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The Partnered Core of an Economy and the Partnered Competitive Equilibrium

Frank H. Page, Jr.[a], Myrna Holtz Wooders[b,*]

[a]Department of Finance, Economics, and Legal Studies, University of Alabama, Tuscaloola, AL 35487, USA
[b]Department of Economics, University of Toronto, Toronto M5S 3G7, Canada

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Abstract

An allocation is partnered if it admits no asymmetric dependencies between players. We introduce the partnered core of an economy and the partnered competitive equilibrium. In an economy with unbounded consumption sets and local non-satiation we show that no unbounded arbitrage is sufficient for non-emptiness of the partnered core. With strictly concave utility functions we obtain a Second Welfare Theorem for the partnered competitive equilibrium and show that no unbounded arbitrage is necessary and sufficient for non-emptiness of the partnered core and the existence of a partnered competitive equilibrium.

Keywords: Partnership; General equilibrium; Core; Pareto-optimality; Necessary and sufficient conditions; Arbitrage

JEL classification: C62; C71; D51; D61; G12

1. Introduction

A natural property of a distribution rule for an economy is that it has the ability to prevent 'asymmetric dependencies'. For example, if one player needs to trade with another player to realize his core payoff, but the other player does not need to trade with the first, then there is an asymmetric dependency. An allocation that does not exhibit asymmetric dependencies is called a *partnered allocation*.

An allocation is in the partnered core if it is in the core and if, additionally, there are no asymmetric dependencies between any pair of players. The partnered core of a game with side payments was introduced in Reny et al. (1993) and the partnered core of a game without side payments was introduced in Reny and Wooders (1993a). We introduce the partnered core of an economy with unbounded consumption sets and show that Page's (1987) condition of no unbounded arbitrage is sufficient for the existence of at least one partnered core allocation.

* Corresponding author. Tel.: (416) 978-5255; fax: (416) 978-6713; e-mail: mwooders@chass.utoronto.ca

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No unbounded arbitrage is the condition that no group of agents can engage in unbounded, utility-non-decreasing and individually rational trades. For 'strictly reconcilable economies', where, at most, one agent may have half-lines in his indifference surfaces, we show that no unbounded arbitrage is necessary and sufficient for non-emptiness of the partnered core of the economy. Since the partnered core can refine the core significantly, our results are stronger than analogous results in Page and Wooders (1993).

The competitive payoff is not necessarily partnered. This is a significant weakness of the concept. If a group of players is dependent on another group of players at competitive prices, it is reasonable to suppose that the group in the stronger position, instead of taking prices as given, will attempt to gain a more favorable outcome for its members.[1] Bennett and Zame (1988, Theorem 3), however, demonstrate the important result that with strict convexity of preferences, the competitive outcome has the partnership property. We demonstrate that their result extends to economies with arbitrary consumption sets and possibly non-monotonicities. We also provide a Second Welfare Theorem for a partnered competitive equilibrium; every individually rational allocation is a partnered competitive equilibrium relative to some redistribution of endowments.

Further motivation for the partnered core is provided by Reny and Wooders (1995), who use the concept of partnership to provide an explanation of the division of organizations into not-necessarily-self-sufficient states, as in a 'commonwealth'.

With a strengthening of strict reconcilability to strict concavity of utility functions, we establish that no unbounded arbitrage is necessary and sufficient for the existence of a partnered competitive equilibrium and non-emptiness of the partnered core. For the case of strict concavity of utility functions, our results provide a sharpening of Werner's (1987) result that no unbounded arbitrage is necessary and sufficient for the existence of a competitive equilibrium and of our result in Page and Wooders (1993, 1996) that no unbounded arbitrage is necessary and sufficient for non-emptiness of the core.

Further discussion of related literature on arbitrage is contained in Section 5. Here we remark only that in this paper we require concavity of utility functions and unbounded consumption sets. This is because of our treatment of the partnered core and the partnered competitive equilibrium, rather than the usual notions. In Page and Wooders (1996), however, we allow arbitrary convex consumption sets and in Page and Wooders (1995) we relax the assumption that preferences are representable by concave utility functions.

2. An economy with unbounded consumption sets and non-monotonicities

Let $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ denote an *exchange economy*. Each agent j has *consumption set* $\Re^L$ and *endowment* $\omega_j \in \Re^L$. The jth agent's preferences over $\Re^L$ are specified via a *utility function* $u_j(\cdot): \Re^L \to \Re$.[2]

[1] An example of a non-partnered competitive equilibrium is developed in Reny and Wooders (1993c), where the partnered competitive equilibrium was identified as a distinct concept and assigned its name.

[2] Our assumptions are chosen for brevity and clarity; in other research we relax several of the assumptions of this paper, including the representation of preferences by concave utility functions. See, especially, Page and Wooders (1994a, 1995).

The set of *individually rational allocations* is given by

$$A = \left\{ (x_1, \ldots, x_n) \in \Re^L \times \ldots \times \Re^L : \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \omega_j \text{ and } u_j(x_j) \geq u_j(\omega_j) \text{ for all } j \right\} .$$

Given any individually rational allocation $x = (x_1, \ldots, x_n) \in A$, let $pr_j(x) = x_j$. For each $x_j \in \Re^L$ the *preferred set* is $P_j(x_j) := \{x' \in \Re^L : u_j(x') > u_j(x_j)\}$. We assume that utility functions satisfy *non-satiation at rational allocations*; that is, for each $j \in N$, $P_j(x_j) \neq \emptyset$ for all $x_j \in pr_j(A)$. Note that for concave utility functions the following implication holds:

$$z_j \in P_j(x_j) \text{ implies that } tz_j + (1-t)x_j \in P_j(x_j), \quad \text{for all } t \in (0, 1] .$$

Thus, concavity together with non-satiation at rational allocations implies *local non-satiation at rational allocations*: given any $\epsilon > 0$, $B_\epsilon(x_j) \cap P_j(x_j) \neq \emptyset$ for all j and for all $x_j \in pr_j(A)$, where $B_\epsilon(x_j)$ denotes the open ball of radius ϵ centered at x_j.

An economy is *reconcilable* if for each $j = 1, \ldots, n$, $u_j(\cdot)$ is continuous, concave, and satisfies *non-satiation at rational allocations*. The problem of non-existence of an equilibrium in an economy with unbounded consumption sets is that the preferences of agents may be too dissimilar to be reconciled by a price system. In a reconcilable economy satisfying no unbounded arbitrage, arbitrage opportunities can be eliminated by a price system.

Given prices $p \in \mathscr{B} := \{p' \in \Re^L : \|p'\| \leq 1\}$ the *budget set* for the jth agent is given by $B(p, \omega_j) = \{x \in \Re^L : \langle p, x \rangle \leq \langle p, \omega_j \rangle\}$, where $\langle p, x \rangle = \Sigma_{\ell=1}^{L} p_\ell \cdot x_\ell$. An *equilibrium* for the economy $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ is an $(n+1)$-tuple of vectors $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ such that: (i) $(\bar{x}_1, \ldots, \bar{x}_n) \in A$; (ii) $\bar{p} \in \mathscr{B} \backslash \{0\}$; and (iii) for each j, $\langle \bar{p}, \bar{x}_j \rangle = \langle \bar{p}, \omega_j \rangle$ and $P_j(\bar{x}_j) \cap B(\bar{p}, \omega_j) = \emptyset$.

An allocation $x = (x_1, \ldots, x_n)$ is *Pareto optimal* if there does not exist another allocation $x' = (x'_1, \ldots, x'_n)$ such that, for all j, $u_j(x'_j) \geq u_j(x_j)$ and for at least one j, $u_j(x'_j) > u_j(x_j)$. With local non-satiation at rational allocations and unbounded consumption sets, an allocation $x = (x_1, \ldots, x_n)$ is *Pareto optimal* if and only if there does not exist another allocation $x' = (x'_1, \ldots, x'_n)$ such that $u_j(x'_j) > u_j(x_j)$ for all j.

2.1. No unbounded arbitrage

We introduce Page's condition of no unbounded arbitrage (see Page, 1987; Nielsen, 1989, and other papers by Page and his co-authors). The jth *agent's arbitrage cone* is the closed convex cone[3] containing the origin given by

$$\mathscr{C}P_j = \{y \in \Re^L : \text{for some } x \in \Re^L,\ u_j(x + \lambda y) \text{ is non-decreasing in } \lambda \text{ for } \lambda \geq 0\} .$$

An economy $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ satisfies *no unbounded arbitrage* if

(2.1) whenever $\Sigma_{j=1}^n y_j = 0$ and $y_j \in \mathscr{C}P_j$ for all agents j, it holds that $y_j = 0$ for all agents j.

[3] The arbitrage cone is the recession cone of the preferred set; this, and the fact that the arbitrage cone is closed and convex, follows from results in Rockafellar (1970, Section 8).

For $x \in \Re^L$, define the *increasing cone for the jth agent*[4] by

$$I_j(x) = \{y \in \mathscr{C}P_j\colon \text{for all } \lambda \geq 0 \text{ there exists } \lambda' \text{ such that } u_j(x + \lambda' y) > u_j(x + \lambda y)\}\,.$$

The jth agent satisfies *extreme desirability* if for any $x \in \Re^L$ it holds that $I_j(x) = \mathscr{C}P_j \backslash \{0\}$.[5] The economy $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ satisfies *extreme desirability* if at least $n - 1$ agents satisfy extreme desirability.

An economy is *strictly reconcilable* if it is reconcilable and satisfies extreme desirability. We say that the economy is strictly reconcilable in this case since, in addition to being reconcilable, the existence of a price system that reconciles the diverse wants of agents implies the elimination of arbitrage opportunities.

3. Partnership, the core and equilibrium

3.1. Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of players and let P be a collection of subsets of N. For each i in N let $P_i = \{S \in P\colon i \in S\}$. The collection P is *partnered* (for N) if for each i in N the set P_i is non-empty and for each pair of players i and j in N the following requirement is satisfied: if $P_i \subset P_j$ then $P_j \subset P_i$. The two players i and j are *partners* (or *i is partnered with j*) if $P_i = P_j$. Note that the set of partners of an agent could be just the agent himself or it could be as large as the total agent set.

For any coalition $S \subset N$, the set of *S-allocations* is given by $A(S) = \{(x_j)_{j\in S}\colon \Sigma_{i\in S} x_i = \Sigma_{j\in S}\, \omega_j$ and, for each $j \in S$, $x_j \in \Re^L$ and $u_j(x_j) \geq u_j(\omega_j)\}$. Corresponding to the set of S-allocations is a set of *utility possibilities* given by $U(A(S)) = \{(u_j)_{j\in S}\colon$ for some $(x_j)_{j\in S}$ in $A(S)$ it holds that $u_j = u_j(x_j)$ for each $j \in S\}$. Now for each coalition of agents $S \subset N$ define

$$V(S) = \{(u_1, \ldots, u_n)\colon \text{there exists } (u_j')_{j\in S} \in U(A(S)) \text{ such that } (u_j \leq u_j' \text{ for all } j \in S\}\,.$$

The pair (N, V) is *the game induced by the economy* $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$. The *core of the game* (N, V) is defined as

$$C(N, V) = \{u \in V(N)\colon \text{there does not exist a coalition } S \subset N \text{ and a vector } u' \in V(S) \text{ such that } u_j' \geq u_j \text{ for all } j \in S \text{ with strict inequality for at least one } j\}\,.$$

Note that the notion of the core used here requires that only one player in an improving coalition is better off, as in Debreu and Scarf (1963), for example, rather than the notion of the core used in Reny and Wooders (1993a), which requires that *all* members of an improving coalition be better off. For the class of games derived from unbounded economies where

[4] See Page and Wooders (1996) for a discussion of the increasing cone.

[5] In Monteiro et al. (1995) it is shown that an agent's utility function satisfies extreme desirability if and only if there are no half-lines in the indifference surfaces.

utility functions satisfy non-satiation at rational allocations, the two notions coincide. In particular, if one player in an improving coalition can be made better off, then all players in the coalition can be made better off; we can take small amounts of commodities away from the player who is strictly better off and carry out a redistribution so that all players are better off.

3.2. The partnered core

Let $u \in \Re^n$ and define $\mathscr{S}(u) := \{S \subset N : u \in V(S)\}$. The utility vector $u \in \Re^n$ is *partnered* if the collection of coalitions $\mathscr{S}(u)$ is partnered. Let $P(N, V) = \{u \in \Re^n : \mathscr{S}(u) \text{ is partnered}\}$. The *partnered core* of a game (N, V), denoted by $C^*(N, V)$, is given by

$$C^*(N, V) = P(N, V) \cap C(N, V) .$$

Let $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ be an economy. An allocation $(x_1, \ldots, x_n) \in A$ is in the *partnered core of the economy*, denoted by $C^*((\Re^L, \omega_j, u_j(\cdot))_{j=1}^n)$, if $(u_1(x_1), \ldots, u_n(x_n)) \in C^*(N, V)$.

Theorem 1. No unbounded arbitrage is sufficient for non-emptiness of the partnered core. Let $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ be a reconcilable economy satisfying local non-satiation. If no unbounded arbitrage holds, then there exists an allocation $x^* = (x_1^*, \ldots, x_n^*) \in A$ in the partnered core of the economy and $u^* := (u_1(x_1^*), \ldots, u_n(x_n^*))$ is in the partnered core of the game induced by the economy.

To show non-emptiness of the core in Page and Wooders (1993) we were able to appeal to the result that an equilibrium is in the core. The competitive equilibrium is not, in general, partnered. Thus, our proof of Theorem 1 appeals to the Reny and Wooders (1993a) result that a balanced game has a non-empty partnered core.[6] All proofs are contained in Section 5.

Theorem 2. In a strictly reconcilable economy, no unbounded arbitrage is necessary and sufficient for non-emptiness of the partnered core. Let $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ be a strictly reconcilable economy satisfying local non-satiation. Then the following three statements are equivalent:

(3.1) *The partnered core of the economy, $C^*((\Re^L, \omega_j, u_j(\cdot))_{j=1}^n)$, is non-empty.*

(3.2) *The partnered core of the game induced by the economy, $C^*(N, V)$, is non-empty.*

(3.3) *The economy satisfies no unbounded arbitrage* (2.1).

We could add to the above list, (3.1) to (3.3), the statements that the economy has an equilibrium and the economy has a Pareto-optimal allocation, but these results are shown under more general conditions in Page and Wooders (1996).

[6] Since the partnered core is a relatively new concept, the mathematical tools required for a direct proof of our result have not yet been determined. Further research aimed in this direction is in progress; see Reny and Wooders (1993b) and Kannai and Wooders (1995).

3.3. The partnered competitive equilibrium

Let $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ be an equilibrium for the economy $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$. We say that $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is a *partnered competitive equilibrium* if (a) $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is an equilibrium and (b) $\mathcal{S}(\bar{x}_1, \ldots, \bar{x}_n) := \{S \subset N: \Sigma_{j\in S} \bar{x}_j = \Sigma_{j\in S} \omega_j\}$ is partnered. Bennett and Zame (1988) demonstrate that with strict convexity of preferences, the competitive equilibrium has the partnership property. Our result below extends the Bennett and Zame result to unbounded economies. Our assumptions require that preferences be representable by strictly concave utility functions but this stronger assumption is not necessary for the following result. For $j \in N$, the function $u_j(\cdot)$ *is strictly concave* if for any x, $x' \in \Re^L$, $x \neq x'$, it holds that $u_j(\lambda x + (1-\lambda)x') > \lambda u_j(x) + (1-\lambda)u_j(x')$ for any $\lambda \in (0, 1)$.[7]

Theorem 3. The partnership property of a competitive equilibrium. Let $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ be a reconcilable economy such that, for each agent j, $u_j(\cdot)$ is strictly concave. If $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is an equilibrium for the economy $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ then $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is a partnered competitive equilibrium.

Our next result is a Second Welfare Theorem for the partnered competitive equilibrium[8]

Theorem 4. A Second Welfare Theorem for the partnered competitive equilibrium. Let $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ be a reconcilable economy such that, for each agent j, $u_j(\cdot)$ is strictly concave. If $(\bar{x}_1, \ldots, \bar{x}_n)$ is an individually rational Pareto-optimal allocation for the economy $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$, then there is a price system $\bar{p}$ such that $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is a partnered competitive equilibrium relative to some redistribution of the initial endowment.

The following theorem is an immediate consequence of the fact that strict concavity of utility functions implies extreme desirability, the proof of sufficiency in Werner (1987), and Theorems 2 and 3.

Theorem 5. Necessary and sufficient conditions for the existence of a partnered competitive equilibrium and non-emptiness of the partnered core. Let $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ be a reconcilable economy such that, for each agent j, $u_j(\cdot)$ is strictly concave. Then the three statements of Theorem 2 are equivalent to the statement that (3.4). *The economy has a partnered competitive equilibrium.*

4. Further relationships to the literature

As discussed in detail in Page and Wooders (1994a) a number of authors have studied the necessary and sufficient conditions for the existence of an equilibrium, cf. Hart (1974) and

[7] When utility functions are strictly concave, we also say that preferences are strictly convex.

[8] In Page and Wooders (1996) we provide a Second Welfare Theorem with the usual notion of a competitive equilibrium in an unbounded economy.

Hammond (1983). For strictly concave preferences, our results in this paper refine those of Werner (1987) since we show that with a strict concavity of preferences no unbounded arbitrage is necessary and sufficient for the existence of a *partnered* competitive equilibrium. Chichilnisky (1995) is also related; see Monteiro et al. (1995) for a detailed discussion. Page and Wooders (1993) provide a proof of Werner's claim that the intersection form of no unbounded arbitrage is necessary for the existence of an equilibrium and a short proof of sufficiency based on Page's previous papers. In addition, Page and Wooders (1993) show that the intersection form of no unbounded arbitrage is necessary and sufficient for the existence of an equilibrium in an economy where *all* agents' utility functions satisfy extreme desirability.

Necessary and sufficient conditions for non-emptiness of the core has been a topic of interest since the works of Bondareva (1962) and Shapley (1967). Other papers on this topic include Kaneko and Wooders (1982), Keiding and Thorlund-Petersen (1987) and Keiding (1992). Page and Wooders (1993) show that no unbounded arbitrage is necessary and sufficient for non-emptiness of the core in a strictly reconcilable economy where *all* agents' utilities satisfy extreme desirability.[9] Chichilnisky (1994) is related; again see Monteiro et al. (1995) for detailed discussions.

Since the partnered core and the partnered competitive equilibrium refine the core and the competitive equilibrium, for the case of strictly concave preferences the results in this paper on the existence of an equilibrium are stronger than the previous results in the literature.

On more technical matters, we remark that for our treatment of the partnered core we required local non-satiation and unbounded consumption sets. This enabled us to use the Reny and Wooders (1993a) result that a balanced game has a non-empty core (where the core concept requires that in an improving coalition all agents be better off). The concavifiability of preferences is not required, however, for the results of this paper and the strict concavity of Theorems 3 and 4 can be replaced by strict convexity of preferred sets. See Page and Wooders (1993, 1995) where a uniformity assumption on the arbitrage cones enables us to relax the concavifiabilty of preferences. Additional relevant technical remarks appear in Page and Wooders (1996).

5. Proofs

Proof of Theorem 1. To prove Theorem 1 we need only observe that from the corollary to Theorem 1 of Page and Wooders (1996) it follows that the game derived from the economy is well defined. In particular, condition (1.4) of Reny and Wooders (1993a) is satisfied. Since the game is derived from an economy with concave utilities (indeed, convexity of preferences will suffice), it is balanced. (See, for example, Scarf, 1967, pp. 53–54).

From local non-satiation at rational allocations and the unboundedness of consumption sets, the core of the induced game coincides with the core according to the definition in Reny and Wooders (1993a), where *all* players must be better off in an improving coalition. With a normalization of utilities so that the utility of each agent from his endowment is greater than

[9] We note that an earlier version of Koutsougeras (1995) appears to have been the first paper studying a notion of the core in the economies with short sales.

zero, it then follows by standard methods that the derived game satisfies all of the conditions of Reny and Wooders (1993a) and the partnered core is non-empty. □

Proof of Theorem 2. That (3.3) implies (3.1) follow from Theorem 1. To show that (3.1) implies (3.3) consider the following. Let $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ be a strictly reconcilable economy and let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ be an allocation in the partnered core of the economy. Suppose that no unbounded arbitrage is not satisfied. Then there is an n-tuple of net trades $(y_1, \ldots, y_n) \neq (0, \ldots, 0)$ in $\mathscr{C}P_1 \times \ldots \times \mathscr{C}P_n$ such that $\Sigma_j y_j = 0$.

Moreover, for at least two of these net trade vectors, say $y_{j'}$ and $y_{j''}$, it holds that $y_{j'} \neq 0$ and $y_{j''} \neq 0$. By strict reconcilability, for at least one agent, say j', there is a positive number λ such that $u_{j'}(\bar{x}_{j'} + \lambda y_{j'}) > u_{j'}(\bar{x}_{j'})$ and, since $(y_1, \ldots, y_n) \in \mathscr{C}P_1 \times \ldots \times \mathscr{C}P_n$, it holds that $u_j(\bar{x}_j + \lambda y_j) \geq u_j(\bar{x}_j)$ for all $j \in N$. Thus, the set N of all agents can improve upon the allocation $\bar{x}$ with the allocation $(\bar{x}_1 + \lambda y_1, \ \ldots, \ \bar{x}_n + \lambda y_n)$, contradicting the assumption that $\bar{x}$ is in the partnered core of the economy. The equivalence of (3.1) and (3.2) follows from the definition of the partnered core of the economy. □

Proof of Theorem 3. Let $(\Re^L, \omega_j, u_j(\cdot))_{j=1}^n$ be an economy where for each agent j, $u_j(\cdot)$ is strictly concave. Let $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ be an equilibrium. We first show that given any proper subcoalition $S \subset N$, $S \neq \emptyset$, if for some S-allocation $(x_j)_{j\in S}$ it holds that $u_j(x_j) \geq u_j(\bar{x}_j)$ for all $j \in S$, then for some $N \backslash S$-allocation $(x_j)_{j \in N\backslash S}$ it holds that $u_j(x_j) \geq u_j(\bar{x}_j)$ for all $j \in N\backslash S$. Let S be a proper subcoalition for which there is an S-allocation $(x_j)_{j\in S}$ satisfying $u_j(x_j) \geq u_j(\bar{x}_j)$ for all $j \in S$. Since $\Sigma_{j\in S} x_j = \Sigma_{j\in S} \omega_j$, it holds that $\bar{p} \cdot \Sigma_{j\in S} x_j = \bar{p} \cdot \Sigma_{j\in S} \omega_j$. Since $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is an equilibrium, it holds that $\bar{p} \cdot \Sigma_{j\in S} \bar{x}_j = \bar{p} \cdot \Sigma_{j\in S} \omega_j$. Since utility functions are strictly concave, it follows that $\bar{p} \cdot x_j \geq \bar{p} \cdot \bar{x}_j$ for all $j \in S$ and, if $\bar{p} \cdot x_j = \bar{p} \cdot \bar{x}_j$, then $x_j = \bar{x}_j$ (otherwise, player j could afford a preferred commodity bundle, consisting of a convex combination of x_j and $\bar{x}_j$). But since $\bar{p} \cdot x_j \geq \bar{p} \cdot \bar{x}_j$ for all $j \in S$ and since $\bar{p} \cdot \Sigma_{j\in S} x_j = \bar{p} \cdot \Sigma_{j\in S} \omega_j = \bar{p} \cdot \Sigma_{j\in S} \bar{x}_j$, it follows that $\bar{p} \cdot x_j = \bar{p} \cdot \bar{x}_j$ and $x_j = \bar{x}_j$ for all $j \in S$. But then it follows that $\Sigma_{j\in N\backslash S} \bar{x}_j = \Sigma_{j \in N\backslash S} \omega_j$ and $(\bar{x}_j \colon j \in N\backslash S)$ is an $N\backslash S$-allocation. It is immediate that $\mathscr{S}(\bar{x}_1, \ldots, \bar{x}_n)$ is partnered. Therefore, $(\bar{x}_1, \ldots, \bar{x}_n, \bar{p})$ is a partnered competitive equilibrium. □

Proof of Theorem 4. Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ be an economy where all agents have strictly concave utility functions. Let $x' = (x'_1, \ldots, x'_n)$ be an individually rational and Pareto-optimal allocation. For each agent j define $\omega'_j := \bar{x}_j$ and $\omega' = (\omega'_1, \ldots, \omega'_n)$. We claim that x' is in the core of $(X_j, \omega'_j, u_j(\cdot))_{j=1}^n$. For suppose there is a coalition S that can improve. (Note that the consumptions of the members of the complementary coalition can remain unchanged when S forms its own feasible allocation, since the members of the complementary coalition are consuming their endowments.) Thus, we have a contradiction to the supposition that x' is a Pareto-optimal allocation. Therefore, x' is in the core of the economy $(X_j, \omega'_j, u_j(\cdot))_{j=1}^n$. Since its core is non-empty, the economy $(X_j, \omega'_j, u_j(\cdot))_{j=1}^n$ satisfies no unbounded arbitrage. Thus, the economy $(X_j, \omega'_j, u_j(\cdot))_{j=1}^n$ has an equilibrium, say $(\bar{x}, \bar{p})$. Since $(\bar{x}, \bar{p})$ is an equilibrium, for each agent j it holds that $u_j(\bar{x}_j) \geq u_j(\omega'_j)$. Suppose that for one (or more) agents it holds that $u_j(\bar{x}_j) > u_j(\omega'_j)$. We then have a contradiction to the Pareto-optimality of x'. Thus, $u_j(\bar{x}_j) = u_j(\omega'_j)$. Since $(\bar{x}, \bar{p})$ is an equilibrium, for any x''_j such that $u_j(x''_j) > u_j(\bar{x}_j)$, it holds that

$\bar{p} \cdot x_j'' > \bar{p} \cdot \omega_j'$. Thus, since $u_j(\bar{x}_j) = u_j(\omega_j')$, for any x_j'' such that $u_j(x_j'') > u_j(\omega_j')$, it holds that $\bar{p} \cdot x_j'' > \bar{p} \cdot \omega_j'$. Therefore $(\omega', \bar{p})$ is an equilibrium. From Theorem 3, the equilibrium is partnered.

Proof of Theorem 5. For the proof that (3.3) implies the existence of a competitive equilibrium, see Nielsen (1989) or Werner (1987),[10] or, for a very short proof, Page and Wooders (1996). From Theorem 3, the equilibrium allocations are all partnered. Thus (3.3) implies (3.4). That (3.4) implies (3.1) follows from the observation that the partnered competitive equilibrium is in the partnered core. □

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References

Bennett, E. and W.R. Zame, 1988, Bargaining in cooperative games, International Journal of Game Theory 17, 279–300.

Bondareva, O., 1962, Theory of the core in an *n*-person game, Vestnik, LGU13, 141–142 (in Russian), Leningrad State University, Leningrad.

Chichilnisky, G., 1994, Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium and the core, and limits voting cycles.

Chichilnisky, G., 1995, Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium with or without short sales, Economic Theory 5, 79–107.

Debreu, G. and H. Scarf, 1963, A limit theorem on the core of an economy, International Economic Review 4, 235–246.

Hammond, P.J., 1983, Overlapping expectations and Hart's condition for equilibrium in a securities model, Journal of Economic Theory 31, 170–175.

Hart, O.D., 1974, On the existence of equilibrium in a securities model, Journal of Economic Theory 9, 293–311.

Kaneko, M. and M.H. Wooders, 1982, Cores of partitioning games, Mathematical Social Sciences 3, 313–327.

Kannai, Y. and M.H. Wooders, 1995, A further extension of the KKMS theorem, submitted.

Keiding, H., 1992, Homological methods in core theory, typescript.

Keiding, H. and L. Thorlund-Petersen, 1987, The core of a cooperative game without side payments, Journal of Optimization Theory 54, 273–288.

[10] Owing to the assumption of strict concavity for all agents made in this theorem, the intersection form of no unbounded arbitrage, used in Werner (1987), is equivalent to Page's condition.

Koutsougeras, L., 1995, A two stage core with applications to asset markets and differential information economies, CORE Discussion Paper No. 9525.
Monteiro, P., F.H. Page, Jr. and M.H. Wooders, 1995, Arbitrage, equilibrium, and gains from trade; A counterexample, Journal of Mathematical Economics, in press.
Nielsen, L.T., 1989, Asset market equilibrium with short-selling, Review of Economic Studies 56, 467–474.
Page, F.H., Jr., 1987, On equilibrium in Hart's securities exchange model, Journal of Economic Theory 41, 392–404.
Page, F.H., Jr. and M.H. Wooders, 1993, Arbitrage in markets with unbounded short sales: Necessary and sufficient conditions for nonemptiness of the core and existence of equilibrium, Working Paper No. 9409, University of Toronto.
Page, F.H., Jr. and M.H. Wooders, 1994a, Arbitrage in markets with price dependent preferences; equilibrium and market stability, Working Paper No. 247, Department of Economics, Finance, and Legal Studies, University of Alabama.
Page, F.H., Jr. and M.H. Wooders, 1994b, The partnered core of an economy, Discussion Paper, Autònoma University of Barcelona and University of Tilburg.
Page, F.H., Jr. and M.H. Wooders, 1995, On necessary and sufficient conditions for compactness of feasible outcomes and existence of equilibrium; an extension of existing results, submitted.
Page, F.H., Jr. and M.H. Wooders, 1996, A necessary and sufficient condition for compactness of individually rational and feasible outcomes and the existence of an equilibrium, Economics Letters 52, 153–162.
Reny, P.J. and M.H. Wooders, 1993a, The partnered core of a game without side payments, Journal of Economic Theory, in press.
Reny, P.J. and M.H. Wooders, 1993b, An extension of the KKMS theorem, Working Paper, Department of Finance, Economics, and Legal Studies, University of Alabama.
Reny, P.J. and M.H. Wooders, 1993c, Partnership and competition, notes in manuscript
Reny, P.J. and M.H. Wooders, 1995, Credible threats of secession, partnership, and commonwealths, in: W. Albers, W. Güth, P. Hammerstein, B. Moldovanu, E. Van Damme, eds., Understanding strategic interaction: Essays in Honor of Reinhard Selten, (Springer, Berlin), in press.
Reny, P.J., E. Winter and M.H. Wooders, 1993, The partnered core of a game with side payments, Discussion Paper, University of Western Ontario.
Rockafellar, R.T., 1970, Convex analysis (Princeton University Press, Princeton, NJ).
Scarf, H., 1967, The core of an *n*-person game, Econometrica 35, 50–69.
Shapley, L.S., 1967, On balanced sets and cores, Naval Research Logistics Quarterly 14, 453–460.
Werner, J., 1987, Arbitrage and existence of competitive equilibrium, Econometrica 55, 1403–1418.