a) for no unbounded arbitrage. The notion of the core involved is that typically used in economics. Since existence of a Pareto optimal allocation is weaker than nonemptiness of the core, we focus on Pareto optimal allocations. To relate inconsequential arbitrage to the partnered core, stronger conditions on the economic model are required — specifically, strictly convex preferences; see Page and Wooders (1996b) and, for the partnered core of a game, Reny and Wooders (1996).

(b) Rather than assume [**A-4**], instead we could have made the following assumptions

[**A-7**] *Uniformity of increasing cones across rational allocations*. For all rational allocations $x = (x_1, \dots, x_n) \in A(\omega)$, $I_j(x_j) = I_j(\omega)$ for all agents *j*.

[**A-8**] $\text{bus}(\omega) = \text{ea}(\omega)$.

Together [**A-7**] and [**A-8**] imply [**A-4**].

(c) Rather than assume [**A-6**], instead we could have strengthened assumption [**A-1**] as follows:

[**A-1**]* *for each agent j*, $u_j(\cdot)$ *is continuous and quasi-concave*.

The following variate of Corollary 5 *can be stated*

If $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *satisfies* [**A-1**]*, [**A-2**]*, [**A-3**] *and* [**A-4**]–[**A-5**], *then the following statements are equivalent*

1. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *has a rational Pareto optimal allocation*.
2. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *satisfies inconsequential arbitrage*.
3. *The set of individually rational utility possibilities* $U(\omega)$ *is compact*.
4. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *has an equilibrium*.

6.2. A second welfare theorem

By strengthening assumption [**A-4**] we can prove a second welfare theorem for the exchange economy described above. This result generalizes the second welfare theorem established in Page and Wooders (1996a). The strengthening of [**A-4**] required is the following:

[**A-4**]* *Uniform nonsatiation off the bus*. Given any rational allocation $x = (x_1, \dots, x_n) \in A(\omega)$, if $y \in \text{arb}(\omega)\backslash\text{bus}(\omega)$, then for each rational allocation $x' = (x'_1, \dots, x'_n) \in A(x)$ there is at least one agent *j* such that for some $\lambda_j > 0$, $x'_j + \lambda_j y_j \in P_j(x'_j)$.

Here,

$$A(x) = \left\{ (x'_1, \ldots, x'_n) \in X_1 \times \cdots \times X_n : \sum_{j=1}^{n} x'_j = \sum_{j=1}^{n} x_j \text{ and } x'_j \in \hat{P}_j(x_j) \quad \text{for all} \quad j \right\}.$$

Theorem 6 (A second welfare theorem). *Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying* [**A-1**], [**A-2**]*, [**A-3**], [**A-4**]*, [**A-5**], *and* [**A-6**]. *If the economy satisfies inconsequential arbitrage, then for each rational, Pareto optimal allocation $x = (x_1, \ldots, x_n) \in A(\omega)$ with $x_j \in \operatorname{int} X_j$ for all j, there is a price vector $p \in B\backslash\{0\}$ such that $(x_1, \ldots, x_n, p)$ is an equilibrium relative to some endowment*.

7. Examples

Example 1. In this example bus(ω) is a proper subset of ea(ω), and thus inconsequential arbitrage does not hold. Consider an economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^2$ in which two commodities are traded. Agent 1 has consumption set

$$X_1 = \{(x_{11}, x_{12},) : x_{11} \geq 1 \text{ and } x_{12} \geq 0\}$$

and Leontief preferences which kink along the curve given by $f(x) = \ln x$, for $x \geq 1$. Agent 2 has consumption set

$$X_2 = \{(x_{21}, x_{22}) : x_{21} \leq 0 \text{ and } x_{22} \leq 0\}$$

(i.e. the southwest quadrant) and preferences given by the utility function

$$u_2(x_{21}, x_{22}) = |x_{22}|.$$

Thus, agent 2 has straight line indifference curves as depicted in Fig. 1.

Agent 1 has endowment $\omega_1 = (2, \ln 2)$ while agent 2 has endowment $\omega_2 = -\omega_1$. Now consider the sequence of rational allocations given by $\{(x_1^k, x_2^k)\}_k$ with $(x_2^k, \omega_2) = x_2^k - \omega_2 = -(x_1^k - \omega_1)$ where the sequence of agent is consumption vectors $\{x_1^k\}_k$ moves along the curve $f(x) = \ln x$, for $x \geq 1$ (see Fig. 1). The sequence $\{(\lambda^k x_1^k, \lambda^k x_2^k)\}_k$ with $\lambda^k = 1/\|x_1^k\|$ converges to $y = (y_1, y_2) = ((1, 0), (-1, 0))$. Thus, $y \in \text{arb}(\omega)$ and it is clear from Fig. 1 that $y \in \text{ea}(\omega)$. Note, however, that $y \notin \text{bus}(\omega)$. In this example $\text{bus}(\omega) = \{((0, 0),\ (0, 0))\}$, while

$$\text{ea}(\omega) = \{((0,0), (0,0))\} \cup \{((\gamma, 0), (-\gamma, 0)) : \gamma > 0\}.$$

Example 2. In the following variation on Example 1, bus(ω) is equal to ea(ω), and inconsequential arbitrage is satisfied. Consider an economy $(X_j, \omega_j P_j(\cdot))_{j=1}^2$ in which two commodities are traded. Agent l's preferences and consumption set are as before. But now agent 2 has consumption set

$$X_2 = \{(x_{21}, x_{22}) : x_{21} \leq 0 \text{ and } x_{22} \geq 0\}$$

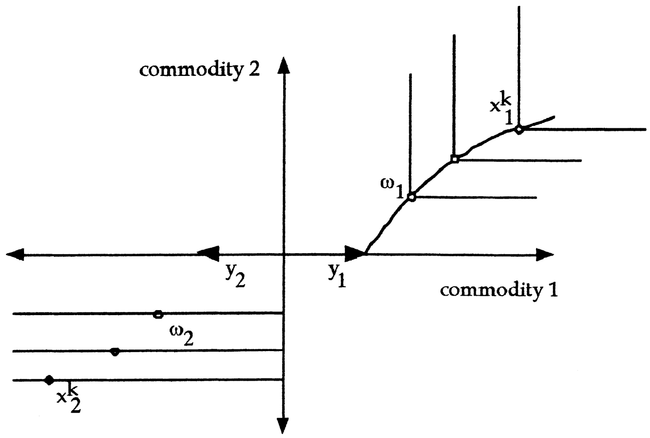

Fig. 1.

(i.e. the northwest quadrant) and preferences again given by the utility function $u_2(x_{21}, x_{22}) = |x_{22}|$. Thus, agent 2 has straight line indifference curves in the northwest quadrant as depicted in Fig. 2.

Agent 1 has endowment $\omega_1 = (2, \ln 2)$ while agent 2 has endowment $\omega_2 = (-2, \ln 2)$. In this example,

$$A(\omega) = \{((2+\gamma, \ln 2), (-2-\gamma, \ln 2)) : \gamma \geq 0\}.$$

Given the simple structure of the set of rational allocations, it follows that all sequences of rational allocations $\{x^k\}_k := \{(x_1^k, x_2^k)\}_k$ are of the form

$$\{(x_1^k, x_2^k)\}_k = \{((2+\gamma^k, \ln 2), (-2-\gamma^k, \ln 2)) : \gamma^k \geq 0\}_k.$$

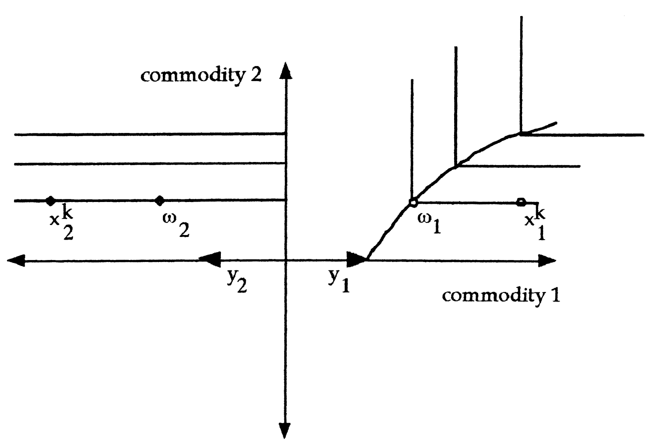

Fig. 2.

Moreover, it is easy to see that the economy satisfies inconsequential arbitrage and that

$$\mathrm{arb}(\omega) = \mathrm{ea}(\omega) = \mathrm{bus}(\omega) = \{((0,0), (0,0))\} \cup \{((\gamma, 0)(-\gamma, 0)) : \gamma > 0\}.$$

8. Other conditions limiting arbitrage: some comparisons

In this section, we compare inconsequential arbitrage to the conditions limiting arbitrage introduced by Hart (1974) and Werner (1987), as well as to the conditions recently introduced by Dana et al. (1999) and Allouch (1999). Conditions limiting arbitrage found in the literature generally fall into three broad categories:

1. *conditions on net trades*, for example, Hart (1974), Page (1987), Nielsen (1989), Page and Wooders (1993, 1996a,b), Allouch (1999) — including the condition of inconsequential arbitrage introduced here;
2. *conditions on prices*, for example, Green (1973), Grandmont (1977, 1982), Hammond (1983), and Werner (1987).
3. *conditions on the set of utility possibilities* (*namely, compactness*), for example, Brown and Werner (1995), Dana et al. (1999).

8.1. A Comparison to Hart and Werner

We begin by showing that the condition of Hart (translated to a general equilibrium setting) and the condition of Werner are equivalent, and more importantly, we show that the Hart/Werner conditions imply inconsequential arbitrage. In order to highlight the extent to which we extend Hart (1974) and Werner (1987), we construct an example (Example 3 below) of an exchange economy in which arbitrage cones are not globally uniform. However, increasing cones are uniform across rational allocations (i.e. [**A**-**7**] holds) and $\mathrm{bus}(\omega) = \mathrm{ea}(\omega)$ (i.e. [**A**-**8**] holds). Thus, there is nonsatiation off $\mathrm{bus}(\omega)$ (i.e. [**A**-**4**] holds). In addition, inconsequential arbitrage holds (and is necessary and sufficient for existence), while the Hart/Werner conditions do not hold. In order to understand more clearly the limitations of inconsequential arbitrage, we also construct an example (Example 4 below) of an exchange economy in which inconsequential arbitrage does not hold, nor do the Hart/Werner conditions. But an equilibrium does exist.

8.1.1. The equivalence of Hart's condition and Werner's condition

In order to show that Hart's condition (translated to a general equilibrium setting) and Werner's condition are equivalent we must first introduce some new definitions and assumptions.

A set closely related to the jth agent's arbitrage cone at ω_j is the *lineality space* at ω_j given by

$$L(\hat{P}(\omega_j)) := -R(\hat{P}_j(\omega_j)) \cap R(\hat{P}_j(\omega_j)).$$

Note that if $y_j \in L(\hat{P}_j(\omega_j))$, then for all $\lambda \in \mathbb{R}$ and all $x_j \in \hat{P}_j(\omega_j)$, $x_j + \lambda y_j \in X_j$ and $u_j(x_j + \lambda y_j) \geq u_j(x_j)$. Under assumption [**A**-**1**], if $y_j \in L(\hat{P}_j(\omega_j))$, then net trades in the directions y_j or $-y_j$ starting at the consumption vector ω_j are utility constant.

Werner (1987) makes the following assumptions concerning lineality spaces and arbitrage cones (i.e. recession cones) in his model

[**W-1**] *Global uniformity of arbitrage cones*. For each agent *j*,

$$R(\hat{P}_j(x)) = R(\hat{P}_j(\omega_j)) := R_j \quad \text{for all } x \in X_j.$$

[**W-2**] *Existence of useful trades*. For each agent *j*,

$$R_j \backslash L_j \neq \emptyset.$$

Here L_j denotes the lineality space corresponding to the recession cone R_j. Together, [**W-1**] and [**W-2**] play the same role in Werner's (1987) proof of existence as does our assumption of nonsatiation at rational allocations, [**A-3**] in our proof of Theorem 1. Note that Werner's assumption [**W-1**] implies our assumption [**A-5**]. Also note that if agents' utility functions are concave then [**W-1**] holds automatically. Finally, note that under [**W-1**], if $y_i \in L_i$ then for all $\lambda \in \mathbb{R}$ and all $x_j \in X_j$, $x_j + \lambda y_j \in X_j$ and $u_j(x_j + \lambda y_j) = u_j(x_j)$. In the terminology of Werner (1987), under [**W-1**] a 'commodity bundle' $y_j \in L_j$ is *useless*.

Werner (1987) also introduces the *no half-lines condition*.[10] The *j*th agent's utility function satisfies the no half-lines condition if for each $x_j \in X_j$ there *does not exist* a nonzero vector of net trades y_j such that $u_j(x_j + \lambda y_j) = u_j(x_j)$ for all $\lambda \geq 0$.

[**W-3**] *No half lines*. Each agent's utility function satisfies the no half-lines condition.

Note that [**W-3**] implies that the set of exhaustible arbitrages consists of an *n*-tuple the zero vectors (i.e. $\mathrm{ea}(\omega) = \{(0, \ldots, 0)\}$). Thus, [**W-3**] implies that $\mathrm{bus}(\omega) = \mathrm{ea}(\omega) = \{(0, \ldots, 0)\}$. Finally, while not stated explicitly, Werner requires for the proof of his main existence result the following strengthening of our assumption [**A-1**] (see Werner (1987), the proof on page 1417):[11]

[**A-1**]** for each agent *j*, $u_j(\cdot)$ is continuous and strictly quasi-concave.

The *dual cone corresponding* to the arbitrage cone R_j, denoted by $\Lambda(R_j)$, is given by

$$\Lambda(R_j) := \{p \in \mathbb{R}^L : \langle y, p \rangle \geq 0 \quad \text{for all } y \in R_j\}.$$

The *positive dual cone corresponding to* R_j, denoted by $\Lambda^+(R_j)$, is given by

$$\Lambda^+(R_j) := \left\{ p \in \mathbb{R}^L : \langle y, p \rangle > 0 \quad \text{for all } y \in R_j \backslash L_j \right\}.$$

If $p \in \Lambda^+(R_j)$, then the price vector *p* assigns a positive value to any vector of *useful* net trades $y \in R_j \backslash L_j$.

Under assumption [**W-2**] it follows from Theorem 2.1 in Yu (1974) that

$$\Lambda^+(R_j) = \mathrm{ri}\Lambda(R_j)$$

where 'ri' denotes the relative interior.

[10] Werner (1987) states that if each agent's utility function satisfies the no *half-line condition*, then his condition is necessary as well as sufficient for existence. Page and Wooders (1996a) show that if all but one agent's utility function satisfies the no half lines condition, then Werner's condition is necessary and sufficient for existence.

[11] Recall that $u_j(\cdot)$ is strictly quasi-concave if whenever x_j and z_j belong to X_j and $u_j(z_j) > u_j(x_j)$, then

$$u_j(\lambda x_j + (1-\lambda) z_j) > u_j(x_j) \text{ for all } \lambda \in [0,1).$$

Now consider an exchange economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfying assumptions [**A-1**]–[**A-2**] and [**W-1**]–[**W-2**]. This economy satisfies *Hart's condition* if whenever

$$\sum_{j=1}^{n} y_j = 0 \text{ and } y_j \in R_j \text{ for all } j, \text{ then } y_j \in L_j \text{ for all } j. \tag{8}$$

Moreover, the economy satisfies *Werner's condition* if

$$\cap_j \Lambda^+(R_j) \neq \emptyset. \tag{9}$$

An economy satisfies Hart's condition, if all n-tuples of mutually compatible net trades representing potential unbounded arbitrages for the individual agents are useless (i.e. are utility constant). The economy satisfies Werner's condition, if the set of prices assigning a positive value to all n-tuples of useful net trades (i.e. net trades representing potential unbounded arbitrages) is nonempty.

Our next result states that Hart's condition (a condition on net trades) is equivalent to Werner's condition (a condition on prices). Theorem 7 below is obtained by specializing Corollary 16.2.2 in Rockafellar (1970) to cover the exchange economy developed here.

Theorem 7 (Hart's condition is equivalent to Werner's). *Let* $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *be an economy satisfying* [**A-1**]–[**A-2**] *and* [**W-1**]–[**W-2**]. *Then Hart's condition* (8) *is satisfied if and only if Werner's condition* (9) *is satisfied*.

We shall refer to the conditions (8) and (9) above as the Hart/Werner conditions. Two interesting variations on the Hart/Werner conditions are the *overlapping expectations* condition of Hammond (1983) (also see Green (1973) and Grandmont (1977, 1982))[12] and the *no unbounded arbitrage* condition of Page (1987). In an asset market setting, under a set of assumptions mildly stronger than those used in Hart (1974) (including the assumption of no half lines), Hammond shows that his condition is equivalent to Hart's condition. Also in an asset market setting, under a set of assumptions similar to Hammond's (also including the assumption of no half-lines), Page shows that his condition given by

$$\text{if } \sum_{j=1}^{n} y_j = 0 \text{ and } y_j \in R_j \text{ for all } j, \text{ then } y_j = 0 \text{ for all } j. \tag{10}$$

is also equivalent to Hart's condition as well as Hammond's. In general, without the no half-lines assumption, $\{0,\ldots,0\}$ is a proper subset of the lineality space L_j. Thus, in general, Page's condition (10) implies the Hart/Werner conditions. Page's condition has been used by Nielsen (1989) to prove existence in an exchange economy model with short sales slightly more general than Werner's, and more recently, Page and Wooders (1993, 1996a,b) have refined Page's condition and shown it to be necessary and sufficient for compactness of the set of rational allocations, as well as for existence and nonemptiness of the core. Dana

[12] As far as we know, Hammond (1983) introduced the terminology 'overlapping expectations'. The condition overlapping expectations is a much weakened form of Green's 'common expectations'.

et al. (1999) and Allouch (1999) provide an excellent discussions of the different notions of arbitrage found in the general equilibrium literature. Page and Wooders (1999) also provide a discussion of notions of arbitrage, including a discussion of the various notions of arbitrage found in the literature on asset markets.

8.1.2. Hart/Werner conditions and inconsequential arbitrage

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an exchange economy satisfying assumptions [**A-1**]–[**A-2**] and [**W-1**]–[**W-2**]. Given the discussion above concerning lineality spaces, it is easy to see that

$$L_1 \times \cdots \times L_n \subseteq \mathrm{bus}(\omega).$$

This fact together with Theorem 7 imply that if the economy satisfies the Hart/Werner conditions, then any arbitrage $y = (y_1, \ldots, y_n) \in \mathrm{arb}(\omega)$ is contained in $L_1 \times \cdots \times L_n$. Thus, Hart's net trades condition or Werner's price condition implies inconsequential arbitrage. If, in addition, the economy satisfies no half-lines, [**W-3**], then $\mathrm{bus}(\omega) = \{0, \ldots, 0\}$, and therefore, $L_1 \times \cdots \times L_n = \mathrm{bus}(\omega)$. Thus, assuming no half lines, inconsequential arbitrage implies the Hart/Werner conditions. Stated formally, we have the following

Theorem 8 (Hart/Werner imply inconsequential arbitrage). *Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an exchange economy satisfying assumptions* [**A-1**]–[**A-2**] *and* [**W-1**]–[**W-2**].

$$\left.\begin{array}{c} \textit{If } \sum_{j=1}^m y_j = 0 \textit{ and } y_j \in R_j \textit{ for all } j, \\ \textit{implies}, \ y_j \in L_j \textit{ for all } j, \\ \text{or} \\ \textit{if } \cap_j \Lambda^+(R_j) \neq \emptyset, \end{array}\right\} \textit{then } \mathrm{arb}(\omega) \subseteq \mathrm{bus}(\omega).$$

If, in addition, the economy satisfies [**W-3**], *no half lines, then*

$$\cap_j \Lambda^+(R_j) \neq \emptyset \ \textit{ if and only if } \mathrm{arb}(\omega) \subseteq \mathrm{bus}(\omega).$$

8.1.3. A summary of results

In this subsection, we summarize our results and those of Werner.

A summary of sufficiency results:

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]**–[**A-2**]*. Werner (1987): Under [**W-1**]–[**W-2**], $\cap_j \Lambda^+(R_j) \neq \emptyset \Rightarrow$ existence of equilibrium.

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]*–[**A-2**]*. Page, Wooders and Monteiro: Under [**A-3**], $\mathrm{arb}(\omega) \subseteq \mathrm{bus}(\omega) \Rightarrow$ existence of equilibrium.

Thus, inconsequential arbitrage is sufficient for existence *without strict quasi-concavity and without uniformity of arbitrage cones* (see Example 3 below). [13]

A summary of characterization results: Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]**–[**A-2**]*. Werner (1987): Under [**W-1**]–[**W-3**], $\cap_j \Lambda^+(R_j) \neq \emptyset \Leftrightarrow$ existence.

[13] Werner's assumption **A5** is weaker than our assumption that $\omega_j \in \mathrm{int} X_j$ for all j (see our assumption [**A-2**]* and Werner's assumption **A5** on page 1410 of Werner (1987)). In fact, our assumption [**A-2**] implies Werner's **A5**.

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]*–[**A-2**]*. Page, Wooders, and Monteiro: Under [**A-3**] and [**A-4**]–[**A-5**], $\text{arb}(\omega) \subseteq \text{bus}(\omega) \Leftrightarrow$ existence.

Thus, inconsequential arbitrage is necessary and sufficient for existence, even *in the presence of half-lines* (see Example 3 below).

8.1.4. Examples

Example 3. Here we construct an example of an exchange economy satisfying assumptions [**A-1**]–[**A-3**] in which arbitrage cones are uniform across rational allocations (i.e. [**A-5**] holds) — but *not* globally uniform. In addition, increasing cones are uniform across rational allocations (i.e. [**A-7**] holds) and $\text{bus}(\omega) = \text{ea}(\omega)$ (i.e. [**A-8**] holds). Thus, there is nonsatiation off the bus(ω) (i.e. [**A-4**] holds). Finally, in this example it is easy to see that inconsequential arbitrage holds (and is necessary and sufficient for existence), while the Hart/Werner conditions fail.

Consider an economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^2$, where for each j, $X_j = \mathbb{R}^2$ Agent 1 has endowment $\omega_1 = (2, -1)$ while agent 2 has endowment $\omega_2 = -\omega_1$. Agent 1's utility function is given by

$$u_1(x_1, x_2) = \begin{cases} x_1 & \text{if} \quad x_1 \le 0 \quad \text{or} \quad x_2 \ge -1 \\ -\dfrac{x_1}{x_2} & \text{if} \quad x_1 \ge 0 \quad \text{and} \quad x_2 \le -1, \end{cases}$$

while agent 2 has utility function given by

$$u_2(x_1, x_2) = x_1 + 2x_2.$$

Fig. 3 summarizes the situation. Note that agents 1's utility function has arbitrage cones (i.e. recession cones) that are not *globally* uniform. Thus, this example is not covered by Werner's (1987) model.

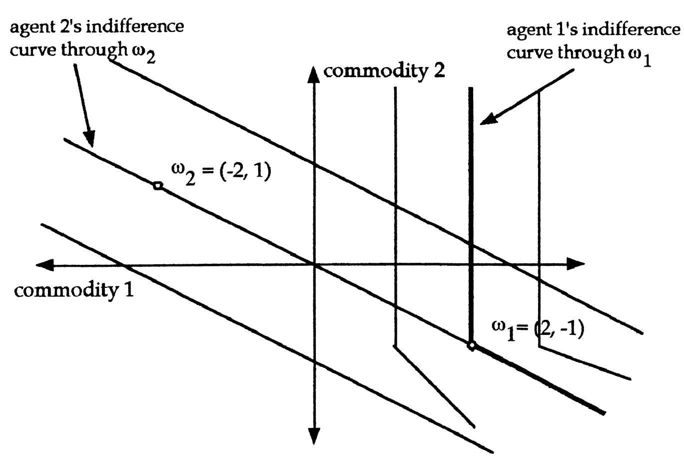

Fig. 3.

In this example,

$$A(\omega)=\{((2,-1)+\gamma\cdot(2,-1),(-2,1)+\gamma\cdot(-2,1)):\gamma\geq 0\}.$$

By inspection of Fig. 3, it is easy to see that increasing cones are uniform across rational allocations (i.e. [**A-7**] holds). In particular,

$$\begin{cases}\boldsymbol{I}_1((2,-1)+\gamma\cdot(2,-1))=\boldsymbol{I}_1((2,-1)),\\ \boldsymbol{I}_2((-2,1)+\gamma\cdot(-2,1))=\boldsymbol{I}_2((-2,1))\end{cases}\quad\text{for all }\gamma\geq 0\ .$$

Moreover, it is easy to see that arbitrage cones are uniform across rational allocations (i.e. [**A-5**] holds) and that local nonsatiation at rational allocations holds (i.e. [**A-3**] holds).

Given the simple structure of the set of rational allocations, it follows that all sequences of rational allocations $\{x^k\}_k:=\{(x_1^k,x_2^k)\}_k$ are of the form

$$\{(x_1^k,x_2^k)\}_k=\{((2,-1)+\gamma^k\cdot(2,-1),(-2,1)+\gamma^k\cdot(-2,1)):\gamma^k\geq 0\}_k.$$

Moreover, it is easy to see that the economy satisfies inconsequential arbitrage and that

$$\mathrm{arb}(\omega)=\mathrm{ea}(\omega)=\mathrm{bus}(\omega)=\{((0,0),(0,0))\}\cup\{(\gamma\cdot(2,-1),-\gamma\cdot(-2,1)):\gamma>0\}.$$

Finally, note that lineality space (the set of useless net trades) for agent 1 is

$$L(\hat{P}_1(2,-1))=\{(0,0)\},$$

while for agent 2, the lineality space is given by

$$L(\hat{P}_2(-2,1))=\{\gamma\cdot(-2,1):\gamma\in\mathbb{R}\}.$$

Thus, in this example, the Hart/Werner conditions are not satisfied.

Example 4. In our last example, we modify Example 3 by changing the endowment of agent 1. As a result, increasing cones are no longer uniform across rational allocations (i.e. [**A-7**] fails), equality of bus(ω) and ea(ω) fails (i.e. [**A-8**] fails), and nonsatiation off the bus(ω) fails (i.e. [**A-4**] fails). Moreover, inconsequential arbitrage does not hold, nor do the Hart/Werner conditions. But an equilibrium does exist. Fig. 4 illustrates the problem.

Agent 1's endowment is now ω_1' (rather than $\omega_1=(2,-1)$). While agent 2's endowment is as before. Consider the sequence $\{(\lambda^k x_1^k,\lambda^k x_2^k)\}_k\in\mathrm{arbseq}^{\omega'}((y_1,y_2))$ given by

$$\{(\lambda^k x_1^k,\lambda^k x_2^k)\}_k=\left\{\left(\frac{1}{k}(\omega_1'+k\cdot y_1),\frac{1}{k}(\omega_2+k\cdot y_2)\right)\right\}_k$$

where $\omega'=(\omega_1',\omega_2)$ and y_1 and y_2 are given in Fig. 4.[14] It is easy to see that

$$y=(y_1,y_2)\notin\mathrm{bus}(\omega').$$

[14] Observe that $y_2=-y_1$.

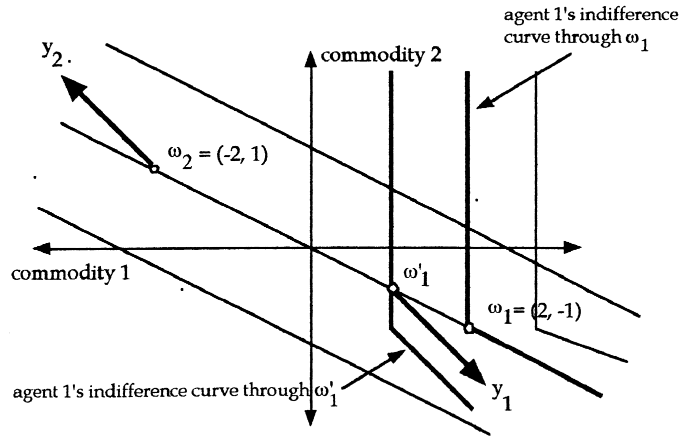

Fig. 4.

Thus, inconsequential arbitrage does not hold. [15] By inspection of Fig. 4 it can be verified that increasing cones are no longer uniform across rational allocations, and that bus(ω') is a proper subset of ea(ω'). In fact, in this example

$$\text{bus}(\omega') = \{((0,0),\ (0,0))\},$$

while

$$\text{ea}(\omega) = \{((0,0),\ (0,0))\} \cup \{(\gamma \cdot (2,-1),\ \gamma \cdot (-2,1)) : \gamma > 0\}.$$

Moreover, [**A-4**], nonsatiation off bus(ω') fails. For example, take

$$y' = (y_1',\ y_2') = ((2,-1),\ (-2,1)) \in \text{arb}(\omega') \backslash \text{bus}(\omega').$$

It is also clear from Fig. 4, that for rational allocation $((\omega_1' + y_1'), (\omega_2 + y_2'))$

$$\left.\begin{aligned} u_1(\omega_1' + y_1' + \lambda y_1') &= u_1(\omega_1' + y_1') \\ u_2(\omega_2' + y_2' + \lambda y_2') &= u_2(\omega_2' + y_2') \end{aligned}\right\} \quad \text{for all } \lambda > 0.$$

Finally, note that this economy has an equilibrium $(\bar{x}_1, \bar{x}_2, \bar{p})$, for example

$$\bar{p} = \frac{1}{||(1,2)||} \cdot (1,2),$$

$$\bar{x}_1 = \omega_1' + y_1',$$

$$\bar{x}_2 = \omega_2 + y_2'.$$

[15] Note that if agent 1's indifference curves had the same shape as his indifference curve through ω_1 then inconsequential arbitrage would be restored. However, if all of agent 1's indifference curves had the same shape as his indifference curve through ω_1' then inconsequential arbitrage would again fail to hold (i.e. global uniformity is not enough to guarantee that the economy satisfies inconsequential arbitrage).

8.2. A comparison to Dana et al. and Allouch

We begin with some terminology. We say that the preference mapping $x_j \to P_j(x_j)$ defined on X_j is *strictly quasi-concave* if given any $x_j \in X_j$ and $x'_j \in P_j(x_j)$

$$(1-\lambda)x_j + \lambda x'_j \in P_j(x_j) \quad \text{for all } \lambda \in (0,1].$$

Note that if the preference mapping $x_j \to P_j(x_j)$ is specified via a utility function as in expression (1) above, then strict quasi-concavity of the underlying utility function implies strict quasi-concavity of the corresponding preference mapping. Next, we say that the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies *global nonsatiation at rational allocations* if for any rational allocation $(x_1, \ldots, x_n) \in A(\omega)$, $P_j(x_j) \neq \emptyset$ for all agents j.

Thus, if the economy satisfies only global nonsatiation at rational allocations rather than local nonsatiation at rational allocations, then for any given rational allocation $(x_1, \ldots, x_n)$ and any given agent j, x_j need not be contained in the boundary of $P_j(x_j)$.[16]

8.2.1. Dana et al

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]–[**A-2**]. Dana et al. (1999) show (Theorem 1) that if, in addition, the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies global nonsatiation at rational allocations, and if agents' utility functions, $u_j(\cdot)$, are stricly quasi-concave, then compactness of utility possibilities $\Rightarrow$ the existence of a quasi-equilibrium.

Note that global nonsatiation at rational allocations and strict quasi-concavity of utility functions together imply our assumption [**A-3**], local nonsatiation at rational allocations. Also note that it follows from part 1 of our Theorem 2 that for economies satisfying [**A-1**]–[**A-2**], inconsequential arbitrage implies compactness of utility possibilities (also see Proposition 5.2 in Allouch (1999)). Thus, inconsequential arbitrage is a stronger condition than compactness of utility possibilities.

8.2.2. Allouch

Allouch (1999) introduces a new condition, *bounded arbitrage*, defined as follows

The economy satisfies bounded arbitrage if for all sequences of rational allocations $\{x^n\}_n \subset A(\omega)$ there exists

- a subsequence $\{x^{n_k}\}_k$,
- a rational allocation $z \in A(\omega)$, and
- a sequence $\{z^k\}_k \subset X_1 \times \cdots \times X_n$ converging to z such that

$$z_j^k \in P'_j(x_j^{n_k}).$$

Here, following Gale and Mas-Colell (1975), the augmented preference correspondence $x_j \to P'_j(x_j)$ is given by

$$P'_j(x_j) := \{x'_j \in X_j : x'_j = (1-\lambda)x_j + \lambda x'_j \quad \text{for } 0 < \lambda \leq 1,\ x'_j \in P_j(x_j)\}.$$

[16] If we assume that each agent's utility function is continuous, rather than upper semicontinuous, then we can show that inconsequential arbitrage implies the existence of a quasi-equilibrium under the weaker assumption of global nonsatiation at rational allocations.

Thus, by bounded arbitrage, for every sequence of rational allocations there is a subsequence that is augmented preference-dominated by a sequence converging to a rational allocation. Note that bounded arbitrage *implies* global nonsatiation at rational allocations.

Allouch shows (Proposition 5.1) that

if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies [**A-1**]–[**A-2**] and
global nonsatiation at rational allocations, and
if agents' preference mappings $x_j \to P_j(x_j)$
are strictly quasi-concave, then
bounded arbitrage $\Leftrightarrow$ the set of utility possibilities is compact.

Moreover, Allouch shows (Proposition 5.2) that if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies [**A-1**]–[**A-2**], and

if agents' preference mappings $x_j \to P_j(x_j)$
are strictly quasi-concave, then
inconsequential arbitrage $\Rightarrow$ the set of utility possibilities is compact.

In fact, Allouch's Proposition 5.2 continues to hold *without strict quasi-concavity*. Thus, we have

if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies [**A-1**]–[**A-2**], then
inconsequential arbitrage $\Rightarrow$ the set of utility possibilities is compact.

Without strict quasi-concavity Allouch's Proposition 5.1 must be weakened as follows:

If the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies [**A-1**]–[**A-2**] and
global nonsatiation at rational allocations, then
bounded arbitrage $\Rightarrow$ the set of utility possibilities is compact.

However, if we strengthen global nonsatiation at rational allocations to local non-satiation at rational allocations (i.e. [**A-3**]), then Allouch's Proposition 5.1 is restored. Thus, we have

if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies [**A-1**]–[**A-3**], then
bounded arbitrage $\Leftrightarrow$ the set of utility possibilities is compact.

Allouch's main existence result (Theorem 3.1), translated to the model here, is the following:

if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies [**A-2**], where
$P_j(x_j) := \{x \in X_j : u_j(x) > u_j(x_j)\}$, and
$\hat{P}_j(x_j) := \{x \in X_j : u_j(x) \geq u_j(x_j)\}$
with $u_j(\cdot) : X_j \to \mathbb{R}$ quasi-concave,
and if
(i) $\hat{P}_j(\omega_j)$ is closed, and
(ii) $x_j \to P_j(x_j)$ is lower semicontinuous on $\hat{P}_j(\omega_j)$, then
bounded arbitrage $\Rightarrow$ the existence of a quasi-equilibrium.

Comparing Allouch's Theorem 3.1 to the following restatement of our Theorem 1, we have shown that

if the economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ satisfies
local nonsatiation at rational allocations and [**A-2**], where
$P_j(x_j) := \{x \in X_j : u_j(x) > u_j(x_j)\}$, and

$$\hat{P}_j(x_j) := \{x \in X_j : u_j(x) \geq u_j(x_j)\}$$
with $u_j(\cdot) : X_j \to \mathbb{R}$ quasi-concave,
and if
$x_j \to \hat{P}_j(x_j)$ is closed-valued on X_j, then
inconsequential arbitrage $\Rightarrow$ the existence of a quasi-equilibrium.

It should be noted that, in fact, Allouch (1999) proves the existence of a quasi-equilibrium in a model more general than the model developed here. For example, Allouch assumes that agents' preferences are given via a partial preorder, $\prec_j$, and does not require that the induced preference mappings, $x_j \to \{x'_j \in X_j : x_j \prec_j x'_j\}$, be everywhere convex-valued nor that the induced weak preference mappings, $x_j \to \{x'_j \in X_j : x_j \preceq_j x'_j\}$, be everywhere closed-valued. Similar extensions of our results are also possible.

8.3. A fundamental equivalence result

As before, let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ denote an exchange economy where the jth agent's preferences $P_j(\cdot)$ over X_j are specified via a utility function $u_j(\cdot) : X_j \to \mathbb{R}$, and suppose that the economy satisfies the following assumptions for each agent $j = 1, \ldots, n$:

[**A-1**]* $u_j(\cdot)$ is continuous and quasi-concave;

[**A-2**]* $\omega_j \in \mathrm{int} X_j$ and X_j is closed and convex;

[**A-3**] *Local nonsatiation ar rational allocations.* $\forall (x_1, \ldots, x_n) \in A(\omega), \forall j, P_j(x_j) \neq \emptyset$ and $\mathrm{cl} P_j(x_j) = \hat{P}_j(x_j)$;

[**W-1**] *Global uniformity of arbitrage cones*. $\forall_j$, $R(\hat{P}_j(x)) = R(\hat{P}_j(\omega_j)) := R_j$ for all $x \in X_j$;

[**W-2**] *Existence of useful trades*. $\forall_j$, $R_j \backslash L_j \neq \emptyset$.

[**W-3**] *No half-lines*. Each agent's utility function satisfies the no half lines condition.

We have the following equivalence result

Theorem 9 (A fundamental equivalence). *Let* $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *be an economy satisfying* [**A-1**]*, [**A-2**]*, [**A-3**]. *If the economy satisfies inconsequential arbitrage, then the economy has an equilibrium. Moreover, if the economy satisfies the conditions of uniformity, existence of useful trades, and no half-lines,* [**W-1**]–[**W-3**], *then the following statements are equivalent*:

1. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *satisfies inconsequential arbitrage*.
2. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *satisfies the Hart/Werner conditions*.
3. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *satisfies Allouch's bounded arbitrage condition*.
4. *The set of utility possibilities*, $U(\omega)$, *is compact*.
5. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *has a rational Pareto optimal allocation*.
6. $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *has an equilibrium*.

The equilivalence of (1), (4), (5), and (6) follows from our Corollary 5 and the fact that assumption [**W-1**] and [**W-3**] imply our assumptions [**A-4**] and [**A-5**]. The equivalence of (1) and (2) follows from our Theorem 8. By Allouch's Proposition 4.3, (2) implies (3), and by Allouch's Proposition 5.1, (3) implies (4).

9. Proofs

9.1. Bounded economies

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an exchange economy satisfying [**A-1**]–[**A-3**], and let $C(k)$ be a closed ball in $\mathbb{R}^L$ of radius r_k, with $r_k > 1$, centered at the origin such that for each agent j, $\omega_j \in \mathrm{int}C(k)$ where int denotes interior. We shall assume that $r_k \to \infty$ as $k \to \infty$. Now define

$$X_{kj} := X_j \cap C(k),$$

$$P_{kj}(x_j) := P_j(x_j) \cap C(k),$$

$$\hat{P}_{kj}(x_j) := \hat{P}_j(x_j) \cap C(k),$$

$$B_k(\omega_j, p) = \{x_j \in X_{kj} : \langle x_j, p\rangle \leq \langle \omega_j, p\rangle\},$$

$$F_k(\omega_j, p) = \{x_j \in X_{kj} : \langle x_j, p\rangle \leq \langle \omega_j, p\rangle\},$$

and consider the k-bounded economy

$$E_k := (X_{kj}, \omega_j, P_{kj}(\cdot))_{j=1}^n.$$

The set of rational allocations for the k-bounded economy E_k is given by

$$A_k(\omega) = \left\{ (x_1, \ldots, x_n) \in X_{k1} \times \cdots \times X_{kn} : \sum_{j=1}^n x_j = \sum_{j=1}^n \omega_j \text{ and } x_j \in \hat{P}_{kj}(\omega_j) \quad \text{for all } j \right\}.$$

An *equilibrium* for the economy E_k is an $(n+1)$-tuple of vectors $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ such that
1. $(\bar{x}_1, \ldots, \bar{x}_n) \in A_k(\omega)$;
2. $\bar{p} \in \mathcal{B}\backslash\{0\}$;
3. for each j, $\langle \bar{x}_j, \bar{p}, \rangle = \langle \omega_j, \bar{p}\rangle$ and $P_{kj}(\bar{x}_j) \cap B_k(\omega_j, \bar{p}) = \emptyset$.[17]

A *quasi-equilibrium* for the economy E_k is an $(n+1)$-tuple of vectors $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ such that
1. $(\bar{x}_1, \ldots, \bar{x}_n) \in A_k(\omega)$;
2. $\bar{p} \in B\backslash\{0\}$;
3. for each j, $\bar{x}_j \in B_k(\omega_j, p)$ and $P_{kj}(\bar{x}_j) \cap F_k(\omega_j, \bar{p}) = \emptyset$.

9.2. A lemma

The following lemma is crucial for the proof of Theorem 1.

[17] Recall that $\mathcal{B} := \{p' \in \mathbb{R}^L : ||p'|| \leq 1\}$.

Lemma. *Let* $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ *be an exchange economy satisfying* [**A-1**]–[**A-2**]. Let $\{x^k\}_k = \{(x_1^k, \ldots, x_n^k)\}_k \subset A_k(\omega)$ *be a sequence of individually rational allocations such that for each* k, $||x_j^k|| = r_k$ *for some agent* j (*i.e.* $\sum_{j=1}^n \|x_j^k\| \to \infty$ *as* $k \to \infty$) *and consider the sequence* $\{\lambda^k x^k\}_k := \{(\lambda^k x_1^k, \ldots, \lambda^k x_n^k)\}_k$ *where* $\lambda^k = (\sum_{j=1}^n ||x_j^k||)^{-1}$. *Then, the following statements are true*:

1. *For any cluster point* $y = (y_1, \ldots, y_n)$ *of the sequence* $\{(\lambda^k x_1^k, \ldots, \lambda^k x_n^k)\}_k$, *it holds that*

$$\sum_{j=1}^n y_j = 0 \quad \text{and} \quad \sum_{j=1}^n ||y_j|| = 1.$$

2. *For any cluster point* $y = (y_1, \ldots, y_n)$ *and subsequence* $\{(\lambda^{k'} x_1^{k'}, \ldots, \lambda^{k'} x_n^{k'})\}_{k'}$ *such that* $(\lambda^{k'} x_1^{k'}, \ldots, \lambda^{k'} x_n^{k'}) \to (y_1, \ldots, y_n)$, *it holds that*

$$\text{for } k' \text{ sufficiently large, } x_j^{k'} - \varepsilon y \in \mathrm{int}C(k') \quad \text{for all } \varepsilon \in (0,1].$$

Proof of Lemma. First, recall that any cluster point $y = (y_1, \ldots, y_n)$ of the sequence $\{(\lambda^k x_1^k, \ldots, \lambda^k x_n^k)\}_k$ is such that, for each agent j, $y_j \in R(\hat{P}_j(\omega_j))$. Part 1 of the Lemma is a restatement of Lemma 3.3 in Page (1987). To prove Part 2, first note that if $y_j = 0$; then, $||x_j^k|| < r_k$ infinitely often (i.e. $x_j^k \in \mathrm{int}C(k)$ for infinitely many k). In particular, if $||x_j^k|| = r_k$ for all k, then

$$\frac{1}{n} = \frac{r_k}{n \cdot r_k} \leq \frac{||x_j^k||}{\sum_{j'=1}^n ||x_j^k||} \quad \text{for all } k.$$

Thus, if $y_j = 0$, we have a contradiction, since

$$\frac{x_j^k}{\sum_{j'=1}^n ||x_j^k||} \to y_j.$$

Because there is a finite number of agents, we can extract a subsequence

$$\{(\lambda^{k'} x_1^{k'}, \ldots, \lambda^{k'} x_n^{k'})\}_{k'}$$

such that

$$(\lambda^{k'} x_1^{k'}, \ldots, \lambda^{k'} x_n^{k'}) \to (y_1, \ldots, y_n),$$

and such that

$$\text{if } y_j = 0 \text{ then } ||x_j^{k'}|| < r_{k'} \quad \text{for all } k' \text{ sufficiently large.}$$

To show that, for k' sufficiently large, $x_j^{k'} - \varepsilon y \in \mathrm{int}C(k')$, for all $\varepsilon \in (0, 1]$, consider the following:

$$\begin{aligned}||x_j^{k'} - \varepsilon y|| &\leq ||x_j^{k'} - \varepsilon\lambda^{k'} x_j^{k'}|| + ||\varepsilon\lambda^{k'} x_j^{k'} - \varepsilon y_j||\\ &= (1 - \varepsilon\lambda^{k'})||x_j^{k'}|| + ||\varepsilon\lambda^{k'} x_j^{k'} - \varepsilon y_j||\\ &= ||x_j^{k'}|| + (||\varepsilon\lambda^{k'} x_j^{k'} - \varepsilon y_j|| - ||\varepsilon\lambda^{k'} x_j^{k'}||)\\ &= ||x_j^{k'}|| + \varepsilon(||\lambda^{k'} x_j^{k'} - y_j|| - ||\lambda^{k'} x_j^{k'}).\end{aligned}$$

If $y_j = 0$, then $||x_j^{k'} - \varepsilon y|| < r_{k'}$ by the arguments above. Suppose now that $y_j \neq 0$. Since $||\lambda^{k'} x_j^{k'} - y_j|| \to 0$ and $||\lambda^{k'} x_j^{k'}|| \to ||y_j|| > 0$, we have for all k' sufficiently large $(||\lambda^{k'} x_j^{k'} - y_j|| - ||\lambda^{k'} x_j^{k'}) < 0$. Since already $||x_j^{k'}|| \leq r_{k'}$, we can conclude that, for all k' sufficiently large, $||x_j^{k'} - \varepsilon y|| < r_{k'}$ for all $\varepsilon \in (0, 1]$. □

9.3. *Proofs of Theorems*

9.3.1. *Proof of Theorem 1 (inconsequentiality implies exhaustibility)*

Proof of Part 1. Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]–[**A-2**]. To show that $\text{bus}(\omega) \subseteq \text{ea}(\omega)$, let $y \in \text{arb}(\omega)$ such that $y \notin \text{ea}(\omega)$ (i.e. y is not exhaustible). This implies that, for some rational allocation $x \in A(\omega)$ and some agent j, $y_j \in \boldsymbol{I}_j(x_j)$. Thus, there exists a strictly increasing sequence of positive numbers $\{t_j^k\}_k$ with $t_j^k \uparrow \infty$, such that $\{u_j(x_j + t_j^k y_j)\}_k$ is increasing. Moreover, we may suppose without loss of generality that

$$t_j^k = \min\{s \geq 0 : u_j(x_j + sy_j) = u_j(x_j + t_j^k y_j)\}.$$

Thus, if $\varepsilon > 0$,

$$u_j(x_j + t_j^k y_j - \varepsilon y_j) < u_j(x_j + t_j^k y_j) \quad (*)$$

for large k. Consider the sequence $\{(x_1 + t_j^k y_1), \ldots, (x_j + t_j^k y_j), \ldots, (x_n + t_j^k y_n)\}_k$. This sequence is contained in $\text{arbseq}^{\omega}(y)$ (take $\lambda^k = (1/t_j^k)$). Given $(*)$, this implies that $y \notin \text{bus}(\omega)$. □

Proof of Part 2. From Part 1 we have,

$$\text{bus}(\omega) \subseteq \text{ea}(\omega) \subseteq \text{arb}(\omega).$$

If the economy satisfies inconsequential arbitrage, then

$$\text{bus}(\omega) = \text{ea}(\omega) = \text{arb}(\omega).$$ □

9.3.2. *Proof of Theorem 2 (inconsequentiality and boundedness of utilities)*

Proof of Part 1. Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]–[**A-2**] and inconsequential arbitrage. We want to show that there exists an integer k_0 such that, for each $u \in U(\omega)$, there exists a rational allocation $x \in C^{k_0}(\omega)$ such that, for each agent j, $u_j \leq u_j(x_j)$. Suppose not. Then, for each $k = 1, 2, \ldots$, there exists $u^k \in U(\omega)$ such that, for each $x \in C^k(\omega)$, $u_j^k > u_j(x_j)$, for some agent j. Consider the sequence $\{u^k\}_k \subset U(\omega)$. Since for each k, $u^k \in U(\omega)$, for each k, there exists $x^k \in A(\omega)$ such that $u_j(x_j^k) \geq u_j^k$ for all j. Now consider the sequence $\{x_j^k\}_k$. If $\{\sum_j \|x_j^k\|\}_k$ is bounded, then $x^k \in C^k(\omega)$ for some k and we have a contradiction.

Suppose then that $\sum_j \|x_j^k\| = r_k \to \infty$. Let

$$H^k(\omega) := \left\{ (x_1, \ldots, x_n) \in A(\omega) : \sum_j ||x_j|| \leq r_k \right\}.$$

Note that $H^k(\omega) \neq \emptyset$, since $x^k = (x_1^k, \ldots, x_n^k) \in H^k(\omega)$. Let

$$B^k(\omega) := \{(x_1, \ldots, x_n) \in H^k(\omega) : u_j(x_j) \geq u_j(x_j^k) \text{ for all } j\}.$$

Note that $B^k(\omega) \neq \emptyset$, since $x^k = (x_1^k, \ldots, x_n^k) \in B^k(\omega)$. $B^k(\omega)$ is compact because $H^k(\omega)$ is compact and the utility functions $u_j(\cdot)$ are upper semicontinuous. For each k, choose $z^k \in B^k(\omega)$ so that

$$\sum_j ||z_j^k|| = \min \left\{ \sum_j ||z_j|| : z = (z_1, \ldots, z_n) \in B^k(\omega) \right\}.$$

We have for all j and for all k

$$u_j(z_j^k) \geq u_j(x_j^k) \geq u_j^k.$$

We claim that $\{\sum_j \|z_j^k\|\}_k$ is bounded. Suppose not. Without loss of generality, suppose

$$\left(\frac{z_1^k}{\sum_j ||z_j^k||}, \ldots, \frac{z_n^k}{\sum_j ||z_j^k||} \right) \to (y_1, \ldots, y_n) \in \text{arb}(\omega).$$

By inconsequential arbitrage, there exists $\varepsilon > 0$, and k_1 such that for all $k \geq k_1$,

$$\text{(a)} \quad (z_1^k - \varepsilon y_1, \ldots, z_n^k - \varepsilon y_n) \in A(\omega),$$

and

$$\text{(b)} \quad u_j(z_j^k - \varepsilon y_j) \geq u_j(z_j^k) \geq u_j(x_j^k) \quad \text{for all } j.$$

Moreover, by Lemma 4.1 (ii) in Allouch (1999),

$$\text{(c)} \quad \sum_j ||z_j^k - \varepsilon y_j|| < \sum_j ||z_j^k|| \leq r_k.$$

Hence

$$(z_1^k - \varepsilon y_1, \ldots, z_n^k - \varepsilon y_n) \in B^k(\omega),$$

and we have a contradiction:

$$\sum_j ||z_j^k - \varepsilon y_j|| < \sum_j ||z_j^k|| = \min\left\{\sum_j ||z_j|| : z = (z_1, \ldots, z_n) \in B^k(\omega)\right\}.$$

Thus, we conclude that $\{\sum_j \|z_j^k\|\}_k$ is bounded. But now this implies that, for some k_2,

$$z^k \in C^k(\omega) \quad \text{for all } k \geq k_2.$$

But recall that we have, for all k and all j, $u_j(z_j^k) \geq u_j^k$. Thus, we have a contradiction. □

Proof of Part 2. Let $C_j^{k_0}(\omega)$ be the projection of $C^{k_0}(\omega)$ onto X_j. Note that because $C^{k_0}(\omega)$ is a compact subset of $X_1 \times \cdots \times X_n$, the set $C_j^{k_0}(\omega)$ is a compact subset of X_j. For each j, let

$$\bar{z}_j \in \arg\max\{u_j(z_j) : z_j \in C_j^{k_0}(\omega)\}.$$

By Part 1 of the Theorem, for all $x \in A(\omega)$, there exists $z \in C^{k_0}(\omega)$ such that

$$u_j(x_j) \leq u_j(z_j) \quad \text{for all } j.$$

We have for each agent j, $u_j(z_j) \leq u_j(\bar{z}_j)$ for all $z \in C^{k_0}(\omega)$, and thus, we have for each agent j, $\bar{z}_j \in C_j^{k_0}(\omega)$ such that

$$u_j(x_j) \leq u_j(\bar{z}_j) \quad \text{for all } x_j \in A_j(\omega).$$ □

9.3.3. Proof of Theorem 3 (inconsequential arbitrage implies existence of a quasi-equilibrium)

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]–[**A-3**] and inconsequential arbitrage.

Part 1. Consider the k-bounded economy $E_k := (X_{kj}, \omega_j, P_{kj}(\cdot))_{j=1}^n$, defined above and note the following:

1. Given assumption [**A-3**], local nonsatiation, it follows from Part 2 of Theorem 2 that, for all k sufficiently large,
   1.1. E_k satisfies *local nonsatiation at rational allocations*; [18]
   1.2. the mapping $x_j \to \hat{P}_{kj}(x_j)$ has nonempty, closed, convex values and the preference relation defined via the mapping $x_j \to \hat{P}_{kj}(x_j)$ is complete and transitive on X_{kj}.
2. For each agent j, $\omega_j \in X_{kj}$ and X_{kj} is convex and compact.

Given these observations, it follows from Theorem 1 in Bergstrom (1976) that, for each k, E_k has a quasi-equilibrium $(x_1^k, \ldots, x_n^k, p^k)$. [19]

[18] Take k large enough so that, for all j, $\bar{z}_j \in \text{int}C(k)$, where the $\bar{z}_j$ are those vectors shown to exist in Part 2 of Theorem 2.

[19] See Gay (1979) for a discussion of Bergstrom's result and the problems caused by the absence of completeness.

Part 2. We now want to show that, if $(x_1^k, \dots, x_n^k, p^k)$ is a quasi-equilibrium for the economy E_k with $x_j^k \in \operatorname{int}C(k)$ for each j, then it is a quasi-equilibrium for the original economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$.

Let $(x_1^k, \dots, x_n^k, p^k)$ be a quasi-equilibrium for the economy $E_k := (X_{kj}, \omega_j, P_{kj}(\cdot))_{j=1}^n$ with $x_j^k \in \operatorname{int}C(k)$ for each j. We have then $p^k \in \mathcal{B}\backslash\{0\}$, and since $A_k(\omega) \subset A(\omega)$, we also have $(x_1^k, \dots, x_n^k) \in A(\omega)$. Finally, we have for all j

$$x_j^k \in B_k(\omega_j, p^k) \quad \text{and} \quad P_{kj}(x_j^k) \cap F_k(\omega_j, p^k) = \emptyset.$$

Thus, $x_j^k \in B_k(\omega_j, p^k)$ for all j. We want to show then that

$$P_{kj}(x_j^k) \cap F_k(\omega_j, p^k) = \emptyset \text{ implies that } P_j(x_j^k) \cap F(\omega_j, p^k) = \emptyset.$$

Suppose not. Then for some agent j, there exists $z' \in P_j(x_j^k) \cap F(\omega_j, p^k)$. Given quasi-concavity (see [**A-1**]), local nonsatiation, [**A-3**], and $x_j^k \in \operatorname{int}C(k)$, there is a $\delta > 0$ such that

$$\delta z' + (1-\delta)x_j^k \in \operatorname{int}C(k)$$

and

$$\delta z' + (1-\delta)x_j^k \in P_j(x_j^k)$$

By the definition of $F(\omega_j, p^k)$, we have

$$\langle \delta z' + (1-\delta)x_j^k, p^k \rangle < \langle \omega_j, p^k \rangle$$

so that

$$\delta z' + (1-\delta)x_j^k \in F(\omega_j, p^k).$$

Thus

$$\delta z' + (1-\delta)x_j^k \in P_{kj}(x_j^k) \cap F_k(\omega_j, p^k),$$

a contradiction. We must conclude, therefore, that

$$P_j(x_j^k) \cap F(\omega_j, p^k) = \emptyset.$$

Thus, $(x_1^k, \dots, x_n^k, p^k)$ is a quasi-equilibrium for the original economy $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$.

Part 3. Suppose that the sequence of quasi-equilibria $\{(x_1^k, \dots, x_n^k, p^k)\}_n$ for the k-bounded augmented economies $\{E_k\}_k$ is such that, for each k, there is an agent j with x_j^k on the boundary of $C(k)$. Thus, we are assuming that the case treated in Part 2 above does not arise. We will now show that this does not matter: inconsequential arbitrage allows us to construct a quasi-equilibrium that fits the case treated in Part 2. So suppose that the sequence $\{(x_1^k, \dots, x_n^k, p^k)\}_k$ of quasi-equilibria is such that, for each k, x_j^k on the boundary of $C(k)$ for some j. This implies that, $\sum_{j=1}^n \|x_j^k\| \to \infty$ as $k \to \infty$. Let $y = (y_1, \dots, y_n)$ be a cluster point of the sequence $\{(\lambda^k x_1^k, \dots, \lambda^k x_n^k)\}_k$ where $\lambda^k = (\sum_{j=1}^n \|x_j^k\|)^{-1}$. Thus,

$y = (y_1, \dots, y_n) \in \mathrm{arb}(\omega)$. By the Lemma, $\sum_{j=1}^{n} y_j = 0$, and $\sum_{j=1}^{n} \|y_j\| = 1$, and for any subsequence $\{(\lambda^{k'} x_1^{k'}, \dots, \lambda^{k'} x_n^{k'})\}_{k'}$ with $(\lambda^{k'} x_1^{k'}, \dots, \lambda^{k'} x_n^{k'}) \to (y_1, \dots, y_n)$, we have for all k' sufficiently large, $x_j^{k'} - \varepsilon y_j \in \mathrm{int}C(k')$ for $\varepsilon \in (0, 1]$. By inconsequential arbitrage, given $(y_1, \dots, y_n) \in \mathrm{arb}(\omega)$, there exists $\varepsilon > 0$ such that, for all k' sufficiently large,

$$x_j^{k'} - \varepsilon y_j \in X_{k'j} \quad \text{and} \quad P_{k'j}(x_j^{k'} - \varepsilon y_j) \subseteq P_{k'j}(x_j^{k'}) \quad \text{for all } j.$$

Our proof will be complete if we can show that $(x_1^{k'} - \varepsilon y_1, \dots, x_n^{k'} - \varepsilon y_n, p^{k'})$ is a quasi-equilibrium for the k-bounded economy $E_{k'}$.

Given that

$$\sum_{j=1}^{n} y_j = 0,$$

the fact that $(x_1^{k'}, \dots, x_n^{k'}, p^{k'})$ is a quasi-equilibrium for $E_{k'}$ implies that
1. $(x_1^{k'} - \varepsilon y_1, \dots, x_n^{k'} - \varepsilon y_n) \in A_{k'}(\omega)$,
2. $p^{k'} \in \mathcal{B} \backslash \{0\}$, and
3. $P_{k'j}(x_j^{k'}) \cap F_{k'}(\omega_j, p^{k'}) = \emptyset$.

Given (3), the fact that $P_{k'j}(x_j^{k'} - \varepsilon y_j) \subseteq P_{k'j}(x_j^{k'})$ implies that $P_{k'j}(x_j^{k'} - \varepsilon y_j) \cap F_k (\omega_j, p^{k'}) = \emptyset$.

To complete the proof, it remains only to show that, for agents $j = 1, 2, \dots, n$, $x_j^{k'} - \varepsilon y_j \in B_{k'}(\omega_j, p^{k'})$. Given that $y = (y_1, \dots, y_n) \in \mathrm{bus}(\omega)$ and that $x_j^{k'} - \varepsilon y_j \in X_{k'j}$, this will be true provided $\langle x_j^{k'} - \varepsilon y_j, p^{k'} \rangle \le \langle \omega_j, p^{k'} \rangle$. We will show that, for agents $j = 1, 2, \dots, n$, $\langle y_j, p^{k'} \rangle = 0$. Since $\sum_{j=1}^{n} y_j = 0$, it suffices to show that, for $j = 1, 2, \dots, n$, $\langle y_j, p^{k'} \rangle \le 0$. Suppose not. Let $\langle y_{j''}, p^{k'} \rangle > 0$ for some agent j''. For this agent,

$$\langle x_{j''}^{k'} - \varepsilon y_{j''}, p^{k'} \rangle < \langle \omega_{j''}, p^{k'} \rangle.$$

By local nonsatiation of the k-bounded augmented economies for k' large, there exists $z_{j''} \in P_{k'j''}(x_{j''}^{k'} - \varepsilon y_{j''})$ sufficiently near $x_{j''}^{k'} - \varepsilon y_{j''}$ such that

$$\langle z_{j''}, p^{k'} \rangle < \langle \omega_{j''}, p^{k'} \rangle.$$

This contradicts the fact that $P_{k'j}(x_j^{k'} - \varepsilon y_j) \cap F_k(\omega_j, p^{k'}) = \emptyset$ for all $j = 1, 2, \dots, n$. We must conclude therefore that, for $j = 1, 2, \dots, n$, $\langle y_j, p^{k'} \rangle = 0$, and thus, we must conclude that, for $j = 1, 2, \dots, n$, $\langle x_j^{k'} - \varepsilon y_j, p^{k'} \rangle \le \langle \omega_j, p^{k'} \rangle$, implying that, for $j = 1, 2, \dots, n$, $x_j^{k'} - \varepsilon y_j \in B_{k'}(\omega_j, p^{k'})$. Thus, $(x_1^{k'} - \varepsilon y_1, \dots, x_n^{k'} - \varepsilon y_n, p^{k'})$ is a quasi-equilibrium for the k-bounded economy $E_{k'}$ such that, for $j = 1, 2, \dots, n$, $x_j^{k'} - \varepsilon y_j \in \mathrm{int}C(k')$, and thus, $(x_1^{k'} - \varepsilon y_1, \dots, x_n^{k'} - \varepsilon y_n, p^{k'})$ is a quasi-equilibrium for the original economy E.

Finally, if for each agent j,

$$\inf_{x \in X_j} \langle x, p^{k'} \rangle < \langle \omega_j, p^{k'} \rangle$$

and

$$P_j(x_j^{k'} - \varepsilon y_j) \text{ is open relative to } X_j,$$

then $(x_1^{k'} - \varepsilon y_1, \ldots, x_n^{k'} - \varepsilon y_n, p^{k'})$ is an equilibrium for the original economy.

9.3.4. Proof of Theorem 4 (equivalence of existence of a Pareto optimal allocation, inconsequential arbitrage, and compactness of utility possibilities)

(1) $\Rightarrow$ (2): let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]–[**A-2**], [**A-4**]–[**A-5**], and let $(\bar{x}_1, \ldots, \bar{x}_n) \in A(\omega)$ be a rational, Pareto optimal allocation. Suppose now that inconsequential arbitrage is *not* satisfied. Thus, for some $y \in \text{arb}(\omega)$, $y \notin \text{bus}(\omega)$. By [**A-4**], nonsatiation off the bus(ω), there is at least one agent j' such that, for some $\lambda_{j'} > 0$, $u_{j'}(\bar{x}_{j'} + \lambda_{j'} y_{j'}) > u_{j'}(\bar{x}_{j'})$. By [**A-5**], we have the uniformity of arbitrage cones across rational allocations:

$$u_j(\bar{x}_j + \lambda_{j'} y_j) \geq u_j(\bar{x}_j) \quad \text{for all } j.$$

But

$$((\bar{x}_1 + \lambda_{j'} y_1), \ldots, (\bar{x}_{j'} + \lambda_{j'} y_{j'}), \ldots, (\bar{x}_n + \lambda_{j'} y_n)) \in A(\omega),$$

contradicting the Pareto optimality of $(\bar{x}_1, \ldots, \bar{x}_n)$.

(2) $\Rightarrow$ (3): this implication follows immediately from Part 1 of Theorem 2.

(3) $\Rightarrow$ (1): obvious.

9.3.5. Proof of Theorem 6 (second welfare theorem)

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**], [**A-2**]*, [**A-3**], [**A-4**]*, [**A-5**], and [**A-6**]. Also, suppose the economy satisfies inconsequential arbitrage. Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in A(\omega)$ be a rational, Pareto optimal allocation with $\bar{x}_j \in \text{int} X_j$ for all j. Consider the economy $(X_j, \bar{x}_j, P_j(\cdot))_{j=1}^n$. This economy satisfies [**A-1**]–[**A-2**]* and must also satisfy inconsequential arbitrage, otherwise, given [**A-4**]* and [**A-5**], we could arrive at a contradiction of the Pareto optimality of $\bar{x}$ via the arguments used in the proof of Theorem 4. Given [**A-1**], [**A-2**]*, [**A-3**], and [**A-6**], $\bar{x}_j \in \text{int} X_j$ for all j implies via Theorem 3 that $(X_j, \bar{x}_j, P_j(\cdot))_{j=1}^n$ has an equilibrium, say $x_1', \ldots, x_n', p'$. For each agent j, $u_j(x_j') \geq u_j(\bar{x}_j)$. In fact, we must have $u_j(x_j') = u_j(\bar{x}_j)$ for all j, otherwise, we would contradict the Pareto optimality of $(\bar{x}_1, \ldots, \bar{x}_n)$. Since $x_1', \ldots, x_n', p'$ is an equilibrium, for any x_j'' such that $u_j(x_j'') > u_j(x_j') = u_j(\bar{x}_j)$, $\langle x_j'', p' \rangle > \langle \bar{x}_j, p' \rangle$. Thus, $(\bar{x}_1, \ldots, \bar{x}_n, p')$ is an equilibrium.

9.3.6. Proof of Theorem 7 (equivalence of Hart and Werner)

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]–[**A-2**] and [**W-1**]–[**W-2**]. As noted in the text, the proof of Theorem 5 follows directly from Corollary 16.2.2 in Rockafellar (1970).

9.3.7. Proof of Theorem 8 (Hart/Werner imply inconsequential)

Let $(X_j, \omega_j, P_j(\cdot))_{j=1}^n$ be an economy satisfying [**A-1**]–[**A-2**] and [**W-1**]–[**W-2**]. The fact that the Hart/Werner conditions imply inconsequential arbitrage is an immediate consequence of the fact that

$$L_1 \times \cdots \times L_n \subseteq \mathrm{bus}(\omega).$$

Under the additional assumption of no half lines, [**W-3**], the fact that inconsequential arbitrage implies the Hart/Werner conditions is an immediate consequence of the fact that with no half lines,

$$L_1 \times \cdots \times L_n = \mathrm{bus}(\omega) = \{0, \ldots, 0\}.$$

Acknowledgements

The authors thank Leo Hurwicz, Manuel Santos, Jan Werner, and an anonymous referee for many valuable comments. We are also grateful to J.-F. Mertens, Michel Le Breton and seminar participants at CORE and Paris I for many helpful discussions and comments. We are especially grateful to Cuong Le Van and Nizar Allouch whose careful reading of the original manuscript led to substantial improvements. Page gratefully acknowledges financial support from the RGC grant program at the University of Alabama. Wooders gratefully acknowledges the support of the Social Sciences and Humanities Research Council of Canada.

References

Allouch, N., 1999. Equilibrium and no market arbitrage. Working Paper. CERMSEM, Universite de Paris I.

Bergstrom, T.C., 1976. How to discard ‘free disposability’ — at no cost. Journal of Mathematical Economics 3, 131–134.

Brown, D., Werner, J., 1995. Arbitrage and existence of equilibrium in infinite asset markets. Review of Economic Studies 62, 101–114.

Chichilnisky, G., 1997. A topological invariant for competitive markets. Journal of Mathematical Economics 28, 445–469.

Dana, R.A., Le Van, C., Magnien, F., 1997. General equilibrium in asset markets with or without short-selling. Journal of Mathematical Analysis and Applications 206, 567–588.

Dana, R.-A., Le Van, C., Magnien, F., 1999. On different notion of arbitrage and existence of equilibrium. Journal of Economic Theory 87, 169–193.

Gale, D., Mas-Colell, A., 1975. An equilibrium existence theorem for a general model without ordered preferences. Journal of Mathematical Economics 2, 9–15.

Gay, A., 1979. A note on Lemma 1 in Bergstrom’s ‘How to Discard Free Disposability’ at no cost. Journal of Mathematical Economics 6, 215–216.

Grandmont, J.M., 1977. Temporary general equilibrium theory. Econometrica 45, 535–572.

Grandmont, J.M., 1982. Temporary General Equilibrium Theory. Handbook of Mathematical Economics, Vol. II. North-Holland, Amsterdam.

Green, J.R., 1973. Temporary general equilibrium in a sequential trading model with spot and futures transactions. Econometrica 41, 1103–1123.

Hammond, P.J., 1983. Overlapping expectations and Hart’s condition for equilibrium in a securities model. Journal of Economic Theory 31, 170–175.

Hart, O.D., 1974. On the existence of equilibrium in a securities model. Journal of Economic Theory 9, 293–311.

Hurwicz, L., 1996. A note on the second fundamental theorem of classical welfare economics. Department of Economics, University of Minnesota (typescript).

Hurwicz, L., Richter, M.K., 2000. The second fundamental theorem of classical welfare economics. Department of Economics, University of Minnesota (typescript).

Monteiro, P.K., Page Jr., F.H., Wooders, M.H., 1998. Increasing cones, recession cones, and global cones. Optimization, in press.

Monteiro, P.K., Page Jr., F.H., Wooders, M.H., 1999. Arbitrage and global cones: another counterexample. Social Choice and Welfare 16, 337–346.

Monteiro, P.K., Page Jr., F.H., Wooders, M.H., 2000. Increasing cones, recession cones, and global cones. Optimization 47, 211–234.

Nielsen, L.T., 1989. Asset market equilibrium with short-selling. Review of Economic Studies 56, 467–474.

Page Jr., F.H., 1987. On equilibrium in Hart's securities exchange model. Journal of Economic Theory 41, 392–404.

Page Jr., F.H., Wooders, M.H., 1993. Arbitrage in markets with unbounded short sales: necessary and sufficient conditions for nonemptiness of the core and existence of equilibrium. Working Paper Number 9409. University of Toronto.

Page Jr., F.H., Wooders, M.H., 1996a. A necessary and sufficient condition for compactness of individually rational and feasible outcomes and existence of an equilibrium. Economics Letters 52, 153–162.

Page Jr., F.H., Wooders, M.H., 1996b. The partnered core and the partnered competitive equilibrium. Economics Letters 52, 152.

Page Jr., F.H., Wooders, M.H., 1999. Arbitrage with price-dependent preferences: equilibrium and market stability. In Topics in Mathematical Economics and Game Theory: Essays in honor of Robert J. Aumann (Myrcna H. Wooders, Ed.). Fields Institute Communications 28, 189–212.

Page Jr., F.H., Wooders, M.H., Monteiro, P.K., 2000. Inconsequential arbitrage. Working Paper 561. Department of Economics, University of Warwick.

Reny, P.J., Wooders, M.H., 1996. The partnered core of a game without side payments. Journal of Economic Theory 70, 298–311.

Rockafellar, R.T., 1970. Convex Analysis. Princeton University Press, Princeton, NJ.

Werner, J., 1987. Arbitrage and existence of competitive equilibrium. Econometrica 55, 1403–1418.

Yu, P.L., 1974. Cone convexity, cone extreme points, and undominated solutions in decision problems with multiobjectives. Journal of Optimization Theory and Applications 14, 319–377.