Increasing cones, recession cones and global cones

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Abstract. In this paper, we discuss and compare various cones used in the economics literature to analyze arbitrage in general equilibrium models with short selling.

1. Introduction

Since Hart (1974), a subject of ongoing interest in economic theory has been conditions ensuring existence of economic equilibrium in models allowing unbounded short sales. When unbounded short sales are allowed, in contrast to Arrow-Debreu-McKenzie general equilibrium models, consumption sets are unbounded below. To illustrate the problem this creates for existence of economic equilibrium, suppose that two agents have diametrically opposed preferences. For example, one agent may want to buy large amounts of one commodity and sell another commodity short while the other agent may prefer to do the opposite. In such a situation, there are unbounded arbitrage opportunities and no equilibrium exists. To ensure existence of equilibrium, arbitrage opportunities must be limited.

A important concept in models with unbounded short sales is the arbitrage cone of an agent. An agent's arbitrage cone is simply the recession cone corresponding to the set of commodity bundles preferred to the agent's endowment. In asset market models[1] the arbitrage cone has played a major role in de…nitions of conditions bounding arbitrage opportunities and ensuring existence of equilibrium. See, for example,

[1] An asset market model is a general equilibrium model with preferences determined by agents' probability beliefs over asset returns and risk-aversion characteristics.

Hart (1974), Milne (1981), Hammond (1983), and Page (1982,1987). More recently, in the context of general equilibrium models, the arbitrage cone has been used for the same purpose (see, for example, Werner (1987), Nielsen (1989), Page and Wooders (1996a,1996b) and Dana, Magnien, and Le Van (1996)).

Besides sufficient conditions, necessary conditions for existence of equilibrium have also been of recurring interest. See, for example, Hart (1974), Milne (1980), Hammond (1983) and Page (1982), who give necessary conditions for existence of equilibrium in asset market models, and see Werner (1987) and Page and Wooders (1993, 1996a,b) who give necessary conditions for existence in general equilibrium models. Page and Wooders (1993) define the strict increasing cone and give necessary and sufficient conditions for existence of equilibrium based on this notion. The strict increasing cone is the set of vectors in the arbitrage cone along which utility increases forever. The increasing cone of Page and Wooders (1996a,1996b) and Monteiro, Page and Wooders (1997a) is the set of directions in the arbitrage cone along which utility eventually increases, thus, thick indifference surfaces are allowed.

As the literature referenced above concludes, it is not the possibility of unbounded trades but rather utility-increasing unbounded trades that presents difficulties for the existence of equilibrium.[2] For example, Werner's (1987) condition ensuring existence of equilibrium can be satisfied even when there exists mutually compatible, arbitrarily large trades, as long as for each agent these trades are in the lineality space corresponding to the set of commodity bundles preferred to agents' endowment - and thus as long as these trades are utility non-increasing. The insights of this literature, and in particular, of Werner (1987), perhaps motivated the definition of global cone in Chichilnisky (1993). The global cone is the set of rays emanating from the agent's endowment along which, not only does utility increase forever, but increases beyond the utility level of any other vector in the agent's consumption set. Since it imposes more requirements than either of the arbitrage cone or the increasing cone, it is immediate that the global cone is contained in the arbitrage cone and also in the increasing cone. What is less clear, is whether or not the global cone is significantly different (say, for example, by more than a closure) from the arbitrage cone and the increasing cone. In this paper, we provide examples showing that the closure of the global cone may be a strict subset of the increasing cone. Thus, except in special circumstances (see Dana, Magnien, and Le Van (1996) for example), conditions based on the global cones may not adequately limit arbitrage opportunities. In fact, Monteiro, Page and Wooders (1997) provide counterexamples to Chichilnisky's claimed results on existence of equilibrium. Nevertheless, the differences between the global cone and the increasing cone shed light on the structure of arbitrage cones and the problem of existence of equilibrium.

[2] This was especially stressed in Page (1989).

In this paper we establish several relationships between the arbitrage cone, increasing cones, and the global cone: with monotonicity of preferences or strict quasi-concavity, the strict increasing cone and the increasing cone are equal. Under an additional assumption, that the arbitrage cones are invariant with respect to their origins (the endowments), the closure of the increasing cone is equivalent to the arbitrage cone. When there are uniform bounds, from above and below, on the norm of the gradients to indifference surfaces, then increasing cones are invariant with respect to their origins. In addition, we provide several examples illustrating that these cones are distinct concepts. In particular, the closure of the global cone may be a proper subset of the increasing cone. We conclude with a discussion of various conditions limiting arbitrage in the literature.

2. Some definitions

Some definitions are required. Let K be a closed convex subset of $\mathbb{R}^L$. We say that $y \in \mathbb{R}^L$ is a direction of recession of K if for any $x \in K$, $x + \lambda y \in K$ for $\lambda \geq 0$. Denote by $R(K)$ the set of all recession directions of K. The set $R(K)$ is called the recession cone of K. Now define $L(K) := -R(K) \cap R(K)$. The set $L(K)$ is called the lineality space of K.

Let S be any subset of $\mathbb{R}^L$. The positive dual cone corresponding to S, denoted by $\alpha^+(K)$, is given by

$$\alpha^+(S) := \{p \in \mathbb{R}^L : \langle y, p\rangle > 0 \text{ for all } y \in S\}.$$

3. The economic model

Let $(X_j; \omega_j; u_j(\cdot))_{j=1}^n$ denote an exchange economy. Each agent j has consumption set $X_j \subset \mathbb{R}^L$ and endowment ω_j. The j^{th} agent's preferences over X_j are specified via a utility function $u_j(\cdot) : X_j \to \mathbb{R}$. It is assumed that for each $j \in N := \{1, \ldots, n\}$, X_j is closed and convex and $\omega_j \in \operatorname{int} X_j$ where "int" denotes "interior". The preferred set of the j^{th} agent at $x \in X_j$ is given by

$$P_j(x) := \{x' \in X_j : u_j(x') > u_j(x)\}$$

and the weak preferred set at $x \in X_j$ is given by

$$\hat{P}_j(x) := \{x' \in X_j : u_j(x') \geq u_j(x)\}.$$

The set of feasible and individually rational allocations is given by

$$F := \{(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n : \sum_{j=1} x_j = \sum_{j=1} \omega_j \text{ and } u_j(x_j) \geq u_j(\omega_j) \text{ for all } j\}.$$

3.1. Arbitrage cones, increasing cones, and global cones. The j^{th} agent's arbitrage cone at endowments $\omega_j \in X_j$ is the closed convex cone containing the origin given by

$$R(\widehat{P}_j(\omega_j)) := \left\{ y \in \mathbb{R}^L : \text{for } x \in \widehat{P}_j(\omega_j) \text{ and } \lambda \geq 0, x + \lambda y \in \widehat{P}_j(\omega_j) \right\} \tag{1}$$

The arbitrage cone $R(\widehat{P}_j(\omega_j))$ is the recession cone corresponding to the j^{th} agent's weak preferred set $\widehat{P}_j(\omega_j)$ at the agent's endowment ω_j; this, and the fact that an arbitrage cone is closed and convex, follows from results in Rockafellar (1970, Section 8).

The strict increasing cone at $x \in X_j$ is defined by [3]

$$I_j^s(x) := \{ y \in R(X_j) : x + \lambda y \in P_j(x + \mu y) \text{ for all } \mu \geq 0 \text{ and } \lambda > \mu \}. \tag{2}$$

The strict increasing cone is the set of rays emanating from $x \in X_j$ along which utility increases forever. In Page and Wooders (1996a,b), the definition of the increasing cone is modified to allow thick indifference surfaces and given by

$$I_j(x) := \{ y \in R(X_j) : \text{ for all } \mu \geq 0, \text{ there exists } \lambda > \mu \text{ such that } x + \lambda y \in P_j(x + \mu y) \}. \tag{3}$$

The point x is called the origin of the increasing cone.[4]

The global cone at endowments $\omega_j \in X_j$ is defined by

$$A_j(\omega_j) := \left\{ y \in \mathbb{R}^L : \forall x \in X_j, \exists \lambda > 0 \text{ such that } (\omega_j + \lambda y) \in P_j(x) \right\}, \tag{4}$$

as stated in the introduction, the global cone is the set of rays emanating from the endowment along which not only does utility increase forever, but it increases beyond the utility level of any other vector in the consumption space.

3.2. Assumptions. We shall maintain the following assumptions throughout: for each agent $j = 1, \ldots, n$,

[3] See also Page (1982), where the increasing cone was introduced to show necessary and sufficient conditions for the existence of equilibrium in an asset market model.

[4] An equivalent definition of the increasing is the following:

$$I_j(x) := \{ y \in R(X_j) \setminus \{0\} : \nexists \max_{\lambda \geq 0} u_j(x + \lambda y) \}.$$

(A-1) $u_j(\cdot)$ is continuous and quasi-concave.[5]

(A-2) X_j is closed and convex.

(A-3) $\omega_j \in \mathrm{int} X_j$ where int denotes interior.

We shall also maintain the following nonsatiation assumption:

(A-4) Global nonsatiation at rational allocations. For any rational allocation $(x_1, \ldots, x_n) \in F$; $P_j(x_j) \neq \emptyset$:

For some of our examples and results, we shall appeal to the following additional assumptions concerning agent utility functions. To spare the reader unhelpful notation, unless more than one agent is involved, we leave off the subscripts denoting the agent. Let $u(\cdot) : X \to R$ be the agent's utility function:

(A-5) $u(\cdot)$ is strictly monotonic. For x, x' in X with $x < x'$, $u(x) < u(x')$.[6]

(A-6) $u(\cdot)$ is strictly quasi-concave. For x, x' in X with $x \neq x'$ and $\lambda \in (0, 1)$; $u(\lambda x + (1 - \lambda)x') > \min\{u(x); u(x')\}$:

(A-7) $u(\cdot)$ satisfies global uniformity of increasing cones, UIC. For $x; x' \in X$;

$$I(x) = I(x') := I:$$

(A-8) $u(\cdot)$ is such that there exists useful trades. For $x \in X$

$$R(\hat{P}(x)) \setminus L((P(x)) \neq \emptyset;$$

(A-9) $u(\cdot)$ satisfies global uniformity of arbitrage cones, UAC. For $x; x' \in X$;

$$R(\hat{P}(x)) = R(\hat{P}(x')) := R:$$

(A-10) $u(\cdot)$ satisfies no half lines, NHL. There does not exist $x \in X$ and $y \in R(X)$, $y \neq 0$, such that $u(x) = u(x + \lambda y)$ for all $\lambda \geq 0$ (i.e., indifference curves contain no half-lines).

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[5] The utility function $u_j(\cdot) : X_j \longrightarrow R$ is quasi-concavity if for $x; x'$ in X_j and $\lambda \in [0; 1]$; $u_j(\lambda x + (1 - \lambda)x') \geq \min\{u_j(x); u_j(x')\}$:

[6] Note that for $x = (x_1; \ldots; x_L)$ and $y = (y_1; \ldots; y_L)$ in R^L, $x < y$ if and only if $x_i \leq y_i$ holds for all i and $x_i < y_i$ for at least one i.

(A-11) $u(\cdot)$ is C^1. The utility function $u(\cdot)$ is continuously differentiable.

(A-12) $u(\cdot)$ satisfies uniform boundedness of norms of gradients, UNG. For $u(\cdot)$ continuously differentiable, there exists positive numbers K and ε such that $x \in X$

$$\varepsilon < \|\nabla u(x)\| < K.$$

(A-13) $u(\cdot)$ satisfies the closed gradients condition, CG. The set of gradient directions along any indifference surface is closed.[7]

Remark 1. For differentiable utility functions, boundedness of norm of the gradients, (A-12), implies

(A-12)* (a), for all $r, s \in \mathbb{R}$ there exists $N(r, s) \in \mathbb{R}$ such that $x \in u^{-1}(r) \Rightarrow$ there exists $z \in u^{-1}(s)$ with $\|x - z\| \le N(r, s)$; and

(A-12)*(b), $N(\cdot, \cdot)$ is bounded on bounded sets.

The proof is as follows:
Consider the differential equation

$$\begin{aligned} z'(w) &= \nabla u(z(w)) \\ z(0) &= x. \end{aligned}$$

It has a solution for $w \ge 0$. Thus if $r = u(x)$ and $s = u(y)$ then since $u(z(w)) - u(x) = \int_0^w \nabla u(z(w)) \cdot z'(w)\,dw = \int_0^w |\nabla u(z(w))|^2 dw \ge \varepsilon^2 w$ we have that if w is such that $u(z(w)) = s$, then defining $y = z(w)$, $s - r \ge \varepsilon^2 w$. From $|y - x| = |z(w) - x| \le |z'(w)|w \le Kw$ we finally obtain $s - r \ge \varepsilon^2 \frac{|y - x|}{K}$ and $N(r, s) = \frac{|s - r|K}{\varepsilon^2}$.

4. Equivalence of the closures of the increasing cone and the arbitrage cone

In this section we first show that under the conditions of continuity and strict monotonicity or strict quasi-concavity, the strict increasing cone is equal to the increasing cone. We then show that under the condition of uniform boundedness of norms of gradients, the closure of the increasing cone is equal to the arbitrage cone.

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[7] Let $L(x) = \{x' \in \mathbb{R}^L : u(x') = u(x)\}$ denote the indifference surface containing $x \in X$, and let $\mathrm{grad}(x) = \{\frac{\nabla u(x')}{\|\nabla u(x')\|} : x' \in L(x)\}$. Assumption (A-13) requires that $\mathrm{grad}(x)$ be a closed set for $x \in X$.

Proposition 1. Let $u(\cdot) : X \to \mathbb{R}$ be a continuous, quasi-concave utility function (i.e., (A-1) holds). If $u(\cdot)$ satisfies strict monotonicity (A-5) and/or strict quasi-concavity (A-6), then

$$I^s(x) = I(x) \text{ for all } x \in X.$$

Proof. First, note that $I^s(x) \subseteq I(x)$. Also, note that if $y \in I(x) \setminus I^s(x)$ then for some $\lambda \geq 0$,

$$u(x + \lambda y) = u(x + (\lambda + \delta)y) \text{ for all } \delta \in [0, \bar{\delta}], \bar{\delta} > 0. \quad (*)$$

We claim that under strict monotonicity and/or strict quasi-concavity (*) implies that

$$u(x + \lambda y) = u(x + (\lambda + \delta)y) \text{ for all } \delta \geq 0, \quad (**)$$

and thus (**) implies that $y \notin I(x)$. Suppose $u(\cdot)$ satisfies strict monotonicity, but (**) does not hold. Then for some $x_{\delta'} := x + (\lambda + \delta')y$, with $\delta' > \bar{\delta}$, $u(x_{\delta'}) > u(x_0)$, where $x_0 = x + \lambda y$. Thus, we have

$$\begin{gathered} u(x_0) = u(x_{\bar{\delta}}) < u(x_{\delta'}), \text{ where} \\ x_0 = x + \lambda y, \ \lambda \geq 0 \\ x_{\bar{\delta}} = x + (\lambda + \bar{\delta})y, \ \bar{\delta} > 0 \\ x_{\delta'} = x + (\lambda + \delta')y, \ \delta' > \bar{\delta}. \end{gathered}$$

By continuity and strict monotonicity, we can find $z_{\delta'} \in X$, such that $z_{\delta'} << x_{\delta'}$, and $u(z_{\delta'}) = u(x_0)$. Note that $u(z_{\delta'}) < u(x_{\delta'})$. Now consider the line segments

$$\begin{gathered} S_x := \{x_v : x_v = (1 - v)x_0 + v x_{\delta'} \text{ for some } v \in [0, 1]\} \\ S_z := \{z_t : z_t = (1 - t)x_0 + t z_{\delta'} \text{ for some } t \in [0, 1]\}. \end{gathered}$$

For $t^* > 0$ sufficiently small, there is a $z_{t^*} \in S_z$ and a corresponding $x_{v^*} \in S_x$ such that

$$\begin{gathered} z_{t^*} << x_{v^*}, \text{ and} \\ x_{v^*} = x + (\lambda + \delta^*)y) \text{ for some } \delta^* \in (0, \bar{\delta}]. \end{gathered}$$

By quasi-concavity, $u(z_{t^*}) \geq u(x_0)$ and by (*), $u(x_{v^*}) = u(x_0)$. But since $z_{t^*} << x_{v^*}$, we have a contradiction of strict monotonicity.

Now suppose $u(\cdot)$ satisfies strict quasi-concavity. By strict quasi-concavity,

$$u(x_0) < u(x_v) \text{ for all } v \in [0, 1].$$

However, for v^* sufficiently small

$$x_{v^*} = x + (\lambda + \delta^*)y) \text{ for some } \delta^* \in (0, \bar{\delta}],$$

and thus by (*), $u(x_{v^*}) = u(x_0)$, a contradiction. ■

Remark 2. Note that if $u(\cdot)$ satisfies strict quasi-concavity, then $u(\cdot)$ satisfies (A-10), no half lines, automatically. Thus, it follows from Theorem 1 in Monteiro, Page, and Wooders (1997) that if $u(\cdot)$ is strictly quasi-concave, then

$$I(x) = R(\widehat{P}(x)) \backslash \{0\}, \text{ for all } x \in X.$$

The assumption (A-9) of global uniformity is required in Werner (1987) to show that his condition limiting arbitrage is sufficient, as well as necessary, for existence of equilibrium. Page and Wooders (1996a,b) require this assumption only to show that their notion limiting arbitrage is necessary for existence of equilibrium.

Proposition 2. Let $u(\cdot) : X \to R$ be a continuous, quasi-concave utility function satisfying strict monotonicity (A-5). If $u(\cdot)$ satisfies global uniformity of arbitrage cones, (A-9), then $\mathrm{cl}\, I(x) = R(\widehat{P}(x))$ for all $x \in X$. Here, "cl" denotes closure.

Proof. Suppose $y \in R(\widehat{P}(x))$. Global uniformity implies that $x' + ty \in \widehat{P}(x')$ for every $t > 0$, $x' \in X$. For any integer $n \geq 1$ define $y^n = y + \frac{1}{n}(1, \dots, 1)$. For any $t > 0$, $u(x + (t+1)y^n) > u(x + ty^n + y) \geq u(x + ty^n)$. Thus $y^n \in I(x)$. Hence $y \in \mathrm{cl}\, I(x)$. ■

Remark 3. Weak monotonicity is not sufficient in the proposition above. Consider for example, $u(\cdot) : R^2 \to R$ given by $u(x_1, x_2) = \max\{0, \min\{x_1, x_2\}\}$. This utility function is continuous, quasi-concave with increasing cone $I(x_1, x_2) = R^2_{++}$ for all $(x_1, x_2) \in R^2$, but $R(\widehat{P}(0)) = R^2$.

If the agent's utility function is concave, then global uniformity of arbitrage, (A-9), holds automatically (see Rockafellar (1970), section 8).

According to our next proposition, if the agent's utility function is only quasi-concave but satisfies uniform boundedness of norms of gradients (A-12), then $u(\cdot)$ will again satisfy global uniformity of arbitrage cones (A-9).

Proposition 3. Let $u(\cdot) : X \to R$ be a continuous, quasi-concave utility function. If $u(\cdot)$ satisfies uniform boundedness of norms of gradients, (A-12) then $u(\cdot)$ also satisfies global uniformity of arbitrage cones, (A-9).

Proof. First, recall that (A-12) implies (A-12)*(a) and (b) (see Remark 1). Now suppose $y \in R(\widehat{P}(\omega))$. Then there exist a sequence $\{x^n\}_n \subseteq \widehat{P}(\omega)$ and $\lambda^n \downarrow 0$ such that $\lambda^n x^n \to y$. If $x \in X$ is given then if $x^n \in \widehat{P}(x)$ for infinitely many n then $y \in R(\widehat{P}(x))$. If for some n', $n \geq n'$ implies that $x^n \notin \widehat{P}(x)$ (i.e., $u(x^n) < u(x)$), define $r = u(\omega)$ and $s = u(x)$. By (A-12)*(a), there is a sequence $\{\hat{x}^n\}_n$ with $u(\hat{x}^n) = u(x)$ such that $|x^n - \hat{x}^n| \leq N(u(x^n), s)$. By (A-12)*(b), $N(\cdot, \cdot)$ is bounded on bounded sets, so that $\lambda^n N(u(x^n); s) \to 0$. Thus $|\lambda^n x^n - \lambda^n \bar{x}^n| \to 0$ implying that $\lambda^n \bar{x}^n \to y$. This in turn implies that $y \in R(\hat{P}(x))$. Since ω and x are chosen arbitrarily, we conclude that the arbitrage cones are all equal. ■

Remark 4. If $u(\cdot)$ is also satisfies strict monotonicity, then by proposition 2, we can conclude that $\mathrm{cl} I(x) = R(\hat{P}(x))$ for all $x \in X$.

5. Examples

5.1. Example 1: The increasing cone is not invariant with respect to its origin. We begin with an example that can be described simply in pictures. In this example the agent's utility function is continuous and quasi-concave and the agent's choice set X is given by R^2. Thus (A-1) and (A-2) hold. However, global uniformity of increasing cones (A-7) fails to hold.

There are two commodities and the agent has Leontief preferences with indifference curves kinked along the curve

$$x_2 = -e^{-x_1}:$$

The Figures 1-4 below summarize the situation:

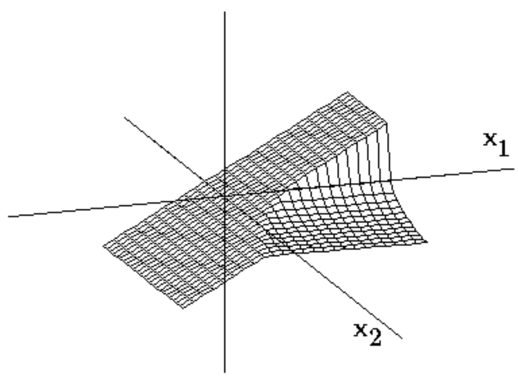

Figure 1
Utility Surface

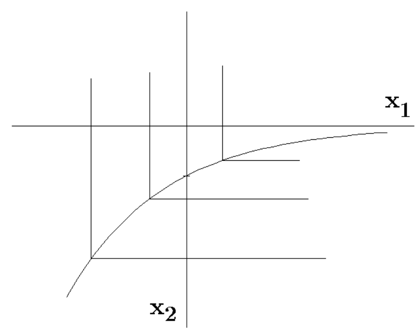

Figure 2
Indifferences Curves

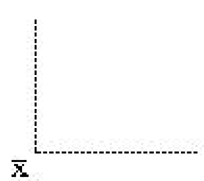

Figure 3
$I(\bar{x}) = \{(y_1; y_2) : y_1 > 0 \text{ and } y_2 > 0\}$
when $\bar{x} = (\bar{x}_1; \bar{x}_2)$ with $\bar{x}_2 < 0$

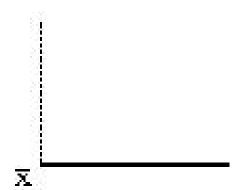

Figure 4
$I(\bar{x}) = \{(y_1; y_2) : y_1 > 0 \text{ and } y_2 \geq 0\}$
when $\bar{x} = (\bar{x}_1; \bar{x}_2)$ with $\bar{x}_2 \geq 0$

It can be easily verified that for this example, the increasing cone $I(\bar{x})$ and the global cone $A(\bar{x})$ are equivalent.

5.2. Example 2. The increasing cone may properly contain the global cone. Let α and β be two functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(X) > \alpha(X) > 0$, $\beta'(X) > 0 > \alpha'(X)$ for every $x \in \mathbb{R}$.[8]

The agent's utility function, $u(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$u(x, y) = \begin{cases} X & \text{if } (X - x)\beta(X) = y \text{ and } y \geq 0 \\ X & \text{if } (X - x)\alpha(X) = y \text{ and } y < 0 \end{cases}$$

The function $u(\cdot)$ is differentiable for $y \neq 0$, with indifference curves given by two half lines intersecting at $(X, 0)$ where X is the utility level. For a fixed X the half-lines are

$$\{(x, y) \in \mathbb{R}^2; y \geq 0; y = (X - x)\beta(X)\},$$
$$\{(x, y) \in \mathbb{R}^2; y < 0; y = (X - x)\alpha(X)\}.$$

Hence the utility function satisfies the closed gradient condition, (A-13). Moreover, the utility function is quasi-concave and strictly monotonic.

First, we show that in this example, as the above, the increasing cones are not globally uniform (i.e., (A-7) does not hold).

Proposition 4. *The increasing cone $I(\omega)$ is not invariant with respect to changes in endowment ω.*

Proof. It suffices to consider $\omega = (a, 0)$. Recall that

$$I(\omega) = \{v \in \mathbb{R}^2 \setminus \{0\}; \nexists \max_{r \geq 0} u(\omega + rv)\}.$$

Suppose first $v = (v_1, v_2)$, $v_1 \geq 0$. If $v_2 \leq 0$, $u((a, 0) + r(v_1, v_2)) \leq u(a, 0) = a$ and there is a maximum. If $v_2 > 0$, $u((a, 0) + r(v_1, v_2)) = u(a + rv_1, rv_2) = X$ where $rv_2 = (X - a - rv_1)\beta(X)$. Hence $r = \frac{(X-a)\beta(X)}{v_2 + v_1\beta(X)}$. If $v_1 = 0$ then $X \to \infty$ if $r \to \infty$. Now suppose $v_1 < 0$. If $\beta(a) \geq \frac{-v_2}{v_1}$, then for every $X > a$, $\frac{(X-a)\beta(X)}{v_2 + v_1\beta(X)} < 0$. Hence there is a maximum whenever $\beta(a) \geq \frac{-v_2}{v_1}$. Finally suppose $\beta(a) < \frac{-v_2}{v_1}$. Then for every $X > a$, such that $\beta(X) < \frac{-v_2}{v_1}$ the function $\frac{(X-a)\beta(X)}{v_2 + v_1\beta(X)}$ is increasing and positive. If X_0 is such that $\beta(X_0) = \frac{-v_2}{v_1}$ then since $u > a$, $\lim_{X \to X_0} \frac{(X-a)\beta(X)}{v_2 + v_1\beta(X)} = \infty$. And if $\beta(X) < \frac{-v_2}{v_1}$ for every X then $\lim_{X \to \infty} \frac{(X-a)\beta(X)}{v_2 + v_1\beta(X)} = \infty$. Thus

$$I(\omega) = \{v \in \mathbb{R}^2; v_1 < 0; \beta(a) < -\frac{v_2}{v_1}\} \cup \{v \in \mathbb{R}^2; v_1 \geq 0; -v_2 < v_1\alpha(a)\}. \quad (5)$$

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[8] For example take $\alpha(X) = -\arctan(X) + \pi/2$ and $\beta(X) = \arctan(X) + 3\pi/2$. Thus: $\alpha'(X) = -\frac{1}{1+X^2} < 0 < \frac{1}{1+X^2} = \beta'(X)$ and $\beta(X) > -\pi/2 + 3\pi/2 = \pi > -\arctan(X) + \pi/2 > 0$.

■

Our next Proposition shows that for this example, even though the closed gradient condition, (A-13), is satisfied, the increasing cone may properly contain the closure of the global cone. Thus, the increasing cone and the global cone can differ significantly, depending on the assumptions of the economic model.

Proposition 5. The increasing cone $I(\omega)$ may properly contain the closure of the global cone, $\overline{A(\omega)}$.

Proof. It suffices to demonstrate this for $\omega = 0$. Also, to simplify the proof we suppose $\alpha(\infty) = 0$. Suppose $v \in \mathbb{R}^2 \setminus \{0\}$. If $v_2 = 0$ and $r > 0$, we have $u(rv_1, 0) = rv_1 \to \infty$ if $v_1 > 0$. If $v_1 = 0$ and $v_2 > 0$ then $u(0, rv_2) = X$ where $X\beta(X) = rv_2$. Hence if $r \to \infty$, $X \to \infty$ and therefore $v \in A_u(0)$. Finally suppose $v_1 > 0 > v_2$. Then $u(rv) = X$, where $(X - rv_1)\alpha(X) = rv_2$. So we have that $\frac{X\alpha(X)}{v_1\alpha(X)+v_2} = r$. Now since the denominator is negative if X is large we have that $u(rv)$ is bounded. Hence we have that $A(0) = \mathbb{R}^2_+ \setminus \{0\}$. By comparison of this expression with 5 it is clear that $I(0) \supsetneq \overline{A(0)}$. ■

To make the example more concrete, let

$$\beta(X) = \arctan(X) + \frac{\pi}{2}$$
$$\alpha(X) = -\arctan(X) + \frac{3\pi}{2}.$$

With these specifications for $\beta(\cdot)$ and $\alpha(\cdot)$, the indifference curve corresponding to any utility level X is given by

$$y = \begin{cases} (X - x)\cdot\left(\arctan(X) + \frac{3\pi}{2}\right) & \text{if } x \le X \\ (X - x)\cdot\left(-\arctan(X) + \frac{\pi}{2}\right) & \text{if } x > X. \end{cases}$$

Figure 5 below depicts the indifference curves corresponding to utility levels $X = 1$,

$X = 2$, $X = 3$, and $X = 4$.

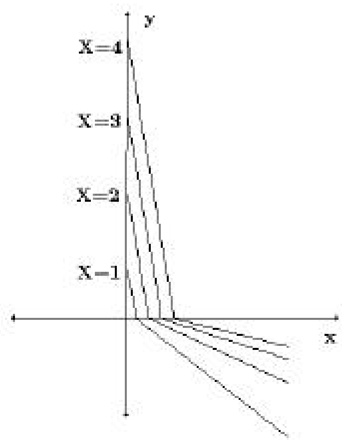

Figure 5

5.3. Example 3. A differentiable example in the spirit of example 2 above. Define $b : \mathbb{R} \to \mathbb{R}$, by $b(a) = a + \beta(a) - \alpha(a)$. Define $H : \mathbb{R}^2 \to \mathbb{R}$ by

$$H(x;a) = \begin{cases} \frac{\beta^2(a)-\alpha^2(a)}{2} - \beta(a)(x-a) & \text{if } x \geq a \\ \frac{\beta^2(a)-\alpha^2(a)}{2} - \beta(a)(x-a) + \frac{(x-a)^2}{2} & \text{if } a \leq x \leq b(a) \\ -\alpha(a)(x-b(a)) & \text{if } b(a) \leq x. \end{cases}$$

First note that H is a continuous function. For every x; $H(x;\cdot)$ is strictly increasing and onto $\mathbb{R}$. We finally define the utility function, $u : \mathbb{R}^2 \to \mathbb{R}$ implicitly by $H(x;u(x;y)) = y$. To prove that u is continuously differentiable it will be enough, using the implicit function theorem to check that H is continuously differentiable. This we now proceed to check. It suffices to show that H has continuous partial derivative $\frac{\partial H}{\partial x}(x;a)$ and $\frac{\partial H}{\partial a}(x;a)$ for $(x;a) \in \mathbb{R}^2$: Thus

$$\frac{\partial H}{\partial x}(x;a) = \begin{cases} -\beta(a) & \text{if } x \leq a \\ -\beta(a) + x - a & \text{if } b(a) \geq x \geq a \\ -\alpha(a)(x-b(a)) & \text{if } b(a) \leq x. \end{cases}$$

and

$$\frac{\partial H}{\partial a}(x;a) =$$

$$\begin{cases} \beta(a)\beta'(a) - \alpha(a)\alpha'(a) - \beta'(a)(x-a) + \beta(a) & \text{if } x \leq a \\ \beta(a)\beta'(a) - \alpha(a)\alpha'(a) - \beta'(a)(x-a) + \beta(a) - (x-a) & \text{if } b(a) \geq x \geq a \\ -\alpha'(a)(x-b(a)) + \alpha(a)b'(a) & \text{if } b(a) \leq x. \end{cases}$$

are continuous and hence H is continuously differentiable. Since

$$\frac{\partial H}{\partial x}(x;a) < 0 < \frac{\partial H}{\partial a}(x;a)$$

it follows that

$$\frac{\partial u}{\partial x} > 0 \text{ and } \frac{\partial u}{\partial y} > 0.$$

6. Cone conditions, existence of equilibrium, and compactness of the utility possibilities.

Given prices $p \in B := \{p' \in \mathbb{R}^L : \|p'\| \leq 1\}$ the budget set for the jth agent is given by $B(\omega_j;p) = \{x \in X_j : \langle x;p\rangle \leq \langle \omega_j;p\rangle\}$. An equilibrium for the economy $(X_j;\omega_j;u_j(\cdot))_{j=1}^n$ is an $(n+1)$-tuple of vectors $(\bar{x}_1;\ldots;\bar{x}_n;\bar{p})$ such that:

(i) $(\bar{x}_1;\ldots;\bar{x}_n) \in A(\omega)$;

(ii) $\bar{p} \in B\setminus\{0\}$; and (iii) for each j; $\langle \bar{x}_j;\bar{p}\rangle = \langle \omega_j;\bar{p}\rangle$ and (iii)$P_j(\bar{x}_j) \cap B(\omega_j;\bar{p}) \neq \emptyset$.[9]

An economy $(X_j;\omega_j;u_j(\cdot))_{j=1}^n$ satisfies Werner's condition if

$$\bigcap_j \alpha^+\left(R(\widehat{P}_j(\omega_j))\setminus L(\widehat{P}_j(\omega_j))\right) \neq \emptyset.^{10} \tag{6}$$

Werner (1987) shows that his condition is sufficient for existence of equilibrium. Werner obtains his result under very mild assumptions. In particular, Werner does not require any nonsatiation condition except that there exist useful trades (i.e., (A-8)). Werner also notes that under the no half lines assumption, (A-10), his condition is necessary and sufficient.

An economy $(X_j;\omega_j;u_j(\cdot))_{j=1}^n$ satisfies the Page and Wooders condition of no unbounded arbitrage if:

$$\begin{array}{l}\text{whenever } \sum_{j=1}^n y_j = 0 \text{ and } y_j \in R(\widehat{P}_j(\omega_j)) \text{ for all agents } j,\\ \text{it holds that } y_j = 0 \text{ for all agents } j.\end{array} \tag{7}$$

---

[9] Recall that

$$P_j(x) := \{x' \in X_j : u_j(x') > u_j(x)\}.$$

[10] Recall that

$$\alpha^+(S) := \{p \in \mathbb{R}^L : \langle y;p\rangle > 0 \text{ for all } y \in S\}.$$

Page and Wooders (1993,1996a,b) demonstrate that no unbounded arbitrage is sufficient for existence of equilibrium, and that the condition is necessary and sufficient, provided at most one agent has half-lines in his indifference surfaces. Under the same condition, Page and Wooders (1993,1996a,b) show that no unbounded arbitrage is necessary and sufficient for nonemptiness of the core and, in fact, necessary and sufficient for nonemptiness of the partnered core and for the existence of a single Pareto optimal point.

The condition of Werner (1987) was modified by Chichilnisky (1993), who replaced the arbitrage cone minus the lineality space by her global cone to obtain the condition:

$$\cap_j \ast^{+}(A_j(\omega_j)) \neq \emptyset. \tag{8}$$

A counterexample in Monteiro, Page and Wooders (1997) shows that this condition is inadequate for existence of equilibrium. The reason why it is inadequate is suggested by the prior results and by the examples presented in this paper. The global cone $A_j(\omega_j)$ may be too small to limit all unbounded, utility-increasing arbitrages. In more recent papers, Chichilnisky again modifies the intersection condition of Werner (1987), but based on the increasing cone rather than the global cone. Specifically, the condition is:

$$\cap_j \ast^{+}(I_j(\omega_j)) \neq \emptyset. \tag{9}$$

Since, in the no-half-lines case, this condition is dual to the condition of Page and Wooders (1993,1996a,b), it follows from their results that an equilibrium exists. See Monteiro, Page and Wooders (1999) for references, further discussion, and a counterexample to a proposition on which Chichilnisky bases her claimed existence result using the above condition.

Since the increasing cone $I_j(\omega_j)$ and the global cone $A_j(\omega_j)$ can differ depending on the economic model (see examples 2 and 3 above), the conditions 8 and 9 can differ. As remarked above, the failure of Chichilnisky's condition is due to the fact that the global cone may be too small. This is illustrated by the following example, taken from Monteiro, Page and Wooders (1997). For this example, the global cone of agent 1 $A_1(\omega_1)$ is depicted in Figure 6 and the increasing cone $I_1(\omega)$ is depicted in

Figure 7.

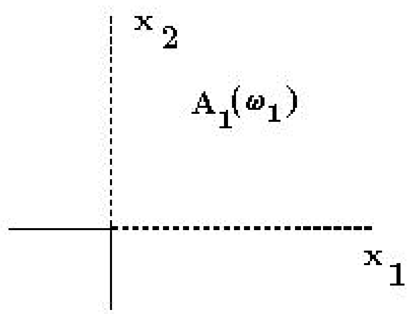

Figure 6

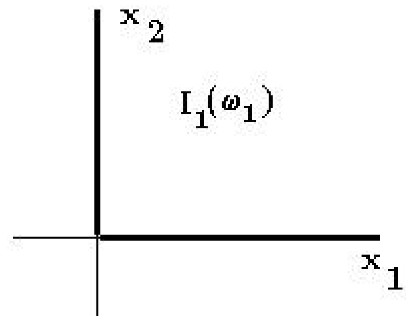

Figure 7

Here the global cone $A_1(\omega_1)$ is given by the positive orthant, while the increasing cone $I_1(\omega_1)$ is given by the non-negative orthant minus the origin. A specific utility function with global cone and increasing cone depicted in Figures 5 and 6 is given by

$$u_1(x_1; x_2) = x_1 + x_2 + 2 - \sqrt{(x_1 - x_2)^2 + 4}.$$

We refer the reader to Monteiro, Page and Wooders (1997) for details.

6.1. Noncompactness of the utility possibilities set. Another question that arises in the literature on arbitrage is whether the set of utility possibilities is closed (see Dana, Le Van and Magnien (1996) for informative results and references). The following example shows that the condition $\cap_j \ast^{+}(A_j(\omega_j)) \neq \emptyset$ does not imply compactness of the set of utility possibility set.

Example Consider the economy $(X_j; \omega_j; u_j(\cdot))_{j=1}^2$; with $X_j = \mathbb{R}^2$ and $\omega_i = 0$ for $i = 1; 2$: Agent one's preference are given by the utility function

$$u_1(x; y) = \begin{cases} x_1 & \text{if } x_1 < 0 \text{ or } x_2 \geq -1 \\ -\frac{x_1}{x_2} & \text{if } x_1 \geq 0 \text{ and } x_2 < -1; \end{cases}$$

while agent two has utility function given by

$$u_2(x_1; x_2) = x_1 + 2x_2.$$

Figure 8 below summarizes the model.

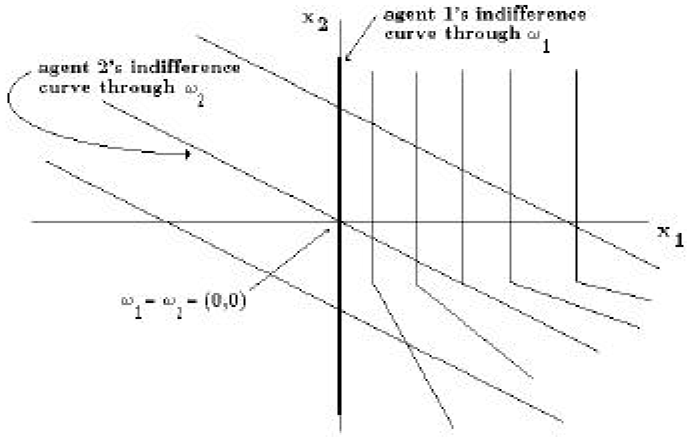

Figure 8

The increasing cones are

$$I_1(0) = \{v \in R^2; v_1 > 0; v_2 \geq 0\}$$
$$I_2(0) = \{v \in R^2; v_1 + 2v_2 > 0\}.$$

Note that

$$(1; 2) \in \alpha^+(I_1(0)) \cap \alpha^+(I_2(0)).$$

Thus, the condition $\cap_j \alpha^+(I_j(\omega_j)) \neq \emptyset$ holds. The set of utility possibilities, however, is not compact. To see this, consider the utility allocation $(u_1(-X_n); u_2(X_n))$ where $X_n = (-n; n)$: Then $u_2(X_n) = -n + 2n = n \to \infty$ and $u_1(-X_n) = -\frac{n}{-n} = 1 > 0$: Even though the set of utility possibilities is not compact, this economy has an equilibrium with price $p = (1; 2)$ and allocation $X_1 = (2; -1)$, $X_2 = (-2; 1)$.

Remark 5. The utility function u_1 also does not have an increasing cone satisfying global uniformity. For example $I_1((1/2; -1/2) = \{v \in R^2; v_1 > 0; v_2 > -v_1\}$.

References

[1] Chichilnisky, G. (1993) "Intersecting families of sets and the topology of cones," Bulletin of the American Mathematical Society 29, 189-207.

[2] Dana, R.A., C. Le Van and F. Magnien (1996) "On different notions of arbitrage and existence of equilibrium," CEPREMAP Working Paper 1996, revised 1997.

[3] Hammond, P. (1983) "Overlapping expectations and Hart's conditions for equilibrium in a securities models," Journal of Economic Theory 31, 170-175.

[4] Hart, O.D. (1974) "On the existence of equilibrium in a securities model," Journal of Economic Theory 9, 293-311.

[5] Milne, F. (1981) "Short-selling, default risk and the existence of equilibrium in a securities model," International Economic Review 21, 255-267.

[6] Monteiro, P.K., F.H. Page Jr., and M.H. Wooders (1999) "Arbitrage and global cones; Another counterexample," Social Choice and Welfare (in press).

[7] Monteiro, P., F.H. Page Jr., and M.H. Wooders (1997) "Arbitrage, equilibrium, and gains from trade; a counterexample," Journal of Mathematical Economics.

[8] Nielsen, L.T. (1989) "Asset market equilibrium with short-selling," Review of Economic Studies 56, 467-474.

[9] Page, F.H. Jr. (1982) "Information, arbitrage, and equilibrium," Department of Finance, The University of Texas at Austin, Working paper 81/82-2-51.

[10] Page, F.H. Jr. (1987) "On equilibrium in Hart's securities exchange model," Journal of Economic Theory 41, 392-404.

[11] Page, F.H. Jr. (1989) "Utility arbitrage and asset prices," Graduate School of Business, Indiana University Working Paper No. 401.

[12] Page, F.H. Jr. (1996) "Arbitrage and asset prices," Mathematical Social Sciences (forthcoming).

[13] Page, F.H. Jr. and M.H. Wooders (1993) "Arbitrage in markets with unbounded short sales: necessary and su¢cient conditions for nonemptiness of the core and existence of equilibrium," University of Toronto Working Paper March 1994 Number 9409.University of Alabama Department of Finance Working Paper No 247.

[14] Page, F.H. Jr. and M.H. Wooders (1996) "A necessary and su¢cient condition for compactness of individually rational and feasible outcomes and existence of an equilibrium" Economics Letters 52 (1996) 153-162.

[15] Page, F.H. Jr. and M.H. Wooders (1996) "The partnered core and the partnered competitive equilibrium," Economics Letters 52 (1996) 143-152.

[16] Rockafellar, R.T. (1970) Convex Analysis, Princeton University Press.

[17] Werner, J. (1987) "Arbitrage and existence of competitive equilibrium," Econometrica 55, 1403-1418.