JOURNAL OF ECONOMIC THEORY **70**, 298–311 (1996)
ARTICLE NO. 0091

The Partnered Core of a Game without Side Payments*

Philip J. Reny

Department of Economics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

and

Myrna Holtz Wooders

Department of Economics, University of Toronto, Toronto, Ontario M5S 1A1

Received February 10, 1994; revised March 22, 1995

A payoff for a game is partnered if it admits no asymmetric dependencies. We introduce the partnered core of a game without side payments and show that the partnered core of a balanced game is nonempty. The result is a strengthening of Scarf's Theorem on the nonemptiness of the core of a balanced game without side payments. In addition, it is shown that if there are at most a countable number of points in the partnered core of a game then at least one core point is *minimally* partnered, meaning that no player requires any other player in particular to obtain his part of the core payoff. *Journal of Economic Literature* Classification Number: C71.

1. INTRODUCTION: THE PARTNERSHIP PROPERTY

A natural property of a solution concept is that it admits no "asymmetric dependencies." A solution displays an asymmetric dependency if one player needs the presence of a second player to realize his payoff in the solution, but the second player does not need the presence of the first. When a player i is dependent on another player j in this sense, but j is not dependent on i, then j is in a position to attempt to obtain a larger share of the surplus from i. Consider, for example, a two-person divide-the-dollar bargaining game. Any division giving the entire dollar to one participant displays an asymmetric dependency; the player receiving the dollar is dependent on the player receiving zero. The player receiving zero is not compelled to join the

* This paper is a revision of University of Toronto Department of Economics Working Paper No. 9404. The authors gratefully acknowledge the financial support of the S.S.H.R.C. The first author also thanks the Faculty of Arts and Sciences of the University of Pittsburgh for financial support. The authors also express their thanks to Elaine Bennett for comments on an earlier version of this paper.

0022-0531/96 $18.00

two-person coalition to receive his part of the payoff. In contrast, to achieve the payoff of 50 cents for each player the two-person coalition is compelled to form—the players are partnered.

A payoff is in the partnered core if it is in the core, and if additionally there are no asymmetric dependencies between any pair of players. In this paper we introduce the partnered core of a game without side payments and show that balancedness guarantees the existence of at least one partnered core payoff. Thus, we obtain a refinement of Scarf's [13] Theorem. It should be noted that our method of proof makes substantial use of the elegant and powerful techniques developed in Shapley and Vohra [15], and also relies on an ingenious construction found in Bennett and Zame [6].

For games without side payments, the partnered core may significantly refine the core. This is in contrast to the case of games with side payments. Reny *et al.* [11] establish that for games with side payments the relative interior of the core is contained in the partnered core.[1] In Section 2 below, we provide an example of a balanced game without side payments whose core is convex, yet no point in its relative interior is partnered. Moreover, in the example the partnered core refines the core from a continuum of points to a single point.

A partnership consists of a group of players who are mutually dependent upon one another. More precisely, consider a collection of subsets of a player set with the property that all the subsets in the collection containing player i also contain player j. In this case, we say that "i depends on j." If j also depends on i, players i and j are "partners" with respect to the collection. If j does not depend on i, then "i asymmetrically depends on j." A collection of subsets of a player set is partnered if (i) each player is in some subset in the collection and (ii) there are no asymmetric dependencies. For each i, the set of all players who depend on i in a partnered collection of subsets is called a partnership. A payoff for a game is partnered if the collection of coalitions that can attain that payoff for their members, that is, the collection of supporting coalitions, is partnered.

Given a partnered payoff, two players are partners if each depends on the other. The payoff is minimally partnered if no player depends on any other (particular) player. That is, a partnered payoff is minimally partnered if the only partner of each player is the player himself.

The property of minimal partnership appears to be of special importance in the study of competitive economies. This motivates our study of payoffs that are minimally partnered. We show that if a game has at most a countable number of core payoffs, then there is at least one core payoff that is

[1] Recall that the relative interior of a d-dimensional convex subset of R^n is the largest d-dimensional open set contained in the subset.

minimally partnered. Further motivation for the study of partnered and minimally partnered payoffs comes from the result that a competitive allocation is partnered (Bennett and Zame [6]).

A partnership structure, that is, a partition of the player set into partnerships, is distinct from a coalition structure, a partition of the set of players into coalitions, and our concepts create distinctions not found in the literature on coalition structure games (c.f., Aumann and Drèze [3]). A partnership describes a group of players who are *compelled* to join together. In our concluding discussion we provide an example illustrating this distinction.

2. GAMES WITHOUT SIDE PAYMENTS

We consider cooperative games in characteristic form. Let $N = \{1, ..., n\}$ denote a set of players. A nonempty subset of N is called a coalition. For any coalition S let R^S denote the $|S|$-dimensional Euclidean space with coordinates indexed by the elements of S. For $x \in R^N$, x_S will denote its restriction on R^S. To order vectors in R^N we use the symbols $\gg$, $>$, $\geqslant$ with the usual interpretations. The nonnegative orthant of R^N is denoted by R^N_+, and the strictly positive orthant by R^N_{++}. For any set $Y \subseteq R^N$, co Y and ∂Y will denote its convex hull and boundary, respectively. Each coalition S has a feasible set of payoffs or utilities denoted by $V_S \subseteq R^S$. It is convenient to describe the feasible utilities of a coalition as a subset of R^N. For each coalition S let

$$V(S) = \{x \in R^N ; x_S \in V_S\};$$

that is, $V(S)$ is a cylinder in R^N and can alternatively be defined as $V_S \times R^{N \backslash S}$.

A (normalized) game without side payments (or simply a game) is a pair (N, V) where the correspondence $V\colon 2^N \backslash \{\phi\} \mapsto R^N$ satisfies the following properties:

(1.1) $V(S)$ is non-empty and closed for all $S \subseteq N$,

(1.2) $V(S)$ is comprehensive for all $S \in 2^N$ in the sense that $V(S) = V(S) - R^N_+$,

(1.3) $V(S)$ is cylindrical in the sense that if $x \in V(S)$ and $y \in R^N$ such that $y_S = x_S$, then $y \in V(S)$.

(1.4) there exists $v^0 \in R^N$ such that $v^0 \gg 0$ and for every $j \in N$, $V(\{j\}) = \{x \in R^N : x_j \leqslant v^0_j\}$,

(1.5) $V(S)$ is "bounded" for all $S \subseteq N$ in the sense that there exists a real number $q > 0$ such that if $x \in V(S)$ and $x_S \geqslant 0$, then $x_i < q$ for all $i \in S$.

A payoff x is *undominated* if for all $S \subseteq N$ and $y \in V(S)$ it is not the case that $y_S \gg x_S$; it is *feasible* if $x \in V(N)$. The *core* of a game (N, V), denoted $C(N, V)$, consists of the set of all feasible and undominated payoffs.

Let N be a finite set of players and let P be a collection of subsets of N. For each i in N let

$$P_i = \{S \in P : i \in S\}.$$

We say that P has the partnership property (for N) if for each i in N the set P_i is nonempty and for each pair of players i and j in N the following requirement is satisfied:

$$\text{if } P_i \subset P_j \text{ then } P_j \subset P_i.$$

That is, if all the coalitions in P that contain player i also contain player j then all the coalitions that contain j also contain i. Two players i and j are partners (or i is partnered with j) if $P_i = P_j$. Clearly, the relation "is partnered with" is an equivalence relation. Consequently, let $\mathscr{P}[i]$ denote the equivalence class containing i's partners, and call $\mathscr{P}[i]$ a partnership. For any collection P with the partnership property we say that P is minimally partnered if, for each player i, $\mathscr{P}[i] = \{i\}$.

Let (N, V) be a game and let x be a payoff for (N, V). A coalition S is said to support the payoff x if $x \in V(S)$. Let $\mathscr{S}(x)$ denote the set of coalitions supporting the payoff x. The payoff x is called a partnered payoff if the collection $\mathscr{S}(x)$ has the partnership property. The payoff x is minimally partnered if it is partnered and if the set of supporting coalitions is minimally partnered. Note that partnered payoffs need not be feasible.

Let $P(N, V)$ denote the set of all partnered payoffs for the game (N, V). The partnered core is denoted by $C^*(N, V)$ and is defined by

$$C^*(N, V) = P(N, V) \cap C(N, V).$$

For any $S \subseteq N$ let e^S denote the vector in R^N whose ith coordinate is 1 if $i \in S$ and 0 otherwise. For ease in notation we denote $e^{\{i\}}$ by e^i.

Let Δ denote the unit simplex in R^N. For every $S \subseteq N$ define

$$\Delta^S = \text{co}\{e^i \colon i \in S\}.$$

For each $S \subseteq N$ define

$$m^S = \frac{e^S}{|S|};$$

these are the centers of gravity of the respective sets Δ^S as well as of the sets $\{e^i : i \in S\}$.

Let β be a collection of coalitions. The collection is balanced if there exist nonnegative weights $\{\lambda^S\}_{S \in \beta}$ such that

$$\sum_{S \in \beta} \lambda^S e^S = e^N.$$

Observe that the collection β is balanced if and only if

$$m^N \in \mathrm{co}\{m^S : S \in \beta\}.$$

A game is balanced if for any balanced collection β

$$\bigcap_{S \in \beta} V(S) \subseteq V(N).$$

Scarf [13] showed that a balanced game has a nonempty core. Our first Theorem extends Scarf's result and shows that a balanced game has a nonempty partnered core. The proof is provided in Section 4, along with the proof of our second result.

THEOREM 2.1. *A balanced game has a nonempty partnered core.*

For games with side payments, as shown in Reny *et al.* [11], the nonemptiness of the partnered core is a consequence of the fact that the collection of supporting coalitions for any payoff in the relative interior of the core of a game is strictly balanced and thus all payoffs in the relative interior of the core of a game with side payments are partnered. For games without side payments, the situation is quite different. This is illustrated by the following example.

EXAMPLE 2.2 (A balanced game with no partnered payoff in the relative interior of the core). Consider the game given by

$$\begin{aligned} V(\{1,2,3\}) &= \{(x_1,x_2,x_3) : x_1+x_2+x_3 \leqslant 1\} \\ V(\{1,2\}) &= \{(x_1,x_2,x_3) : x_1+2x_2 \leqslant 1,\ x_1 \leqslant 1,\ x_2 \leqslant \tfrac{1}{2}\} \\ V(\{1,3\}) &= \{(x_1,x_2,x_3) : x_1+2x_3 \leqslant 1,\ x_1 \leqslant 1,\ x_3 \leqslant \tfrac{1}{2}\} \\ V(\{2,3\}) &= \{(x_1,x_2,x_3) : x_2 \leqslant 0,\ x_3 \leqslant 0\} \\ V(\{i\}) &= \{(x_1,x_2,x_3) : x_i \leqslant 0\}, \qquad i=1,2,3. \end{aligned} \tag{2.1}$$

The game is balanced and the core consists of all convex combinations of the two points $(0, \frac{1}{2}, \frac{1}{2})$ and $(1, 0, 0)$. Thus the relative interior of the core is well defined and it consists of all strict convex combinations of the two points. No point in the relative interior, however, is partnered. Indeed, the only partnered core point is the boundary point $(1, 0, 0)$.

3. MINIMALLY PARTNERED CORE POINTS

Minimally partnered core payoffs appear to have special significance. In particular, minimal partnership is a natural outcome of certain competitive environments. Bennett and Zame [6] show that in pure exchange economies with strictly convex and strictly monotone preferences, all competitive equilibria are partnered. It follows that whenever any such economy is replicated, all equilibria become minimally partnered. Theorem 3.1 below shows that unless a game admits a continuum of partnered core points, at least one must be minimally partnered.

THEOREM 3.1. *Let (N, V) be a balanced game. If there are at most countably many points in the partnered core then at least one core point is minimally partnered.*

One cannot dispense with the assumption that the partnered core is countable. For instance, in the two player divide-the-dollar game, none of the continuum of partnered core payoffs is minimally partnered.

Theorem 3.1 yields the following corollary which is established in Reny *et al.* [11] for the special case of games with side payments.

COROLLARY 3.2. *If (N, V) is a balanced game and $C(N, V)=\{x\}$, then x is minimally partnered.*

The result expressed in Theorem 3.1 might lead one to conjecture that any isolated core point is minimally partnered. This is not the case, as illustrated by the following example.

EXAMPLE 3.3 (A balanced game having an isolated but non minimally partnered core payoff. Define a balanced game with the total player set $N=\{1, 2, 3\}$ and with the characteristic function

$$V(\{1, 2, 3\})=C-R^3_+$$

$$V(\{1, 2\})=\{(x_1, x_2, x_3):(x_1, x_2)\leqslant(1, 2)\},$$

$$V(\{2, 3\})=\{(x_1, x_2, x_3):(x_2, x_3)\leqslant(2, 1)\},$$

$$V(\{1, 3\})=\{(x_1, x_2, x_3):(x_1, x_3)\leqslant(0, 0)\}, \text{ and}$$

$$V(\{i\})=\{(x_1, x_2, x_3):x_i\leqslant 0\} \text{ for each } i=1, 2, 3$$

where $C=\{(1, 0, 1)\}\cup\{(\alpha, 2, 0), (0, 2, \alpha):\alpha\in[1, 0]\}$.

The core of the game is C. The set of supporting coalitions for the isolated core point $(1, 0, 1)$ is

$$\mathscr{S}((1, 0, 1))=\{\{1, 2\}, \{2, 3\}, \{2\}, \{1, 2, 3\}\}.$$

Note that $\mathscr{S}((1, 0, 1))$ is not partnered, as player 1 needs player 2 but player 2 needs no one. Consequently, the isolated core point $(1, 0, 1)$ is not even partnered, let alone minimally partnered.

To see that the core is equal to C, suppose x is in the core and $x_2 < 2$. Then it must be that x_1 and x_3 both equal 1. But x_1 and x_3 both equal to 1 implies that $x = (1, 0, 1)$, which is in the core as can be easily verified. Now consider the case in which $x_2 = 2$. If either player 1 or player 3 receives a nonnegative payoff, then x is undominated. Moreover, any such payoff with $1 \geqslant x_1 \geqslant 0$ and $x_3 = 0$, or vice versa, is in the core.

Finally, note that although there are a continuum of core payoffs, only $(0, 2, 0)$ is in the partnered core. (See also example 1, where the partnered core similarly refines the core.) Consequently, by Theorem 3.1, $(0, 2, 0)$ must be minimally partnered, which can be readily verified.

4. PROOFS

4.1. *Proof of Theorem 2.1*

Our proof relies on a result due to Bennett and Zame [6], as well as on an extension of the techniques developed by Shapley and Vohra [15] to prove Scarf's [13] Theorem. In particular, as in Shapley and Vohra [15], we make use of a mapping from Δ to a suitable modification of the boundary of an expanded payoff set. Given q as in (1.5), let $Q = \{x \in R^N : x \leqslant qe^N\}$ and define

$$W = \left(\bigcup_{S \subseteq N} V(S) \right) \cap Q.$$

The following Lemma plays a central role in the proof of Theorem 2.1.

Lemma 4.1. *Let (N, V) be a balanced game. Suppose that for each pair of players i and j there is a continuous function $c_{ij} \colon \partial W \rightarrow R_+$ such that for all $S \subseteq N$,*

$$c_{ij} \text{ is identically zero on } V(S) \cap \partial W \quad \text{whenever } i \notin S \text{ and } j \in S. \tag{4.1}$$

Then there is a core payoff y^* such that for all $i \in N$,

$$\sum_{j \in N} (c_{ij}(y^*) - c_{ji}(y^*)) = 0.$$

Remarks. 1. Putting c_{ij} identically equal to zero gives Scarf's result.

2. The following functions, introduced by Bennett and Zame [6], satisfy (4.1) and are particularly useful for the proofs of our theorems. For each pair of distinct players i, j in N, define the function $c_{ij}\colon R^N \to R_+$ by

$$c_{ij}(y) = \min_{\substack{S: \\ i \notin S \\ j \in S}} \operatorname{dist}(y^S, V(S)),$$

where dist is Euclidean distance. Note that since the distance from a closed set to a point depends continuously on the point, c_{ij} is a continuous function. For convenience set $c_{ii}(y) = 0$ for each i and all y.

3. Intuitively, a function c_{ij} satisfying (4.1) provides, for each payoff, $y \in \partial W$, a measure of j's indebtedness to i, or i's credit against j. For example, if $y \in \partial W$, and j does not need i for y, then $y \in V(S) \cap \partial W$ for some $S \subseteq N$ with $j \in S$ and $i \notin S$. Consequently, (4.1) implies that $c_{ij}(y) = 0$, aptly reflecting the fact that i has no credit against j for y. Thus, c_{ij} takes on positive values only when j needs i for the given payoff.

Proof of the Lemma. We break the proof into three steps.

Step 1. [Find y^*, a candidate for satisfying the conclusion of the Lemma, namely $\sum_{j \in N} (c_{ij}(y^*) - c_{ji}(y^*)) = 0$ for all $i \in N$ and $y^* \in C(N, V)$.]

For any $S \subseteq N$ and $y \in \partial W$, let

$$\eta_i^S(y) = \sum_{j \in S} [c_{ij}(y) - c_{ji}(y)].$$

Since $c_{ij}(y)$ is i's credit against j and $c_{ji}(y)$ is j's credit against i we can think of $[c_{ij}(y) - c_{ji}(y)]$ as i's net credit against j (for y) and thus we can think of $\eta_i^S(y)$ as i's net credit against the members of S. Also let $\eta^S(y)$ denote the vector in R^N whose ith coordinate is $\eta_i^S(y)$ if $i \in S$, and is zero otherwise.

Following Shapley and Vohra [15], for each $x \in \Delta$, define $f\colon \Delta \to \partial W$ by

$$f(x) = \{y \in \partial W : y = tx \text{ for some } t \geqslant 0\}$$

As shown by Shapley and Vorha, f is well defined, single-valued and upper hemicontinuous. (This result relies on all the defining properties (1.1)–(1.5) of a game.) Thus, f is continuous taken as a function. Let

$$\eta^* = \max_{\substack{S \subseteq N \\ x \in \Delta \\ i \in N}} |\eta_i^S(f(x))| \tag{4.2}$$

and define

$$\Omega = \left\{ \omega \in R^n : |\omega_i| \leqslant \eta^* \text{ and } \sum_{i=1}^{n} \omega_i = 0 \right\}. \tag{4.3}$$

Observe that Ω is compact and convex.

Define the continuous function h: $\Delta \times \Delta \times \Omega \to \Delta$ by

$$h_i(x, p, \omega) = \frac{x_i + \max[p_i - 1/n, \omega_i, 0]}{1 + \sum_{j \in N} \max[p_j - 1/n, \omega_j, 0]} \tag{4.4}$$

for each $i \in N$.

Define the correspondence, G: $\Delta \to \Delta \times \Omega$ by

$$G(x) = \{(m^S, \eta^S(f(x)) : f(x) \in V(S)\} \tag{4.5}$$

Observe that for each $x \in \Delta$, $G(x)$ is nonempty. In addition, G is upper-hemicontinuous since $\eta^S(y)$ and $f(x)$ are continuous functions.

Consider the correspondence $h \times \text{co}\, G$: $\Delta \times \Delta \times \Omega \to \Delta \times \Delta \times \Omega$. Since $h \times \text{co}\, G$ is nonempty-valued, convex-valued, and upper-hemicontinuous, by Kakutani's theorem it admits a fixed point, (x^*, p^*, ω^*). Consequently,

$$(p^*, \omega^*) \in \text{co}\{(m^S, \eta^S(f(x^*))) : f(x^*) \in V(S)\}), \tag{4.6}$$

and

$$x_i^* \sum_j \max\left[[p_j^* - \frac{1}{n}, \omega_j^*, 0\right] = \max\left[p_i^* - \frac{1}{n}, \omega_i^*, 0\right] \tag{4.7}$$

for all $i \in N$.

By (4.6) there exist nonnegative real numbers $(\alpha_S)_{S \subseteq N}$ satisfying

$$\sum_{S \subseteq N} \alpha_S = 1,$$

$$\omega^* = \sum_{S \subseteq N} \alpha_S \eta^S(f(x^*)), \tag{4.8}$$

$$p^* = \sum_{S \subseteq N} \alpha_S m^S, \tag{4.9}$$

and

$$\alpha_S > 0 \Rightarrow f(x^*) \in V(S). \tag{4.10}$$

Let $y^* = f(x^*)$ be the desired candidate.

Step 2. [Show that $\sum_j \max[p_j^* - 1/n, \omega_j^*, 0] \neq 0$, implies that $x_m^* = 0$, for some m.]

Let $M = \{m : p_m^* = \min_j \{p_j^*\}\}$. Then $p_m^* \leqslant 1/n$ for each $m \in M$ since p^* is in Δ. Now if $p_j^* > p_m^*$, then by (4.9) there exists $\alpha_S > 0$ such that $j \in S$ and $m \notin S$. But (4.10) then implies that $y^* \in V(S)$. Hence, $c_{mj}(y^*) = 0$. Therefore for each $m \in M$, for all $S \subseteq N$,

$$\eta_m^S(y^*) = \sum_{j \in S} [c_{mj}(y^*) - c_{jm}(y^*)]$$

$$= \sum_{\substack{j \in S \\ p_j^* = p_m^*}} [c_{mj}(y^*) - c_{jm}(y^*)] + \sum_{\substack{j \in S \\ p_j^* > p_m^*}} [c_{mj}(y^*) - c_{jm}(y^*)]$$

$$= \sum_{\substack{j \in S \\ p_j^* = p_m^*}} [c_{mj}(y^*) - c_{jm}(y^*)] - \sum_{\substack{j \in S \\ p_j^* > p_m^*}} c_{jm}(y^*),$$

so that

$$\sum_{m \in M} \eta_m^S(y^*) = \sum_{m \in M \cap S} \eta_m^S(y^*), \qquad \text{since if } i \notin S \text{ then } \eta_i^S(y^*) = 0$$

$$= \sum_{m \in M \cap S} \sum_{j \in M \cap S} [c_{mj}(y^*) - c_{jm}(y^*)]$$

$$- \sum_{m \in M \cap S} \sum_{j \in S \setminus M} c_{jm}(y^*)$$

$$= - \sum_{m \in M \cap S} \sum_{j \in S \setminus M} c_{jm}(y^*) \leqslant 0,$$

since each term $c_{jm}(y^*)$ is nonnegative.

Now from (4.8)

$$\omega^* = \sum_{S \subseteq N} \alpha_S \eta^S(y^*).$$

Hence,

$$\sum_{m \in M} \omega_m^* = \sum_{S \subseteq N} \alpha_S \sum_{m \in M} \eta_m^S(y^*) \leqslant 0.$$

Consequently, $\omega_{m'}^* \leqslant 0$ for some $m' \in M$. Finally, since $p_{m'}^* \leqslant 1/n$, (4.7) implies $x_{m'}^* = 0$ if $\sum_j \max[p_j^* - 1/n, \omega_j^*, 0] \neq 0$.

Step 3. [Show that y^* satisfies the conclusion of the Lemma, namely that $y^* \in C(N, V)$ and $\eta_i^N(y^*) = 0$ for all $i \in N$].

We first show that $p_i^* = 1/n$ and $\omega_i^* = 0$ for all $i \in N$. Suppose not. Then

$$\sum_{j \in N} \max\left[p_j^* - \frac{1}{n},\ \omega_j,\ 0 \right] > 0.$$

Let $I = \{i \in N : x_i^* > 0\}$ and let $K = \{k \in N: x_k^* = 0\}$. By step 2, $K \neq \emptyset$. By (4.7), for all $i \in I$, either $p_i^* > 1/n$ or $\omega_i^* > 0$. In either case, (4.8) or (4.9) implies that $\alpha_S > 0$ for some S with $i \in S$. So, by (4.10), $y^* \in V(S)$. Consequently, for all $i \in I$, $y_i^* < q$.

Now, for all $k \in K$, $y_k^* = f_k(x^*) = 0$, by the definition of f. Moreover, since $K \neq \emptyset$, there exists at least one such player $\acute{k}$. But $y^* \in \partial W$ and $y_{\acute{k}}^* = 0$ imply $y_i^* = q$ for some i, a contradiction. We conclude that $p_i^* = 1/n$ and $\omega_i^* = 0$ for all $i \in N$.

Since $p^* = (1/n, 1/n, ..., 1/n)$, (4.6) implies that $\{S : y^* \in V(S)\}$ is balanced. Hence, because the game (N, V) is balanced, $y^* \in V(N)$. Since in addition, $y^* = f(x^*) \in \partial W$, y^* is in the core. It remains to show that $\eta_i^N(y^*) = 0$ for all $i \in N$. But this follows from the equalities below which hold for all $i \in N$.

$$\begin{aligned}
0 = \omega_i^* &= \sum_{S \subseteq N} \alpha_S \eta_i^S(y^*), \qquad \text{by (4.8)} \\
&= \sum_{S \subseteq N} \alpha_S \sum_{j \in S} [c_{ij}(y^*) - c_{ji}(y^*)] \\
&= \sum_{j \in N} \Bigg(\sum_{\substack{S \subseteq N \\ j \in S}} \alpha_S \Bigg) [c_{ij}(y^*) - c_{ji}(y^*)] \\
&= \frac{1}{n} \sum_{j \in N} [c_{ij}(y^*) - c_{ji}(y^*)], \qquad \text{by (4.9)} \\
&= \frac{1}{n} \eta_i^N(y^*). \quad \blacksquare
\end{aligned}$$

Proof of Theorem 2.1. Define c_{ij} as in Remark 2 . By the Lemma, there is a payoff $y^* \in C(N, V)$ such that $\eta_i^N(y^*) = 0$ for all $i \in N$. But as shown in Bennett and Zame [6] (see the proof of their Lemma), for these choices of the c_{ij}, $\eta_i^N(y^*) = 0$ for all $i \in N$ implies that y^* is a partnered payoff. ▌

4.2. Proof of Theorem 3.1

Proof. Again let the functions c_{ij} be as in Remark 2 of the preceeding section. Let x^1, x^2, ... denote the (at most) countably many points in the partnered core. By Theorem 2.1, there is at least one such point. It suffices to show that for some k,

$$c_{ij}(x^k) = 0 \qquad \text{for all} \quad i, j \in N.$$

Let

$$A(x)=\left\{\alpha\in R^{n^2}_{++} : \text{for all } i\in N, \sum_{j\in N}(\alpha_{ij}c_{ij}(x)-\alpha_{ji}c_{ji}(x))=0\right\}.$$

Since for every $\alpha\in R^{n^2}_{++}$ $\alpha_{ij}c_{ij}(\cdot)$ satisfies (4.1), the Lemma implies that $\alpha\in A(x^*)$ for some core point x^*. But as in the proof of Theorem 2.1, this implies that x^* is in the partnered core. Hence, for every $\alpha\in R^{n^2}_{++}$, there is a k such that

$$\alpha\in A(x^k).$$

Consequently,

$$\bigcup_{k=1}^{\infty} A(x^k)=R^{n^2}_{++}$$

The Baire Category Theorem (see, for instance, Friedman [7], p. 106, Theorem 3.4.2) then implies that there exists $\acute{k}$ such that $A(x^{\acute{k}})$ is somewhere dense in $R^{n^2}_{++}$. Consequently, the closure of $A(x^{\acute{k}})$ contains an open set $A^0(x^{\acute{k}})$. For all $i\in N$ and all $\alpha\in A^0(x^{\acute{k}})$, we have $\sum_{j\in N}(\alpha_{ij}c_{ij}(x^{\acute{k}})-\alpha_{ji}c_{ji}(x^{\acute{k}}))=0$. But this implies that $c_{ij}(x^{\acute{k}})=0$ for all pairs i and j. ∎

5. DISCUSSION OF THE LITERATURE

The concept of a partnered collection of sets was introduced in the impressive study of the kernel carried out by Maschler and Peleg [8, 9] and it was further studied in Maschler *et al.* [10]. They, however, used the term "separating collection". We use the term partnered, introduced by Albers [2], as it also has appeared in several papers in the literature and reflects our interest in formation of coalitions of individuals with mutual interests.

The study of undominated partnered payoffs was initiated in Albers [1, 2]. Bennett [4, 5] refines the partnership property by imposing an "equal gains" criterion and obtains further results. Bennet [4, 5] also provide a rich collection of examples. In a non-cooperative model of characteristic function bargaining, Selten [14] showed that undominated partnered payoffs can arise as equilibrium demands of players.

Reny *et al.* [11] introduced the concept of the partnered core of a game with side payments and showed that the partnered core of a game with side payments includes all payoffs in the relative interior of the core (see also Albers [2], Lemma 3.3).

The study of undominated partnered payoffs of games without side payments was initiated in Bennett and Zame [6], where the existence of such payoffs is established. Undominated partnered payoffs need not be feasible however. Thus it may not be possible to achieve such payoffs by the coalition of the total player set (or by a partition of that player set, or by all the sets in a balanced collection of subsets). In contrast, Theorem 2.1 establishes, for balanced games, the existence of partnered payoffs that are both undominated and feasible. Since our result can be applied to balanced cover games, it also establishes the existence of partnered and undominated payoffs for all games satisfying (1.1)–(1.5), balanced or not. Thus, Bennett and Zame's [6] result is a corollary of Theorem 2.1. Moreover, we do not require the assumption of strong comprehensiveness (nonleveledness of payoff sets) made in Bennett and Zame [6].

As an outgrowth of Theorem 2.1 of this paper, we provide, in Reny and Wooders [12], a generalization of the Knaster–Kuratowski–Mazurkiewicz–Shapley Theorem. As in the present paper, ideas from Shapley and Vohra [15] and Bennett and Zame [6] figure prominently in obtaining this generalization.

We conclude with an example illustrating the distinction between coalitions supporting payoffs in the core and the partnership structure of a core payoff. The example is taken from Reny *et al.* [11].

EXAMPLE. There are eight players, $\{1, 2, ..., 8\}$. Partition the player set into four pairs, $p_1 = \{1, 2\}$, $p_2 = \{3, 4\}$, $p_3 = \{5, 6\}$ and $p_4 = \{7, 8\}$. Only distinct pairs of these player pairs are productive. Formally, define the characteristic function w by

$$w(S) = 2 \text{ if } S = p_i \cup p_j \text{ for some } i \neq j \text{ and } w(S) = 0 \text{ otherwise.}$$

Define a superadditive characteristic function v by

$$v(S) = \max \sum_{S' \in p(S)} v(S'),$$

where the maximum is taken over all partitions $p(S)$ of S. The game (N, v) is a balanced game (with side payments). A partition of the total player set into coalitions supporting the core is, for example,

$$\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\},$$

or any other partition of the set of players into two coalitions, each consisting of two distinct pairs, p_i and p_j. The core is given by

$$\{x \in R^8 : x_k \geqslant 0 \text{ for each player } k \text{ and for each pair } p_i, x(p_i) = 1\},$$

where $x(p_i)$ denotes the sum of the payoffs to the two players in p_i.

The partnerships induced by any point in the relative interior of the core are the pairs p_1, p_2, p_3 and p_4, and the partnered core (which coincides with the core's relative interior here) is

$$\{x \in R^8 : x_k \geqslant 0 \text{ for each player } k \text{ and for each pair } p_i,\ x(p_i) = 1\},$$

To realize a payoff in the partnered core the players in the partnerships, p_i, are inseparably united, while in contrast, a number of different coalition structures (three, to be exact) are consistent with the achievement of any such payoff. In this sense, partnerships are more stable and therefore perhaps more basic than coalition structures.

REFERENCES

1. W. Albers, Zwei Logungskonzepte für kooperative Mehrpersonspiele, die auf Anspruchsniveaus der Spieler bastiern, *OR-Verfahren* (*Meth. Oper. Res.*) **21** (1974), 1–13.
2. W. Albers, Core and kernel variants based on imputations and demand profiles, *in* "Game Theory and Related Topics" (O. Moeschlin and D. Pallaschke, Eds.), North Holland, Amsterdam, 1979.
3. R. J. Aumann and J. Drèze, Cooperative games with coalition structures, *Int. J. Game Theory* **3** (1974), 217–237.
4. E. Bennett, "Coalition Formation and Payoff Distribution in Cooperative Games," Ph.D. dissertation, Northwestern University, (1980).
5. E. Bennett, The aspiration approach to predicting coalition formation and payoff distribution in sidepayment games, *Int. J. Game Theory* **12** (1983), 1–28.
6. E. Bennett and W. R. Zame, Bargaining in cooperative games, *Int. J. Game Theory* **17** (1988), 279–300.
7. A. Friedman, "Foundations of Modern Analysis," Dover, New York (1982).
8. M. Maschler and B. Peleg, A characterization, existence proof and dimension bounds for the kernel of a game, *Pacific J. Math.* **18** (1966), 289–328.
9. M. Maschler and B. Peleg, The structure of the kernel of a cooperative game, *SIAM J. Applied Math.* **15** (1967), 569–604.
10. M. Maschler, B. Peleg, and L. S. Shapley, The kernel and bargaining set for convex games, *Int. J. Game Theory* **1** (1971), 73–93.
11. P. J. Reny, E. Winter, and M. H. Wooders, "The Partnered Core of a Game with Side Payments," University of Western Ontario Working Paper, 1993.
12. P. J. Reny and M. H. Wooders, An extension of the Knaster, Kuratowski, Mazurkiewicz, and Shapley lemma, notes in manuscript, 1993.
13. H. E. Scarf, The core of an *n*-person game, *Econometrica* **35** (1967), 50–67.
14. R. Selten, A noncooperative model of characteristic function bargaining, *in* "Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern" (V. Boehm and H. Nacktkamp, Eds.), in Gesellschaft, Recht, Wirtschaft, Band 4, Wissenschaftsverlag Bibliographisches Institute, Mannheim-Wien-Zurich, pp. 131–151 1981.
15. L. S. Shapley and R. Vohra, On Kakutani's fixed point theorem, the K-K-M-S theorem and the core of a balanced game, *Econ. Theory* **1** (1991), 107–116.