Networks and Farsighted Stability*

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Abstract

We provide a new framework for the analysis of network formation and demonstrate the existence of farsightedly consistent directed networks. Our framework extends the standard notion of a network and also introduces the notion of a supernetwork. A supernetwork is made up of a collection of directed networks (the nodes) and represents (via the arcs connecting the nodes) preferences and rules governing network formation. By extending Chwe's 1994 result on the nonemptiness of farsightedly consistent sets, we show that for any supernetwork there exists a nonempty subset of farsightedly consistent directed networks. *Journal of Economic Literature Classification Numbers: A14, D20, J00*

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1 Introduction

Overview

The main contribution of this paper is to provide a new framework for the analysis of network formation. Our framework extends the standard notion of a network and also introduces the notion of a *supernetwork*. All directed networks are composed of nodes and arcs. In most economic applications, nodes represent economic agents, while arcs represent connections or interactions between agents. In a supernetwork, nodes represent the networks in a given collection, while arcs represent coalition moves and coalitional preferences over the networks in the collection. Given any collection of directed networks and any profile of agent preferences over the collection, a supernetwork uniquely represents all the coalitional preferences and all the coalitional moves allowed by the rules governing network formation (i.e., the rules governing the addition, subtraction, or replacement of arcs). We note that our framework promises to have numerous applications, including to learning, to models of bargaining and trade, and to questions of political economy; such applications, however, are beyond the scope of the current paper.

Since the seminal paper by Jackson and Wolinsky [9] there has been a rapidly growing literature on social and economic networks and their stability and efficiency properties (e.g., see Jackson [6] and Jackson and van den Nouweland [7]). As noted in [6], an important issue that has not yet been addressed in the literature on networks and network formation is the issue of *farsighted stability* (see [6], p. 21 and p. 35).[1]

[1] As far as we know, Watts [14] and Page, Wooders, and Kamat [10] are the first papers in

Using our supernetwork framework, we address this issue. In particular, our second contribution is to demonstrate the existence of farsightedly consistent networks. Given the rules governing network formation as represented via the supernetwork, *a directed network (i.e., a particular node in the supernetwork) is said to be farsightedly consistent if no agent or coalition of agents is willing to alter the network (via the addition, subtraction, or replacement of arcs) for fear that such an alteration might induce further network alterations by other agents or coalitions that in the end leave the initially deviating agent or coalition no better off - and possibly worse off.* By extending Chwe's basic result on the nonemptiness of the largest farsightedly consistent set (see [1]), we show that for any supernetwork corresponding to a given collection of directed networks, the set of farsightedly consistent networks is nonempty. Thus, we conclude that any supernetwork possesses nodes (i.e., networks) that are farsightedly consistent.[2]

Directed Networks vs Linking Networks

We focus on directed networks, extending the definition of directed networks found in the literature. In a directed network, each arc possesses an orientation or direction: arc j connecting nodes i and i' must either go from node i to node i' or

the literature to address non-myopic behavior in strategic network formation and [10] is the first to address the issue of farsighted stability in network formation. Since then, other papers have appeared focusing on non-myopic behavior in network formation. Most notable are the papers by Deroian [4], and B. Dutta, S. Ghosal, and D. Ray, "Farsighted Network Formation" (typescript, 2003).

[2]Note that farsighted consistency is quite distinct from the equilibrium notion of subgame perfection used in Currarini and Morelli [2], for example, in that farsighted consistency is a coalitional equilibrium notion rather than a non-cooperative equilibrium notion.

must go from node i' to node i.[3] In an undirected (or linking) network, arc j would have no orientation and would simply indicate a connection or link between nodes i and i'. Under our extended definition of directed networks, nodes are allowed to be connected by multiple arcs. For example, nodes i and i' might be connected by arcs j and j', with arc j running from node i to i' and arc j' running in the opposite direction (i.e., from node i' to node i).[4] Thus, if node i represents a seller and node i' represents a buyer, then arc j might represent a contract offer by the seller to the buyer, while arc j' might represent the acceptance or rejection of that contract offer. Also, under our extended definition loops are allowed and arcs are allowed to be used multiple times in a given network.[5] For example, arc j might be used to connect nodes i and i' as well as nodes i' and i''. However, we do not allow arc j to go from node i to node i' multiple times in the same direction. By allowing arcs to possess direction and be used multiple times and by allowing nodes to be connected by multiple arcs, our extended definition makes possible the application of networks to a richer set of economic environments. Until now, most of the economic literature on networks has focused on linking networks (see for example, Jackson and Wolinsky [9] and Dutta and Mutuswami [5]).

Given a particular directed network, an agent or a coalition of agents can change the network to another network by simply adding, subtracting, or replacing arcs from

[3] We denote arc j going from node i to node i' via the ordered pair $(j,(i,i'))$, where (i,i') is also an ordered pair. Alternatively, if arc j goes from node i' to node i, we write $(j,(i',i))$.

[4] Under our extended definition, arc j' might also run in the same direction as arc j.

[5] A loop is an arc going *from* a given node *to* that same node. For example, given arc j and node i, the ordered pair $(j,(i,i))$ is a loop.

the existing network in accordance with the rules represented by the supernetwork.[6] For example, if the nodes in a network represent agents, then the rule for adding an arc j from node i to node i' might require that both agents i and i' agree to add arc j. Whereas the rule for subtracting arc j, from node i to node i', might require that only agent i or agent i' agree to dissolve arc j. This particular set of rules has been used, for example, by Jackson and Wolinsky [9]. Other rules are possible. For example, the addition of an arc might require that a simple majority of the agents agree to the addition, while the removal an arc might require that a two-thirds majority agree to the removal.[7] Given the flexibility of the supernetwork framework, any set rules governing network formation can be represented.

While here we focus on directed networks, the same methodology can be used to deduce the existence of farsightedly consistent undirected networks (i.e., linking networks - such as the networks considered by Jackson and Wolinsky [9] and Dutta and Mutuswami [5]). An excellent paper on stability *and* efficiency in linking networks is Jackson [6]. Other papers which focus on network formation, but which do not consider the issue of farsightedness are Skyrms and Pemantle [12], Watts [13], and Jackson and Watts [8].[8] In [11] we introduce a new notion of farsighted stability in network formation called the *farsighted basis*, and we show that all supernetworks

[6] Put differently, agents can change one network to another network by adding, subtracting, or replacing ordered pairs, $(j,(i,i'))$, in accordance with certain rules.

[7] Majority addition and two-thirds majority subtraction rules might arise naturally in agenda formation networks where agendas are represented by nodes and moves from one agenda to another are represented by arcs.

[8] For recent surveys of topics in network formation see Demange and Wooders [3].

possess a farsighted basis. A *farsightedly basic network* contained in the farsighted basis of a given supernetwork represents a possible final resting point (or absorbing state) of a network formation process in which agents behave farsightedly.

2 Directed Networks

We begin by giving a formal definition of the class of directed networks we shall consider. Let N be a finite set of nodes, with typical element denoted by i, and let A be a finite set of arcs, with typical element denoted by j. Arcs represent potential connections between nodes, and depending on the application, nodes can represent economic agents or economic objects such as markets or firms.[9]

Definition 1 *(Directed Networks)*

Given node set N and arc set A, a directed network, G, is a subset of $A \times (N \times N)$. We shall denote by $\mathbb{N}(N, A)$ the collection of all directed networks given N and A.

A directed network $G \in \mathbb{N}(N, A)$ specifies how the nodes in N are connected via the arcs in A. Note that in a directed network order matters. In particular, if $(j, (i, i^{'})) \in G$, this means that arc j *goes from* node i *to* node $i^{'}$. Also, note that under our definition of a directed network, loops are allowed - that is, we allow an arc to go from a given node back to that given node. Finally, note that under our definition an arc can be used multiple times in a given network and multiple arcs can go from one node to another. However, our definition does not allow an arc j to go from a node i to a node i' multiple times.

[9] Of course in a supernetwork, nodes represent networks.

The following notation is useful in describing networks. Given directed network $G \subseteq A \times (N \times N)$, let

$$\left.\begin{aligned} G(j) &:= \left\{(i, i^{'}) \in N \times N : (j, (i, i^{'})) \in G\right\}, \\ G(i) &:= \left\{j \in A : (j, (i, i^{'})) \in G \text{ or } (j, (i', i)) \in G\ \right\} \\ G(i, i^{'}) &:= \left\{j \in A : (j, (i, i^{'})) \in G\right\}, \\ G(j, i) &:= \left\{i^{'} \in N : (j, (i, i^{'})) \in G\right\}. \end{aligned}\right\} \tag{1}$$

Thus,

$G(j)$ is the *set of node pairs* connected by arc j in network G,

$G(i)$ is the *set of arcs* going from node i or coming to node i in network G,

$G(i, i^{'})$ is the *set of arcs* going from node i to node i' in network G,

and

$G(j, i)$ is the *set of nodes* which can be reached by arc j from node i in network G.

Note that if for some arc $j \in A$, $G(j)$ is empty, then arc j is not used in network G. Moreover, if for some node $i \in N$, $G(i)$ is empty then node i is not used in network G, and node i is said to be isolated relative to network G.

Suppose that the node set N is given by $N = \{i_1, i_2, \ldots, i_5\}$, while the arc set A is given by $A = \{j_1, j_2, \ldots, j_5, j_6, j_7\}$. Consider the network, G, depicted in Figure 1.

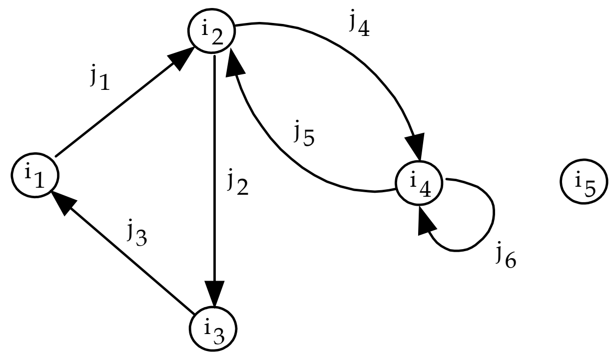


Figure 1: Network G

In network G, $G(j_6) = \{(i_4, i_4)\}$. Thus, $(j_6, (i_4, i_4)) \in G$ is a loop. Also, in network G, arc j_7 is not used. Thus, $G(j_7) = \emptyset$.[10] Finally, note that $G(i_4) = \{j_4, j_5, j_6\}$, while $G(i_5) = \emptyset$. Thus, node i_5 is *isolated* relative to G, and is not part of network G.[11]

[10] The fact that arc j_7 is not used in network G can also be denoted by writing

$$j_7 \notin proj_A G,$$

where $proj_A G$ denotes the projection onto A of the subset

$$G \subseteq A \times (N \times N)$$

representing the network.

[11] If the loop $(j_7, (i_5, i_5))$ were part of network G in Figure 1, then node i_5 would no longer be considered isolated under our definition. Moreover, we would have $G(i_5) = \{j_7\}$. Stated loosely, under our definition of a network a node is isolated relative to a given network, and therefore not part of the given network, if it is not acted upon by any arc in the given network.

Consider the new network, $G' \in \mathbb{N}(N, A)$ depicted in Figure 2.

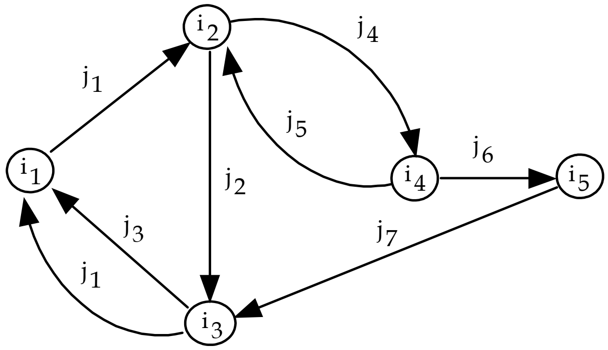


Figure 2: Network G'

In network G', $G'(j_1) = \{(i_1, i_2), (i_3, i_1)\}$. Thus, $(j_1, (i_1, i_2)) \in G'$ and $(j_1, (i_3, i_1)) \in G'$. Note that in network G', node i_5 is no longer isolated. In particular, $G'(i_5) = \{j_6, j_7\}$. Also, note that nodes i_2 and i_4 are connected by two different arcs pointed in opposite directions. Under our definition of a directed network it is possible to alter network G' by replacing arc j_5 from i_4 to i_2 with arc j_4 *from* i_4 *to* i_2. However, it is *not* possible under our definition to replace arc j_5 from i_4 to i_2 with arc j_4 from i_2 to i_4 - because our definition does not allow j_4 to go *from* i_2 *to* i_4 multiple times. Finally, note that nodes i_1 and i_3 are also connected by two different arcs, but arcs pointed in the same direction. In particular, $G(i_3, i_1) = \{j_1, j_3\}$.

Remark:

Under our extended definition of a directed network, a directed graph or digraph can be viewed as a special case of a directed network. A directed graph consists of a pair, (N, E), where N is a nonempty set of nodes or vertices and E is a nonempty set of ordered pairs of nodes. Given node set N, arc set A, and directed network $G \in \mathbb{N}(N, A)$, for each arc $j \in A$, $(N, G(j))$ is a directed graph where, recall from expression (1) above, $G(j)$ is the set of ordered pairs of nodes connected by arc j, given by

$$G(j) := \left\{ (i, i^{'}) \in N \times N : (j, (i, i^{'})) \in G \right\}.$$

Thus, a directed network is a collection of directed graphs where each directed graph is labelled by a particular arc.

3 Supernetworks

3.1 Definition

Let D denote a finite set of agents (or economic decision making units) with typical element denoted by d, and let $\Gamma(D)$ denote the collection of all nonempty subsets (or coalitions) of D with typical element denoted by S.

Given collection of directed networks $\mathbb{G} \subseteq \mathbb{N}(N, A)$, we shall assume that each agent's preferences over networks in $\mathbb{G}$ are specified via a network payoff function,

$$v_d(\cdot) : \mathbb{G} \to \mathbb{R}.$$

For each agent $d \in D$ and each directed network $G \in \mathbb{G}$, $v_d(G)$ is the payoff to agent d in network G. Agent d then prefers network G' to network G if and only if

$$v_d(G') > v_d(G).$$

Moreover, coalition $S' \in \Gamma(D)$ prefers network G' to network G if and only if

$$v_d(G') > v_d(G) \text{ for all } d \in S'.$$

By viewing each network G in a given collection of directed networks $\mathbb{G} \subseteq \mathbb{N}(N, A)$ as a node in a larger network, we can give a precise network representation of the rules governing network formation as well as agents' preferences. To begin, let

$$\mathbb{M} := \{m_S : S \in \Gamma(D)\} \text{ denote the set of move arcs (or } m\text{-arcs for short),}$$

$$\mathbb{P} := \{p_S : S \in \Gamma(D)\} \text{ denote the set of preference arcs (or } p\text{-arcs for short),}$$

and

$$\mathbb{A} := \mathbb{M} \cup \mathbb{P}.$$

Given networks G and G' in $\mathbb{G}$, we shall denote by

$$\begin{tikzcd} G \arrow[r, "m_{S'}"'] & G' \end{tikzcd}$$

Figure 3

(i.e., by an m-arc, belonging to coalition S', going from node G to node G') the fact that coalition $S' \in 2^D$ can change network G to network G' by adding, subtracting, or replacing arcs in network G. Moreover, we shall denote by

$$\begin{tikzcd} G \arrow[r, dashed, "p_{S'}"'] & G' \end{tikzcd}$$

Figure 4

(i.e., by a p-arc, belonging to coalition S', going from node G to node G') the fact that each agent in coalition $S' \in 2^D$ prefers network G' to network G.

Definition 2 *(Supernetworks)*

Given directed networks $\mathbb{G} \subseteq \mathbb{N}(N, A)$, *agent payoff functions* $\{v_d(\cdot) : d \in D\}$, *and arc set* $\mathbb{A} := \mathbb{M} \cup \mathbb{P}$, *a supernetwork,* $\mathbf{G}$, *is a subset of* $\mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ *such that for all* *networks G and G' in $\mathbb{G}$ and for all coalitions $S' \in \Gamma(D)$,*

$$(m_{S'}, (G, G')) \in \mathbf{G} \textit{ if and only if coalition } S' \textit{ can change network } G \textit{ to network } G',$$

$$G' \neq G, \textit{ by adding, subtracting, or replacing arcs in network } G,$$

and

$$(p_{S'}, (G, G')) \in \mathbf{G} \textit{ if and only if } v_d(G') > v_d(G) \textit{ for all } d \in S'.$$

Thus, a supernetwork $\mathbf{G}$ specifies how the networks in $\mathbb{G}$ are connected via coalitional moves and coalitional preferences - and thus provides a *network representation* of agent preferences and the rules governing network formation.

Remarks:

(1)Under our definition of a supernetwork, m-arc loops and p-arc loops are ruled out. Thus, for any network G and coalition S',

$$(m_{S'}, (G, G)) \notin \mathbf{G} \text{ and } (p_{S'}, (G, G)) \notin \mathbf{G}.$$

While m-arc loops are ruled out by definition, the absence of p-arc loops in supernetworks is due to the fact that each agent's preferences over networks are irreflexive.

(2) The definition of agent preferences via the network payoff functions,

$$\{v_d(\cdot) : d \in D\},$$

also rules out the following types of p-arc connections:

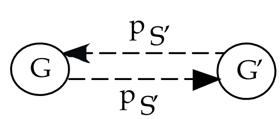


Figure 5

Thus, for all coalitions $S' \in \Gamma(D)$ and networks G and G' contained in $\mathbb{G}$,

$$\text{if } (p_{S'}, (G, G')) \in \mathbf{G}, \text{ then } (p_{S'}, (G', G)) \notin \mathbf{G}.$$

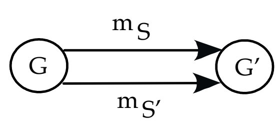
(3) For all coalitions $S' \in \Gamma(D)$ and networks G and G' contained in $\mathbb{G}$, if

$$(p_{S'}, (G, G')) \in \mathbf{G}, \text{ then}$$

$$(p_S, (G, G')) \in \mathbf{G} \text{ for all subcoalitions } S \text{ of } S'.$$

(4) Under our definition of a supernetwork, multiple m-arcs, as well as multiple p-arcs, connecting networks G and G' in supernetwork $\mathbf{G}$ are allowed. Thus, in supernetwork $\mathbf{G}$ the following types of m-arc and p-arc connections are possible:

For coalitions S and S', with $S \neq S'$



and

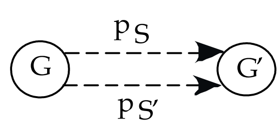


Figure 6

However, multiple m-arcs, or multiple p-arcs, from network $G \in \mathbf{G}$ to network $G' \in \mathbf{G}$ belonging to the *same* coalition are not allowed - and moreover, are unnecessary. Allowing multiple arcs can be very useful in many applications. For example, multiple m-arcs (not belonging to the same coalition) connecting networks G and G' in a given supernetwork $\mathbf{G}$ denote the fact that in supernetwork $\mathbf{G}$ there is more than one way to get from network G to network G' - or put differently, there is more than one way to change network G to network G'.

(5) In many economic applications, the set of nodes, N, used in defining the networks in the collection $\mathbb{G}$, and the set of economic agents D are one and the same (i.e., in many applications $N = D$).

3.2 An Example Illustrating the Connection Between Supernetworks and the Rules Governing Network Formation

We take as our collection of directed networks

$$\mathbb{G} := \{G, G'\},$$

where G and G' are as depicted in Figure 7.

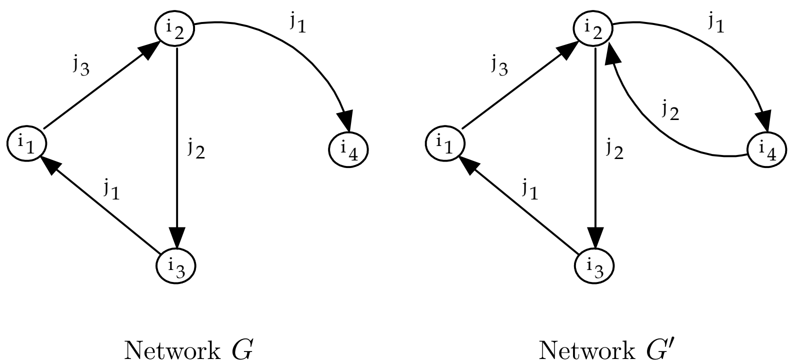


Figure 7

Each node in the set $N = \{i_1, i_2, i_3, i_4\}$ represents an agent (i.e., $N = D$), while each arc in the set $A = \{j_1, j_2, j_3\}$ represents a particular type of interaction between two agents. Thus, $(j, (i, i')) \in A \times (N \times N)$ denotes a type j interaction between agents i and i' in which agent i is the initiating agent while agent i' is the receiving agent.

Written out long hand, network G is given by

$$G = \{(j_1, (i_3, i_1)), (j_1, (i_2, i_4)), (j_2, (i_2, i_3)), (j_3, (i_1, i_2))\},$$

while network G' is given by

$$G' = \{(j_1, (i_3, i_1)), (j_1, (i_2, i_4)), (j_2, (i_2, i_3)), (j_2, (i_4, i_2)), (j_3, (i_1, i_2))\}.$$

Agent preferences over the collection $\mathbb{G} := \{G, G'\}$ are given as follows:

$$v_1(G) = v_1(G'),$$

$$v_2(G) > v_2(G'),$$

$$v_3(G) < v_3(G'),$$

$$v_4(G) > v_4(G').$$

To begin, suppose that the rules governing network formation (i.e., the rules governing the addition and subtraction of arcs) are as follows:

> (1) In order to establish an interaction of any type between two agents (i.e., in order to add an arc of type j_k, $k = 1, 2, 3$) both agents must agree.
>
> (2) In order to terminate an interactions of any type between two agents (i.e., in order to subtract an arc of type j_k, $k = 1, 2, 3$) the initiating agent must agree.

According to the rules above, in order to move from network G to network G', agents i_2 and i_4 must both agree to establish an interaction of type j_2 initiated by agent i_4. Thus, the move from network G to network G' can be represented via a move arc belonging to coalition $S' = \{i_2, i_4\}$ from G to $G^{'}$. In order to move from network $G^{'}$ back to network G, according to the rules agent i_4 must agree to terminate the interaction of type j_2 between agents i_4 and i_2 initiated by agent i_4. Thus, the move from network G' to network G can be represented via a move arc belonging to coalition $S = \{i_4\}$ from $G^{'}$ to G.

Figure 8 below depicts supernetwork $\mathbf{G}_1$ corresponding to agent preferences and the network formation rules above.

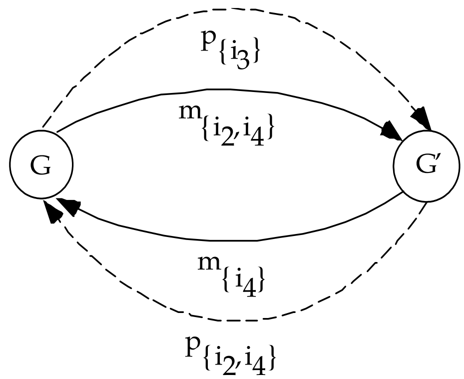


Figure 8: Supernetwork G_1

Written out long hand, supernetwork $\mathbf{G}_1$ is given by

$$\mathbf{G}_1 = \left\{ (m_{\{i_2,i_4\}}, (G, G^{'})), (m_{\{i_4\}}, (G^{'}, G)), (p_{\{i_3\}}, (G, G^{'})), (p_{\{i_2,i_4\}}, (G^{'}, G)) \right\}.$$

The network formation rules above can be described as a mix of bilateral (arc addition) and unilateral (arc subtraction) rules. Assume now that the rules of network formation are purely unilateral and given as follows:

> In order to establish or terminate an interaction of any type between agents, only the initiating agent must agree.

According to the new, purely unilateral rules, the move from network G to network $G^{'}$ can be represented via a move arc belonging to coalition $S = \{i_4\}$ from G to $G^{'}$, while the move from network $G^{'}$ to network G can be represented via a move arc belonging to coalition $S = \{i_4\}$ from $G^{'}$ to G. Figure 9 depicts supernetwork $\mathbf{G}_2$ corresponding to agent preferences and the new network formation rules. Note that in Figure 9 the move arc connecting networks G and $G^{'}$ has arrowheads at both ends indicating that there is an $m_{\{i_4\}}$-arc from network G to network $G^{'}$, as well as an $m_{\{i_4\}}$-arc from network $G^{'}$ to network G.

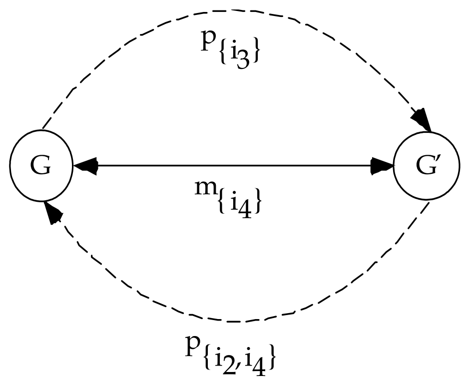


Figure 9: The New Supernetwork G_2

Written out long hand, supernetwork $\mathbf{G}_2$ is given by

$$\mathbf{G}_2 = \left\{ (m_{\{i_4\}}, (G, G^{'})), (m_{\{i_4\}}, (G^{'}, G)), (p_{\{i_3\}}, (G, G^{'})), (p_{\{i_2, i_4\}}, (G^{'}, G)) \right\}.$$

4 Farsightedly Consistent Networks

4.1 Farsighted Dominance and Farsighted Consistency

Given supernetwork $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$, we say that network $G' \in \mathbb{G}$ farsightedly dominates network $G \in \mathbb{G}$ if there is a finite sequence of networks,

$$G_0, G_1, \ldots, G_h,$$

with $G = G_0$, $G' = G_h$, and $G_k \in \mathbb{G}$ for $k = 0, 1, \ldots, h$, and a corresponding sequence of coalitions,

$$S_1, S_2, \ldots, S_h,$$

such that for $k = 1, 2, \ldots, h$

$$(m_{S_k}, (G_{k-1}G_k)) \in \mathbf{G},$$

and

$$(p_{S_k}, (G_{k-1}G_h)) \in \mathbf{G}.$$

We shall denote by $G \lhd\lhd G'$ the fact that network $G' \in \mathbb{G}$ farsightedly dominates network $G \in \mathbb{G}$. Figure 10 below provides a network representation of the farsighted dominance relation in terms of m-arcs and p-arcs. In Figure 10, network G_3 farsightedly dominates network G_0.

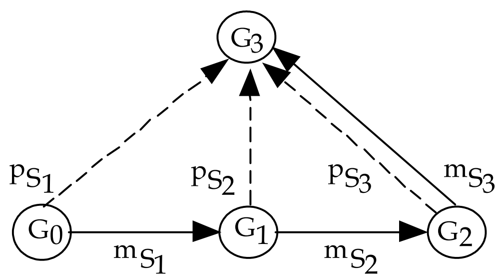


Figure 10: G_3 farsightedly dominates G_0

Note that what matters to the initially deviating coalition S_1, as well as coalitions S_2 and S_3, is the ultimate network outcome G_3. Thus, the initially deviating coalition S_1 will not be deterred even if

$$(p_{S_1}, (G_0, G_1)) \notin \mathbf{G}$$

as long as the ultimate network outcome G_3 is preferred to G_0, that is, as long as G_3 is such that

$$(p_{S_1}, (G_0, G_3)) \in \mathbf{G}.$$

Definition 3 *(Farsightedly Consistent Networks)*

Let $\mathbb{G} \subseteq \mathbb{N}(N, A)$ *be a collection of directed networks and let* $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ *be a supernetwork. A subset* $\mathbb{F}_{\mathbf{G}}$ *of directed networks in* $\mathbb{G}$ *is said to be farsightedly consistent if*

$$\textit{for all } G_0 \in \mathbb{F}_{\mathbf{G}} \textit{ and } (m_{S_1}, (G_0, G_1)) \in \mathbf{G},$$

$$\textit{there exists } G_2 \in \mathbb{F}_{\mathbf{G}}$$

$$\textit{with } G_2 = G_1 \textit{ or } G_2 \rhd\rhd G_1 \textit{ such that,}$$

$$(p_{S_1}, (G_0, G_2)) \notin \mathbf{G}.$$

Thus, a subset of directed networks $\mathbb{F}_{\mathbf{G}}$ is farsightedly consistent if given any network $G_0 \in \mathbb{F}_{\mathbf{G}}$ and any m_{S_1}-deviation to network $G_1 \in \mathbb{G}$ by coalition S_1 (via adding, subtracting, or replacing arcs) there exists further deviations leading to some network $G_2 \in \mathbb{F}_{\mathbf{G}}$ where the initially deviating coalition S_1 is not better off - and possibly worse off.

There can be many farsightedly consistent sets. We shall denote by $\mathbb{F}^{\star}_{\mathbf{G}}$ the largest farsightedly consistent set. Thus, if $\mathbb{F}_{\mathbf{G}}$ is a farsightedly consistent set, then $\mathbb{F}_{\mathbf{G}} \subset \mathbb{F}^{\star}_{\mathbf{G}}$.

4.2 Nonemptiness of the Largest Farsightedly Consistent Set

Extending Chwe's existence and nonemptiness results to the supernetwork framework, we are able to conclude that any supernetwork contains a nonempty set of farsightedly consistent networks.

Theorem 1 *(*$\mathbb{F}^{\star}_{\mathbf{G}} \neq \emptyset$*)*

Let $\mathbb{G} \subseteq \mathbb{N}(N, A)$ *be a collection of directed networks. Given any supernetwork* $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$*, there exists a unique, nonempty, largest farsightedly consistent set*

$\mathbb{F}^{\star}_{\mathbf{G}}$. *Moreover,* $\mathbb{F}^{\star}_{\mathbf{G}}$ *is externally stable with respect to farsighted dominance, that is, if network* G *is contained in* $\mathbb{G} \backslash \mathbb{F}^{\star}_{\mathbf{G}}$*, then there exists a network* G' *contained in* $\mathbb{F}^{\star}_{\mathbf{G}}$ *that farsightedly dominants* G *(i.e.,* $G' \rhd\rhd G$*).*

Proof. The existence of a unique, largest farsightedly consistent set, $\mathbb{F}^{\star}_{\mathbf{G}}$, follows from Proposition 1 in Chwe [1]. Moreover, since the set of networks, $\mathbb{G}$, is finite and since each agent's preferences over networks are irreflexive, nonemptiness follows from the Corollary to Proposition 2 in Chwe [1]. Finally, the external stability of $\mathbb{F}^{\star}_{\mathbf{G}}$ with respect to farsighted dominance follows from Proposition 2 in Chwe [1]. ■

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