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Inessentiality of large groups and the approximate core property: an equivalence theorem*

Myrna Holtz Wooders**

Department of Economics, University of Toronto, 150 St. George Street, Toronto, Ontario, M5S 1A1 Canada

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Summary. Inessentiality of large groups or, in other words, effectiveness of small groups, means that almost all gains to group formation can be realized by partitions of the players into groups bounded in absolute size. The approximate core property is that all sufficiently large games have nonempty approximate cores. I consider these properties in a framework of games in characteristic function form satisfying a mild boundedness condition where, when the games have many players, most players have many substitutes. I show that large (finite) games satisfy inessentially of large groups *if and only if* they satisfy the approximate core property.

1. Large games and economies with inessentiality of large groups

In diverse economic situations small groups of players can realize nearly all gains to group formation. While arbitrarily large groups may form, they can realize little more, per-capita, than small groups. To model such situations I introduce a framework of games in characteristic function form. This framework has the property that the opportunities open to a group of players depend continuously on the "types" or "attributes" of the members of the group and these opportunities are defined independently of the society in which the group is embedded. Also, if there are many players in a game, then most players have many substitutes, a

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consequence of compactness and continuity assumptions. I define two properties: Inessentiality of large groups means that almost all gains to group formation can be realized by partitions of the players into groups bounded in absolute size of membership. The approximate core property is that all sufficiently large games have nonempty approximate cores. I show that large games satisfy the condition of inessentiality of large groups *if and only if* they have the approximate core property. This provides an asymptotic version of the Bondareva (1963) and Shapley (1967) result that games have nonempty cores if and only if they are balanced, with inessentiality of large groups replacing the balancedness condition.

The modelling assumptions underlying our framework are:

(1) *Self-sufficiency:* The opportunities available to a group of players depend only on the group itself, and not on the society in which the group is embedded;
(2) *Superadditivity*: The opportunities realizable by two disjoint groups of players are realizable by their union;
(3) *Substitution*: In games with many players most players have many substitutes:
(4) *Boundedness of balanced payoffs:* There is a uniform bound on core payoffs of balanced covers of games.

The current model and results are restricted to economies with quasi-linear utilities and to games with sidepayments (TU games and economies).[1] However, our modelling assumptions are valid in diverse economic environments. For example, they characterize economies with private goods, including ones with finite or infinite dimensional commodity spaces, nonconvexities, indivisibilities, and nonmonotonicities, as well as economies with local public goods and/or coalition production.

Within our framework I relate the following 2 properties:

(A) *Inessentiality of large groups*: Almost all gains to group formation can be realized by coalitions bounded in absolute size; and
(B) *The approximate core property*: All games with "enough" players, in absolute numbers, have nonempty approximate cores.

My result is:

Theorem. (A) *is necessary and sufficient for* (B).[2]

The result has implications for the theory of perfect competition. Instead of an equivalence for outcomes for solution concepts, as in Debreu–Scarf (1963) or Aumann (1964, 1975) for example, and instead of game-theoretic equivalence of the structures of games and markets as in Shapley–Shubik (1969) and Wooders (1988),

[1] Partial analogues for games without sidepayments (NTU games) have been obtained in Wooders (1990b); these are significantly more subtle and difficult in interpretation than those for TU games.

[2] Assumptions (1)–(4) together are typically not valid in models of economies with pure public goods and in voting games; in these models (A) is also not satisfied and we can easily find examples where the approximate core property does not hold. Condition (4) is technical and appears mild. It is milder, for example, than the property that small subsets of players do not have significant effects on per-capita payoffs of large populations.

I show equivalence of properties of large games. These equivalence results together raise the question of what is a minimal requirement to impose on games to ensure that a large number of participants leads to competitive payoff/price taking behaviour i.e., that the games are game-theoretically equivalent to large exchange economies. An essential requirement is the nonemptiness of approximate cores of games, analogous to the existence of approximate equilibrium of exchange economies. Taking this requirement as a minimal standard, inessentiality of large groups is a necessary and sufficient condition for large games to be analogous to large economies.

The importance of the absence of effective collective action by large groups for competition was emphasized by von Neumann–Morgenstern (1944), and their discussion warrants repetition:

> It is neither certain nor probable that a mere increase in the number of participants will lead *in fine* to the conditions of perfect competition. The classical definitions of free competition all involve further postulates besides the greatness of that number. E.g., it is clear that if certain great groups of pacticipants – for any reason whatsoever – act together, then the great number of participants may not become effective; the decisive exchanges may take place directly between large 'coalitions,' few in number, and not between individuals, many in number, acting independently. Any satisfactory theory will have to explain under what circumstances such big 'coalitions' will or will not be formed – i.e., when the large number of participants will become effective and lead to more or less free competition.

My results emphasize the importance of the smallness of essential group sizes. As von-Neumann–Morgenstern suggest, when large groups are essential, i.e., when significant increased payoff can be achieved by forming very large, rather than small coalitions, then we cannot expect perfect competition. Von Neumann–Morgenstern viewed the "role and size of 'coalitions' as decisive." I too view the role of "coalitions" – "groups" in our terminology – and possible behaviours of individuals within groups and in group formation as fundamental.

I consider the literature further in the concluding section of this paper. Also, I discuss the relationship of this work to previous research and relate the inessentiality of large groups condition to the boundedness conditions used previously.

2. Games and pregames

2.1. Games

A *game* (in characteristic function form, with sidepayments) is a pair (N, v) where N is a finite set (the set of *players*) and v is a function, the *characteristic function (of the game)*, from the set 2^N of subsets of N to the set R_+ of nonnegative real numbers, with the property that $v(\phi) = 0$. Nonempty subsets S of N are called

[3] deleted in proof.

groups (or *coalitions*) and the number $v(S)$ is the *worth* of the group S. If the player set N is understood, we frequently refer to v itself as the game. The game (or v) is *superadditive* if for all disjoint subsets S and S' of N we have

$$v(S \cup S') \geqq v(S) + v(S').$$

A *payoff* for the game (N, v) is a vector x in R^N_+. The payoff is *feasible* if there is a partition of N into (disjoint) coalitions, say $\{S_1, \ldots, S_K\}$, such that

$$x(N) \leqq \sum_{k=1}^{K} v(S_k)$$

where $x(S) = \sum_{i \in S} x_i$, for any $S \subset N$.

For $\varepsilon \geqq 0$, the payoff x is the *ε-core* of (N, v) if

$$\begin{array}{ll} x \text{ is feasible} & \text{and} \\ x(S) \geqq v(S) - \varepsilon|S| & \text{for all coalitions } S \end{array}$$

where $|S|$ denotes the number of elements in the set S. When $\varepsilon = 0$ the ε-core is simply the *core*. The ε-core consists of those feasible payoffs with the property that no group of players could be better off by ε per-capita.

2.2. *Pregames*

I introduce the framework of a pregame to formalize the notion of a large game for which the worth of a group of players depends only on the attributes of its members, and players with similar attributes are approximately substitutes.

Let Ω be a compact metric space, interpreted as a set of player "types" or attributes. A *profile* on Ω, interpreted as a description of a group of players in terms of the types of the members of the group, is a function f from Ω to the set Z_+ of nonnegative integers for which the support of f,

$$\text{support}(f) = \{\omega \in \Omega : f(\omega) \neq 0\}$$

is finite. A profile is simply a list of elements of Ω with each element ω appearing $f(\omega)$ times. For each $\omega \in \Omega$, we interpret $f(\omega)$ as the number of players of type ω or, in other words, with attributes ω, in the group described by f. The set of profiles on Ω is denoted by $P(\Omega)$. We write 0 for the profile which is identically zero, $f \leqq g$ if $f(\omega) \leqq g(\omega)$ for each ω in Ω and for ω_0 in Ω we write χ_{ω_0} for the profile given by

$$\chi_{\omega_0}(\omega) = \begin{bmatrix} 0 & \text{if} & \omega \neq \omega_0, \\ 1 & \text{if} & \omega = \omega_0. \end{bmatrix}$$

By the *norm* of a profile, we mean

$$\|f\| = \sum_{\omega \in \text{support}(f)} f(\omega),$$

which is simply the number of players in a group represented by f. This is a finite sum since f has finite support.

A *pregame* is a pair (Ω, Ψ) where Ω is a compact metric space, called the *space of attributes* and $\Psi : P(\Omega) \to R_+$, called the *characteristic function* (of the pregame), is a function with the following properties:

(2.1) $\Psi(0)=0$,
(2.2) for every profile f in $P(\Omega)$, for any $\varepsilon>0$ there is a $\delta>0$ such that for all ω_1, ω_2 in Ω with dist $(\omega_1, \omega_2)<\delta$, we have $|\Psi(f+\chi_{\omega_1})-\Psi(f+\chi_{\omega_2})|<\varepsilon$ (continuity),
(2.3) $\Psi(f)+\Psi(g)\leqq\Psi(f+g)$ for all profiles f and g (superadditivity), and an additional property
(2.4) to be defined below.

The first condition means that a total of zero players can realize nothing. The second is that players with similar attributes are nearly substitutes. This condition and the compactness of Ω ensure the substitution property discussed in the introduction. The third condition expresses the idea that an option open to a group is to split into several smaller groups.

We frequently refer to the elements of Ω as "types." Players of the same type are substitutes.

To derive a game from a pregame (Ω, Ψ), we specify a finite set N and a function $\alpha: N\to\Omega$, called an *attribute function*. With any subset S of N we can then associate a profile, $\mathrm{prof}(\alpha|S)$, given by

$$\mathrm{prof}(\alpha|S)(\omega)=|\alpha^{-1}(\omega)\cap S|.$$

Now we have a *derived game* (N, v_α) where

$$v_\alpha(S)=\Psi(\mathrm{prof}(\alpha|S))$$

for each $S\subset N$.

"Equal-treatment" payoffs to the players in a derived game will be used frequently. Let (Ω, Ψ) be a pregame and let (N, v_α) be a derived game. A payoff $x\in R^N_+$ has the *equal-treatment property* if $x_i=x_j$ whenever $\alpha(i)=\alpha(j)$, i.e., identical players are treated identically. Let $\{\omega_1,\dots,\omega_T\}=\alpha(N)$. A vector $\bar{x}\in R^T$ *represents* a feasible payoff with the equal-treatment property if x, defined by

$$x_i=\bar{x}_t \quad \text{if} \quad \alpha(i)=\omega_t$$

for each $i\in N$ is a feasible payoff for the game.

When $\Sigma f^k=f$ for some collection of profiles $f, f^1,\dots,f^K$, not necessary distinct, we say that the collection is a *partition of* f and each member of the collection is called a *subprofile* of f. Obviously, a partition of a profile is related to a partition of a set of players. If (N, v_α) is a game derived from (Ω, Ψ), and $\{S_1,\dots,S_K\}$ is a partition of N, then $\{f^k:\mathrm{prof}(\alpha|S_k)=f^k,\ k=1,\dots,K\}$ is a partition of $\mathrm{prof}(\alpha|N)$.

To define the boundedness condition used in this paper, I must first define the balanced cover of the characteristic function Ψ of a pregame (Ω, Ψ). Let h be a profile and let β be a collection of subprofiles of h, i.e., $s\in\beta$ implies s is a profile and $s\leqq h$. The collection is a *balanced collection of subprofiles of* h if there are positive real numbers, γ_s for $s\in\beta$ such that $\sum_{s\in\beta}\gamma_s s=h$. For each profile h, define

$$\bar{\Psi}(h)=\max_{\beta}\sum_{s\in\beta}\gamma_s\Psi(s)$$

where the maximum is taken over all balanced collections β of subprofiles of h

with weights γ_s for $s \in \beta$. The pair $(\Omega, \bar{\Psi})$ is called the *balanced cover pregame* of (Ω, Ψ). It follows immediately from the Bondareva (1963) and Shapley (1967) result that any game determined by $(\Omega, \bar{\Psi})$ has a nonempty core. Since the core of a game in characteristic form is convex, any game determined by $(\Omega, \bar{\Psi})$ has a payoff in its core with the equal treatment property, called a *balanced payoff*. Also, it can be shown that, given $\varepsilon > 0$, if a game (N, v_α) derived from (Ω, Ψ) and with $\mathrm{prof}(\alpha|N) = h$ has a nonempty ε-core, then $\bar{\Psi}(h) - \varepsilon \|h\| \leq \Psi(h)$. It is required that a pregame (Ω, Ψ) satisfies *boundedness of balanced payoffs*,

(2.4) there is a constant A such that for all profiles f, for all balanced payoffs x we have $x_t < A$ for all ω_t in support (f).

3. Equivalence

After introducing the required definitions, I state my result.

A pregame (Ω, Ψ) satisfies *uniform inessentiality of large groups* if, for any $\varepsilon > 0$ there is an $n_1(\varepsilon)$ such that for any profile f there is some integer K and profiles $f^1, \ldots, f^K$ with the properties that

(3.1) $\|f^k\| \leq n_1(\varepsilon)$ for each $k = 1, \ldots, K$,

(3.2) $\sum_{k=1}^{K} f^k = f$, and

(3.3) $\Psi(f) - \sum_k \Psi(f^k) < \varepsilon \|f\|$.

This property means that given the measure of per-capita approximation (the ε), there is a bound $n_1(\varepsilon)$ such that almost all per-capita gains to cooperation can be realized by groups smaller in size than that bound; the games can be approximated by ones with bounded essential coalition sizes.[5] In other words, bounded sized coalitions can nearly exhaust gains to scale of coalition formation.

A pregame (Ω, Ψ) has the *uniform approximate core property* if, for any $\varepsilon > 0$, there is an integer $n_2(\varepsilon)$ such that for all profiles f with $\|f\| \geq n_2(\varepsilon)$, any derived game (N, v_α) with $\mathrm{prof}(\alpha|N) = f$ has a nonempty ε-core.

Theorem. *Let (Ω, Ψ) be a pregame. Then (Ω, Ψ) satisfies inessentiality of large groups if and only if it has the approximate core property.*

In the next section I provide some examples illustrating the role of inessentiality of large groups and in the following section, the proofs. First, however, I state another theorem which is used in the proofs and a lemma. The theorem is a *TU* form of the main result of Wooders (1983) for NTU games.

[4] deleted in proof.

[5] A game, or pregame, has *bounded essential coalition sizes* if all gains to group formation can be realized by groups bounded in absolute size. This does not rule out the formation of large coalitions, but it does mean that the formation of coalitions larger than the bound does not increase per-capita payoff.

Theorem (Wooders 1983). *Let (Ω, Ψ) be a pregame and let $\{(N^n, v_n)\}$ be a sequence of derived games where the profile of N^n equals n times the profile of N^1. Then for any $\varepsilon > 0$ there is an integer $\eta(\varepsilon)$ such that for all $n \geqq n(\varepsilon)$, the ε-core of (N^n, v_n) is nonempty.*

Our proofs also rely on the following lemma. Roughly, the lemma states that when there is only a finite number of player types we can approximate a large player set by a large number of players subsets, all with the same profile, plus a number of "left:overs" who, in total, constitute only a small percentage of the total player set. This allows us to apply the above theorem.

Lemma 1. *Let (Ω, Ψ) be a pregame where $\Omega = \{\omega_1, \ldots, \omega_T\}$. Let $(f^n\}$ be a sequence of profiles on Ω such that $\| f^n \| \to \infty$ as $n \to \infty$ and $(1/\| f^n \|) f^n \to f$ for some function $f: \Omega \to R_+$. Then given any $\varepsilon > 0$ there is a profile h and an integer $n(\varepsilon)$ such that for each $n \geqq n(\varepsilon)$, for some integer r_n and some profile l^n we have*

$$r_n h + l^n = f^n \quad \textit{and} \quad \frac{\| l^n \|}{\| f^n \|} < \varepsilon.$$

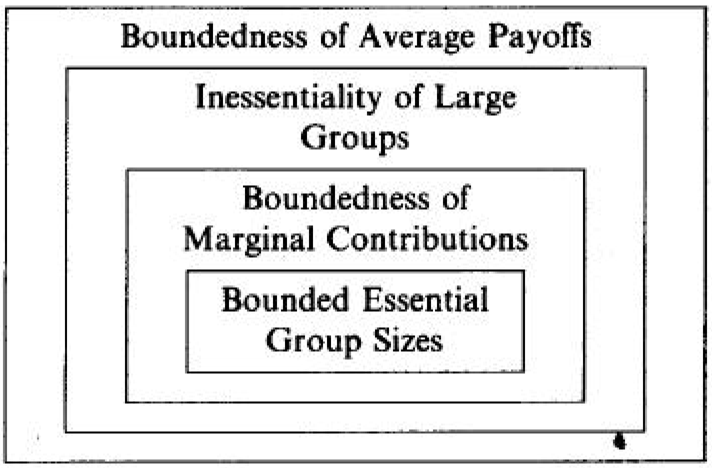
Moreover, when f is rational-valued, we can take $g = mf$ for some integer m such that mf is integer-valued.

Since the lemma is not surprising but has a long proof, we omit the proof and refer the reader to Wooders (1990a) for details.

4. Boundedness conditions and inessentiality of large groups

In this section I relate some boundedness assumptions to the inessentiality-of-large-groups condition. Also I indicate the use of these boundedness conditions in previous work.

We obtain the following relationships, expressed by set inclusions, between games satisfying the various properties.



4.1 Inessentiality of large groups, boundedness of average payoffs, and the uniform approximate core property

A pregame (Ω, Ψ) satisfies *boundedness of average payoffs* if there is a constant C such that for all profiles f we have $\dfrac{\Psi(f)}{\|f\|} \leqq C$.

One might conjecture that boundedness of average payoffs suffices for uniform inessentiality of large groups. Our next example illustrates that it does not.

Example 1. *A "pregame" satisfying boundedness of average payoffs but not uniform inessentiality of large groups.*

The term pregame is in quotation marks here as this example also illustrates a situation where (2.4) does not hold.

There are two types of players, $\Omega = \{\omega_1, \omega_2\}$. The "pregame" characteristic function is given by

$$\Psi(f) = \begin{bmatrix} f(\omega_1) + f(\omega_2) & \text{if} \quad f(\omega_1) > 0 \quad \text{and} \quad f(\omega_2) > 0 \\ 0 & \text{if} \quad f(\omega_1) = 0 \quad \text{or} \quad f(\omega_2) = 0. \end{bmatrix}$$

Let $\varepsilon_0 > 0$ be given and suppose the bound $n_1(\varepsilon_0)$, given in the definition of inessentiality, is B. Let f^n be a profile defined by

$$f^n(\omega_1) = 1$$
$$f^n(\omega_2) = n - 1$$

for each integer n. The "inessentiality" requirement is that for each n,

$$\Psi(f^n) - \sum_k \Psi(f^{nk}) \leqq \varepsilon_0 \|f^n\|$$

for some partition $\{f^{nk}\}$ of f^n where $\|f^{nk}\| \leqq n_1(\varepsilon_0)$ for each nk. But $\sum_k \Psi(f^{nk}) \leqq B$ since only 1 member of the partition will contain the one player of type 1. Therefore, satisfaction of "inessentiality" requires that $n - B \leqq \varepsilon_0 n$ for every n sufficiently large; this is impossible, so we have a contradiction.

We have the result, proven in the next section, that uniform inessentiality of large groups implies boundedness of average payoffs.

Proposition 1. *Let (Ω, Ψ) be a pregame satisfying only (2.1) to (2.3). Suppose (Ω, Ψ) satisfies uniform inessentiality of large groups. Then (Ω, Ψ) satisfies boundedness of average payoffs.*

Boundedness of balanced payoffs (2.4) implies boundedness of average payoffs. I have not established any relationship between boundedness of balanced payoffs and inessentiality of large groups. However, it is apparent that inessentiality of large groups does not imply boundedness of balanced payoffs. The inessentiality condition allows some games where very small percentages of players can receive very large balanced payoffs.

4.2. *Inessentiality of large groups and boundedness of marginal contributions to coalitions*

A pregame satisfies *boundedness of marginal contributions to groups* if there is a constant M such that $\Psi(f+\chi_\omega) \leqq \Psi(f) + M$ for all profiles f and all ω in Ω. The property of boundedness of individual marginal contributions means that there are no types of players who can make unbounded contributions to society.[6]

Boundedness of marginal contributions implies the uniform inessentiality of large groups. This follows from Theorem 1 and the result of Wooders–Zame (1984) that pregames satisfying boundedness of marginal contributions have the uniform approximate core property. A direct proof using Lemma 1 is given in Wooders (1990a).

Proposition 2. *Let (Ω, Ψ) be a pregame satisfying boundedness of marginal contributions. Then (Ω, Ψ) satisfies inessentiality of large groups.*

It can also be shown that boundedness of marginal contributions implies (2.4), boundedness of balanced payoffs.

The example below shows that the converse of Proposition 2 does not hold.

Example 2. *A pregame satisfying inessentiality of large groups but not boundedness of marginal contributions.*

Let (Ω, Ψ) be a pregame with $|\Omega| = 1$, i.e., there is only one type of player. In this case, a profile is equivalent to a nonnegative integer number of players. We define Ψ as follows

$$\begin{aligned}
\Psi(0) &= 0\\
\Psi(1) &= 1\\
\Psi(10) &= 10\\
\Psi(10^{2^k}) &= (10^{2^k})\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^k}\right) \quad \text{for} \quad k = 1, 2, \ldots .
\end{aligned}$$

and $\Psi(\cdot)$ is defined as the least superadditive worth function satisfying the above, for all $n = 1, 2, \ldots$, i.e., for any profile f we define $\Psi(f) = \max \Sigma\Psi(f^k)$ where the max is taken over all partitions of $\{f^k\}$ of f. It is easy to show that, for any k we have $\Psi(10^{2^k}) = \max \Sigma\Psi(10^{2^j})$, for all partitions $\{10^{2^j}\}$ of 10^{2^k}. Note that Ψ satisfies inessentiality of large groups, which in this 1-type case means simply that the *per capita* benefit of increasing the size of a "large" group by 1 additional player, is bounded by $1/10^k$. However, the marginal contribution of a player to a coalition containing $10^{2^k} - 1$ players for any positive integer k is at least

[6] It has been suggested to the author that boundedness of marginal contributions is a condition on individuals rather than on groups, and in that sense more appealing than inessentiality of large groups. The view of this author is that the ability of an individual to contribute to a group depends on the group as much as the individual, so in that sense we view the two conditions as similar.

$10^{2k}/10^k$, which becomes infinite as k becomes large. Also note that $\lim_{n\to\infty} \frac{\Psi(10^{2k})}{10^{2k}} = 1 + \sum_{i=1}^{\infty} \frac{1}{10^i} = 1 + \frac{1}{9}$. For any $\varepsilon > 0$, for all n sufficiently large so that $\frac{\Psi(n)}{n} > 1 + \frac{1}{9} - \varepsilon$, and the $\varepsilon-$core of a derived game with n players is nonempty.

4.3. The boundedness assumptions and nonemptiness of approximate cores

The approach taken in this paper has origins in the literature of exchange economies and the core, especially Shapley–Shubik (1966, 1969), where the authors showed that replica exchange economies with a finite number of types of goods and of types of agents have nonempty approximate cores and also that (finite) games are market games if and only if they are totally balanced. Wooders (1980, 1989) applies extensions of the notions of approximate cores of Shapley–Shubik to economies with local public goods, and introduces boundedness of essential coalition sizes assumptions to ensure that the public goods are "local" rather than "pure". These assumptions enable core convergence results. The sorts of models of large games used in this paper, separate from any specific underlying economy but satisfying some form of inessentiality of large groups and substitution were introduced in Wooders (1983) and have now appeared in several papers. To illustrate the roles of various inessentiality of large groups conditions in these papers, we briefly discuss these papers (in order of undertaking), and some other related literature below. More generally related literature is discussed in the concluding section.

For replica games, one with a finite number of player types and constant proportions of players of each type in each game, boundedness of average payoffs suffices for the nonemptiness of approximate cores of a subsequence of the games, even in the without sidepayments (NTU) case.[7] When we consider games with finite types and just a limiting distribution of player types and use notions of NTU approximate cores where a small set of players is ignored, an approximate core with "left-overs" then, from Lemma 1 and Wooders result, we obtain nonemptiness of approximate cores for all sufficiently large games in the sequence.[8] For TU games, the need for a weaker approximate core concept does not arise; we have nonemptiness of the ε-core for all sufficiently large terms in the sequence with just the assumption of boundedness of average payoffs.

Kaneko–Wooders (1982) focus on bounded essential coalition sizes and highlight the basic properties of large replica games following from the lemma's in Wooders (1983). Specifically, for a sequence of "replications of a given game", with bounded essential coalition sizes, all members of a subsequence have nonempty cores and this subsequence depends *only* on the size of the total player set and not on the particular characteristic function.[9]

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[7] This is shown in Wooders (1983) (see especially Lemmas 3 to 8).

[8] See Wooders (1983) and also Shubik–Wooders (1983).

[9] This is a consequence of the fact that a sufficiently large replication of a minimal balanced collection is a partition. This itself is a consequence of the fact that minimal balanced collections of sets have rational weights, exploited in Wooders (1983). (The concept of a minimal balanced collection was introduced by Bondareva (1963) and Shapley (1967).)

A continuum game with finite coalitions, introduced in Kaneko–Wooders (1986) embodies a limit form of inessentiality of large groups, since coalitions of measure zero can realize all gains to coalition formation.[10] The concept of a measurement-consistent partition of the total player set into finite coalitions is introduced; this concept extends the notion of an assignment or matching to a continuum player set where admissible groups may contain only finite number of members. With measurement-consistent partitions we can aggregate the behaviours of finite coalitions while preserving the proportions given by the measure on the total player set. The property of inessentiality of "large" coalitions (by which I now mean ones of positive measure), along with an assumption of per-capita boundedness, ensures the nonemptiness of the f-core of a continuum game with finite coalitions.[11]

Boundedness of marginal contributions was introduced in Wooders–Zame (1984), where the finite-type assumption of the previous papers was relaxed. The formulation developed there to extend the framework of Wooders (1983), with a finite set of types, to a compact metric space of types is used in this paper.[12] This framework is very appropriate from our purposes. However, the assumption of boundedness of marginal contributions is too restrictive for the purposes of this paper because it is not necessary that players make bounded contributions to large player sets, as we have illustrated.[13] Also, we show there nonemptiness of the individually rational ε-core with the additional condition that for each player there are "enough" similar players.[14] We note that Wooders–Zame contains a number of informative examples.

In Kaneko–Wooders (1990) we consider continuum games with bounded coalition sizes and without the restriction to a finite set of player types of our earlier paper.[15] Nonemptiness of the core, called the f-core, is obtained without any further restrictions. Thus we provide a new continuum limit model of assignment games and their generalizations, including many-sorted assignment games and games with bounded coalition sizes (cf., Gale–Shapley (1962) and Shapley–Shubik (1982)), without the restriction to a finite number of types of our

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[10] See also Hammond–Kaneko–Wooders (1989) and Kaneko–Wooders (1989).

[11] That research has had a major impact on the research of this paper since the continuum with finite coalitions, along with previous results on economies with local public goods and on large games, suggests the equivalence herein.

[12] The NTU framework of Wooders (1983) is slightly more general, even in application to TU games, than suggested here. However, the extent of this additional generality is not very significant and the Wooders–Zame model is exactly an extension of my earlier (1979) TU version of the 1983 paper.

[13] Here the continuity notion used in the definition of the pregame is also less restrictive. Rather than requiring uniform continuity as in Wooders–Zame, we use the pointwise continuity of the pregame to obtain a uniform continuity on a subset of norm-bounded profiles. This suffices since inessentiality of large groups means that we can approximate a pregame by one with norm-bounded essential profiles. (See Note 5.)

[14] Such a result could also be established with the assumption of inessentiality of large groups rather than boundedness of marginal contributions. The result would need some additional assumption such as one ensuring that each player has enough near-substitutes.

[15] This, and Kaneko–Wooders (1986) both treat games without sidepayments. The earlier paper has only a finite number of types.

earlier work.[16] In situations with bounded essential coalition sizes and a continuum of players the properties of large games satisfying inessentiality of large groups emerge sharply.[17] The current paper shows that we can approximate large games satisfying mild conditions, such as boundedness of marginal contributions or inessentiality of large groups, by ones with bounded essential coalition sizes.

Finally, we note that the properties of large games that we have studied, with the exception of the approximate core property, do not appear to have been isolated in the literature on core convergence in exchange economies. However, recent examples showing asymptotic nonequivalence of the core and the competitive allocations appear to depend on violation of inessentiality of large coalitions as noted by Anderson (see Anderson (1990) and also Manelli (1989)).

5. Proofs

For a proof of Lemma 1 see Wooders (1990a).

Proof of Theorem 1.

Inessentiality of large groups implies the approximate core property:

We first show that inessentiality of large groups implies the approximate core property. Suppose (Ω, Ψ) satisfies inessentiality of large groups and, to obtain a contradiction, that (Ω, Ψ) does not satisfy the approximate core property. Let $\varepsilon_0 > 0$ be a real number with the property that for each n there is a profile f^n, $\|f^n\| > n$, such that the game (N^n, v_{α^n}) has an empty $3\varepsilon_0$-core, where $\mathrm{prof}(v_{\alpha^n}|N^n) = f^n$. We denote this game simply by (N^n, v_n) in the remainder of the proof.

For the proof of this part of the theorem, we consider first the finite-type case, and then carry out the general case by approximating Ω by a finite number of player types.

Case 1. Ω Finite

Suppose Ω is finite, $\Omega = \{\omega_1, \ldots, \omega_T\}$. By passing to a subsequence if necessary, we can assume that $(1/\|f^n\|) f^n \to f$, i.e., given $\varepsilon > 0$ there is an n_0 sufficiently large so that for all $n \geqq n_0$, for each $\omega_t \in \Omega$ we have

$$\left| \frac{f^n(\omega_t)}{\|f^n\|} - f(\omega_t) \right| < \varepsilon.$$

[16] A many-sorted assignment game is a composition of assignment games and includes multiple-sided assignment games.

[17] In particular, for TU situations we have equivalence of the f-core and the competitive payoffs of the associated continuum market. This is almost immediate from the fact that the market games of Wooders (1988) satisfy the conditions of Mas-Colell (1975) and he demonstrates an equivalence result.

From inessentiality of large groups, there is a bound B such that for any profile g on Ω, for some partition of g into sub-profiles, say $\{g^k : k = 1, \ldots, K\}$ where $\|g^k\| \leqq B$ for each g^k, we have

$$\Psi(g) - \sum_k \Psi(g^k) < \varepsilon_0 \|g\|.$$

Let $M = \max\{\{\Psi(g) + 2B\varepsilon_0 : \|g\| \leqq B\}, 3\varepsilon_0\}$.

From Lemma 1 there is an integer n_0 and a profile h such that for each $n \geqq n_0$, for some integer r_n and some profile l^n we have

$$r_n h + l^n = f^n \quad \text{and} \quad \frac{\|l^n\|}{\|f^n\|} < \frac{\varepsilon_0}{2M}.$$

Since $\dfrac{r_n\|h\|}{\|f^n\|} \geqq 1 - \dfrac{\varepsilon_0}{2M}$ and $M > 2\varepsilon_0$, there is an $n' \geqq n_0$ such that for all $n \geqq n'$,

$$r_n \|h\| \varepsilon_0 > \|l^n\| M, \quad \text{and} \quad r_n > B.$$

We will use this fact to construct an ε_0-core payoff.

From Wooders (1983) for all sufficiently large n, say $n \geqq n^*$ where, for later convenience we choose $n^* \geqq n'$, any derived game with profile of the total player set equal to $r_n h$ has a nonempty ε_0-core containing a payoff with the equal-treatment property. Let $\bar{x}_n = (\bar{x}_{1n}, \ldots, \bar{x}_{Tn})$ represent such an equal-treatment payoff. Since $r_n \|h\| \varepsilon_0 > \|l_n\| M$, we can construct a payoff z for the game (N^n, v_n) (with the profile of the total player set equal to f^n) and with the properties that if $i \in N^n$, and $\alpha(i) = \omega_t$ is in the support of h, then $z_i \geqq \bar{x}_{tn} - \varepsilon_0$ and if $\alpha(i) = \omega_t$ is not in the support of h, then $z_i \geqq M$. Specifically, let W denote a subset of N^n with profile $r_n h$. For each i in W define

$$z_i = \bar{x}_{tn} - \varepsilon_0 \quad \text{when} \quad \alpha(i) = t, \quad t = 1, \ldots, T.$$

and, for $i \in N^n$, $i \notin W$

$$z_i = M.$$

(It is not necessary to ensure that $z(N^n) = v_n(N^n)$, but of course we could.)

We now show that z is in the $2\varepsilon_0$-core of the game (N^n, v_n). Suppose not. Then for some coalition $S \subset N^n$, letting s denote the profile of S, we have

$$\Psi(s) - \sum_{i \in S} z_i > 2\varepsilon_0 \|s\|.$$

From the inessentiality of large groups, for some partition of S into groups $\{S^k : k = 1, \ldots, K\}$ with profiles $s^1, \ldots, s^K$ respectively and with $\|s^k\| \leqq B$ for each k we have

$$\begin{aligned} &\sum_k \Psi(s^k) - \sum_{i \in S} z_i + \varepsilon_0 \|s\| \\ &\quad \geqq \Psi(s) - \sum_{i \in S} z_i > 2\varepsilon_0 \|s\|. \end{aligned}$$

This implies that for some k'

$$\Psi(s^{k'}) - \sum_{i \in S^{k'}} z_i > \varepsilon_0 \|s^{k'}\|.$$

We cannot have the support of $s^{k'}$ contained in the support of $r_n h$; otherwise we would have a contradiction to the fact that $\bar{x}_n$ is in the ε_0-core of any derived game with profile of the total player set equal to $r_n h$ for any n sufficiently large. We cannot have the profile $s^{k'}$ greater than or equal to $r_n h$ since $\|s^k\| \leqq B < r_n$. Therefore for at least one $i \in S^{k'}$, we must have $z_i \geqq M$. But then

$$\Psi(s^{k'}) - \sum_{i \in S^{k'}} z_i < \Psi(s^{k'}) - M + 2\varepsilon_0(B-1) < 0,$$

since $\Psi(s^{k'}) + 2\varepsilon_0 B \leqq M$, which gives us a contradiction

Case 2. Ω not necessarily finite

We proceed by first constructing another sequence of games derived from a pregame with bounded essential coalition sizes. Let $n_1(\varepsilon_0)$ be the number given by the definition of inessentiality of large groups for the pregame (Ω, Ψ). Define the characteristic function Γ by

$$\Gamma(f) = \max \Sigma\, \Psi(f^{nk})$$

where the maximum is taken over all partitions $\{f^{nk}\}$ of f with $\|f^{nk}\| \leqq n_1(\varepsilon_0)$ for each member f^{nk} of the partition.

The assumption that (N^n, v_n) has an empty $3\varepsilon_0$-core implies that (N^n, γ_n) has an empty $2\varepsilon_0$-core, for each n where γ_n is the characteristic function of the game derived from the attribute function α^n and the function Γ. Note that the set of profiles with bounded norm $(n_1(\varepsilon_0))$ is a compact space (cf., Mas-Colell (1975)).

We proceed by approximating the pregame (Ω, Γ) by one with a finite number of types. Let $\delta \in (0,1)$ be such that if $\omega_1, \omega_2 \in \Omega$ and $\operatorname{dist}(\omega_1, \omega_2) < \delta$, then $|\Gamma(h + \chi_{\omega_1}) - \Gamma(h + \chi_{\omega_2})| < \varepsilon_0$ for every profile h on Ω with $\|h\| \leqq n_1(\varepsilon_0)$. (This is possible from the continuity of Ψ and the compactness of the set of profiles with bounded norms.) Let $\{\Omega_1, \ldots, \Omega_T\}$ be a partition of Ω such that, for each $t = 1, \ldots, T$, if $\omega \in \Omega_t$ and $\omega' \in \Omega_t$ then $\operatorname{dist}(\omega, \omega') < \delta$. Select a point $\omega_t \in \Omega_t$ for each t.

Now consider the sequence $\{f^n\}$. We construct another sequence $\{g^n\}$ with support $(g^n) \subset \{\omega_1, \ldots, \omega_T\}$. Define $g^n(\omega_t) = \sum_{\omega \in \Omega_t} f^n(\omega)$. Observe that

$$|\Gamma(f^n) - \Gamma(g^n)| \leqq \varepsilon_0 \|g^n\|.$$

(This can be demonstrated by successively subtracting profiles χ_ω from f^n, where ω is in the support of f^n, and adding profiles $\chi_{\omega'}$, where ω' is in the support of g^n, with $\operatorname{dist}(\omega, \omega') < \delta$, and applying continuity until we reach the profile g^n.)

For all sufficiently large n, a game with the profile of the total player set equal to g^n and characteristic function derived from Γ has a nonempty ε_0-core. Let $\bar{x}$ represent an equal-treatment payoff in the ε_0-core of such a game. Define a payoff y for the game (N^n, γ_n) by $y_i = \bar{x}_t - \varepsilon_0$ if $\alpha^n(i) \in \Omega_t$ for each $i \in N^n$. Observe that the payoff y is feasible:

$$\sum_{i \in N^n} y_i \leqq \Gamma(g^n) - \varepsilon_0 \|f^n\| \leqq \Gamma(f^n).$$

Also, y cannot be "$3\varepsilon_0$-improved upon". To show this, suppose $S \subset N^n$ and

$$\sum_{i \in S} y_i < \gamma_n(S) - 3\varepsilon_0 |S|.$$

Let s denote the profile of S. Let g denote a profile defined by

$$g(\omega_t) = \sum_{\omega \in \text{support}(s) \cap \Omega_t} s(\omega) \quad \text{for each} \quad \omega_t = t = 1, \ldots, T.$$

Now $\sum_{i \in S} y_i < \gamma_n(S) - 3\varepsilon_0 |S|$ implies that $\sum_t (\bar{x}_t - \varepsilon_0) g(\omega_t) < \gamma_n(S) - 2\varepsilon_0 |S|$. From continuity, $\gamma_n(S) = \Gamma(s) < \Gamma(g) + \varepsilon_0 \| g \|$. We then have $\sum_t (\bar{x}_t - \varepsilon_0) g(\omega_t) = \sum_t \bar{x}_t g(\omega_t) - \varepsilon_0 \| g \| < \Gamma(g) - 2\varepsilon_0 \| g \|$, which is a contradiction to the fact that $\bar{x}$ is in the ε_0-core of a game derived from (Ω, Γ) with profile of the total player set equal to g^n.

The approximate core property implies inessentiality of large groups: Suppose the assertion is false. Then for some $\varepsilon_0 > 0$, for each integer n there is a profile f^n such that for any partition of f^n into subprofiles $\{f^{ni}\}$ with $\| f^{ni} \| \leqq n$ for each i, we have

$$\Psi(f^n) - \sum_i \Psi(f^{ni}) > \varepsilon_0 \| f^n \|.$$

We consider first the case where Ω is a finite set, $\Omega = \{\omega_1, \ldots, \omega_T\}$. We can assume without loss of generality that $\{(1/\| f^n \|) f^n\}$ converges, say to a function f.

For each profile f^n let x^n be a balanced payoff. From the boundedness assumption (2.4), by passing to a subsequence if necessary we can suppose that x^n converges to a payoff x^*.

We now show that for all n sufficiently large $(x^* - 2\varepsilon_0 1_\tau) \cdot f^n \leqq \Psi(f^n)$. This follows from the approximate core property since, for all n sufficiently large, we have

$$\begin{aligned} x^* \cdot f^n - 2\varepsilon_0 \| f^n \| &\leqq x^n \cdot f^n - \varepsilon_0 \| f^n \| \\ &= \bar{\Psi}(f^n) - \varepsilon_0 \| f^n \| \quad (\text{since } x^n \cdot f^n = \bar{\Psi}(f^n)) \\ &\leqq \Psi(f^n) \quad (\text{from the approximate core property}). \end{aligned}$$

Observe that for all n sufficiently large we also have

$$\begin{aligned} \Psi(f^n) &\leqq \bar{\Psi}(f^n) \quad (\text{from the definition of the balanced cover}) \\ &= x^n \cdot f^n \\ &\leqq x^* \cdot f^n + \varepsilon_0 \| f^n \|. \end{aligned}$$

Suppose that n_0 is sufficiently large so that the above inequalities hold for all $n \geqq n_0$.

Let $z = \max_t x_t^* + \varepsilon_0$ (where ε_0 is added into the expression for z simply to ensure that $z \neq 0$). From Lemma 1 we can select a profile h with the property that for all n sufficiently large, say all $n \geqq n_1$, for some integer r_n and profile m^n we have

$$f^n = r_n h + m^n \quad \text{and} \quad \frac{\| m^n \| z}{\| f^n \|} < \varepsilon_0.$$

We can assume that $n_1 \geqq n_0$. This implies that for all $n \geqq n_1$,

$$\begin{aligned} \Psi(f^n) - \Sigma r_n \Psi(h) &\leqq x^* \cdot f^n + \varepsilon_0 \| f^n \| - r_n x^* \cdot h + 2\varepsilon_0 r_n \| h \| \\ &\leqq x^* \cdot m^n + 3\varepsilon_0 \| f^n \| \leqq 4\varepsilon_0 \| f^n \| \end{aligned}$$

since $x^* \cdot f^n + \varepsilon_0 \| f^n \| \geqq \Psi(f^n)$ for all $n \geqq n_0$ and $n_1 \geqq n_0$, $-\Psi(h) \leqq -(x^* \cdot h - 2\varepsilon_0 \| h \|)$ and $\dfrac{m^n \cdot x^*}{\| f^n \|} \leqq \dfrac{\| m^n \| z}{\| f^n \|} < \varepsilon_0$. We now have

$$\frac{\Psi(f^n)}{\| f^n \|} - \frac{\Sigma r_n \Psi(h)}{\| f^n \|} \leqq 4\varepsilon_0.$$

This yields a contradiction since $\| h \| \leqq n$ for all n sufficiently large and a partition containing r_n profiles h and $(f^n(\omega_t) - r_n h(\omega_t))$ profiles χ_t for each t satisfies the required condition. Specifically, for all $n \geqq n_1$, we have

$$\Psi(f^n) - r_n \Psi(h) - \Sigma_t m^n(\omega_t) \Psi(\chi_{\omega_t}) \leqq 4\varepsilon_0.$$

Case 2. Ω not necessarily finite. This case can be treated the same way as the extension of the finite-type case to the general compact metric space of consumer types was treated previously. Thus I leave the details to the reader. □

Proof of Proposition 1. Consider the finite type case, $\Omega = \{\omega_1, \ldots, \omega_T\}$ and suppose the Proposition is false. In particular, let $\{f^n\}$ be a sequence of profiles such that

$$\frac{\Psi(f^n)}{\| f^n \|} > n \quad \text{for each } n.$$

Note that it cannot be that $\| f^n \|$ is bounded. Let ε be given, $0 < \varepsilon < 1$, and let $n_1(\varepsilon)$ be the parameter in the definition of inessentiality of large profiles. Let

$$\frac{A}{2} = \sup_{\| f \| \leqq n_1(\varepsilon)} \{\Psi(f), 2\},$$

Since the set of profiles with norms less than or equal to $n_1(\varepsilon)$ is compact, this supremum exists. Then we have $\Psi(f^n) - \sum_k \Psi(f^{nk}) \leqq \varepsilon \| f^n \|$ for some collection of subprofiles $\{f^{nk} : k = 1, \ldots, K\}$ of f^n with

$$\| f^{nk} \| \leqq n_1(\varepsilon) \quad \text{for each } nk,$$

$$\Sigma f^{nk} = f^n, \quad \text{and} \quad \Psi(f^n) \leqq \varepsilon \| f^n \| + \frac{KA}{2},$$

and therefore $\dfrac{\Psi(f^n)}{\| f^n \|} \leqq \varepsilon + \dfrac{KA}{2\| f^n \|} \leqq \varepsilon + \dfrac{A}{2} \leqq A$ since $KA \leqq A \| f^n \|$ and since A is greater than 2ε. This is a contradiction.

The extension to the case where Ω is a compact metric space can be obtained similarly to this extension in the proof of Proposition 1. □

6. Discussion of the literature and conclusions

The idea that large economies with small effective coalitions, firms, or groups have competitive properties has appeared in other forms in the literature. One example is the work of Novshek–Sonnenschein (1979) and others on economies with production and small-capacity firms with roots going back to Joseph (1932).

Literature on coalition production where productive coalitions become relatively small as the economy grows large includes Böhm (1974) and others. A third sort of model with small effective groups (clubs or communities) for the production and/or consumption of public goods was initiated by Tiebout (1956) and Buchanan (1965).[18] Tiebout (1956) conjectured that when public goods are "local" rather than pure, then competitive forces lead to "market-like" outcomes.[19] Exploiting the feature that in economies with local public goods and sufficient crowding or congestion, all or nearly all gains to group size can be realized by relatively small groups, Wooders (1978, 1980, 1989) demonstrated the "Tiebout Hypothesis" by exact core-equilibrium equivalence in Wooders (1978) (with one private good), and, in (1980, 1989) by core convergence. Tomasiunas (1990) obtains a core convergence result without crowding but when the numbers of agents who can jointly consume the public goods are bounded.

It is perhaps of greatest importance that private goods exchange economies have the property of inessentiality of large groups. The importance of the effectiveness of small coalitions for the study of the core and the competitive equilibrium was clearly recognized by Mas-Colell (1979) who showed an "inessentiality of large coalitions" from the perspective of "improvement" or "blocking" in exchange economies. The inessentiality of large coalitions for both improvement and feasibility was shown by Hammond–Kaneko–Wooders (1989) and Kaneko–Wooders (1989) (see also Kaneko–Wooders (1986, Lemma 3.2)).[20]

The role of inessentiality of large groups in economies with infinite dimensional commodity spaces has not yet been adequately studied. Of course if large groups are essential, convergence of the core to the competitive allocations would be quite surprising (and probably a very delicate phenomenon). Because economies with infinite dimensional commodity spaces are quite subtle, I note that the application of our framework to such a model may be most transparent for economies of the type introduced by Mas-Colell (1975), but with finite player sets and quasi-linear utilities. In this case, when agents are endowed with commodity bundles containing only finite numbers of distinct commodities, my framework can be applied. This is partially because the space of attributes is analogous to Mas-Colell's space of commodity characteristics. (To relax the restriction of quasi-linear utilities, however, may be quite difficult, as the examples of Anderson (1990) suggest.)

I conclude with a caution to the reader. Rather than situations where almost all gains to group formation can be realized by groups bounded in size, it may be

[18] For this author, the importance of the inessentiality of large groups in internalizing externalities emerges more sharply in Tiebout's work than in Buchanan's. Although Buchanan considered clubs as the result of individual optimising behaviour, I have not found any reference to the "market-like" optimality of the outcome.

[19] Another author who appears to have had similar ideas to those of Tiebout and Buchanan is Allais (1942). Allais considered the "market mechanism" primarily in terms of supporting prices in situations with essential groups.

[20] To the author of this paper, the emphasis of the continuum models initiated by Aumann, with coalitions of positive measure, is quite distinct. We intend to discuss this, and Aumann's "Equivalence Principle" elsewhere.

the case that because of group formation costs, for example, only small groups form. Without modelling such group or coalition formation costs explicitly, we may take them into account by allowing only relatively small coalitions to form. In such situations we may still be able to apply the results of this paper to the game with restricted coalition sizes, even though it is not necessarily the case that all gains to coordination of group activities can in fact be realized by the cooperative activities of individual players in small groups and that the competitive equilibrium is Pareto-optimal.

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