

Cores of many-player games; nonemptiness and equal treatment

Myrna Wooders

Received: 25 April 2007 / Accepted: 28 July 2009 / Published online: 30 September 2009
© The Author(s) 2009. This article is published with open access at Springerlink.com

Abstract This paper provides sufficient conditions to ensure nonemptiness of approximate cores of many-player games and symmetry of approximate core payoffs (the equal treatment property). The conditions are: (a) essential superadditivity—an option open to a group of players is to partition into smaller groups and realize the worths of these groups and (b) small group effectiveness (SGE)—almost all gains to collective activities can be realized by cooperation only within members of some partition of players into relatively small groups. Another condition, small group negligibility (SGN), is introduced and shown to be equivalent to SGE. SGN dictates that small groups of players cannot have significant effects on average (i.e., per capita) payoffs of large populations; thus, SGN is an analogue, for games with a finite player set, of the condition *built into* models with a continuum of player that sets of measure zero can be ignored. SGE implies per capita boundedness (PCB), that the supremum of average or per capita payoffs is uniformly bounded above. Further characterization of SGE in terms of its relationship to PCB is provided. It is known that if SGE does not hold, then approximate cores of many-player games may be empty. Examples are developed to show that if SGE does not hold and if there are players of “scarce types” (in other works, players with scarce attributes) then even if there is only a finite number of types of players and approximate cores are non-empty, symmetry may be

The author is indebted to an anonymous referee and to John Ledyard for helpful comments.

This paper is dedicated to the memory of Leonid Hurwicz, a giant among economic theorists and a giant among men.

M. Wooders (✉)
Department of Economics, Vanderbilt University, Nashville, TN, USA
e-mail: myrna.wooders@vanderbilt.edu
URL: www.myrnawooders.com

lost; moreover, *even players of abundant types may be treated asymmetrically* by the core.

Keywords Core · Market games · Approximate cores · Equal treatment · Core convergence · Small group effectiveness · Symmetry · Per capita boundedness · Small group negligibility · Games with a continuum of players

JEL Classification C71 · D71 · C78 · H41 · D40 · D50

1 Introduction

Since the classic early works of general equilibrium theory of [Arrow and Debreu \(1954\)](#); [Arrow and Hurwicz \(1958\)](#), and [McKenzie \(1959\)](#), general equilibrium theory has been at the foundations of economic theory. The theory has primarily focused on economies with only private goods. Economies with (pure) public goods and other externalities have also been important; see, for example, [Lindahl \(1958\)](#), [Samuelson \(1954\)](#), [Foley \(1970\)](#), [Mas-Colell \(1980\)](#) and [Hurwicz \(1999\)](#). Yet, there are many possible divergences of economies from these two classic models; there may be indivisibilities, non-monotonicities, and non-convexities. Also, public goods may be local (subject to exclusion and/or congestion) or individuals may be social and gain enjoyment from consuming and/or producing jointly with others, or there may be issues of matching individuals with other individuals, and so on. Some divergences from the classic models have been studied in the context of specific economic models but there has been less research aimed at more broadly identifying essential characteristics of models that undergird the notion of price-taking economic behavior.

It has been recognized for some time that price-taking behavior requires a large number of economic agents, even in private goods exchange economies. It has also been recognized that in private goods economies with many agents, problems of nonconvexities disappear or become insignificant; some seminal contributions are [Aumann \(1964\)](#), [Shapley and Shubik \(1966\)](#), [Aumann and Shapley \(1974\)](#), and, for economies with production, [Hurwicz and Uzawa \(1977\)](#). More specifically, in a number of situations it has been shown that, in economies with many agents, cores are nonempty, price-taking equilibria exist, and equilibrium outcomes are close to core outcomes, where approximations, if any, become arbitrarily good as the economies become large. For private goods economies, there are many contributions to this literature; a fairly recent contribution is [Hurwicz \(1995\)](#). Study of this literature and other papers suggests that there is some underlying set of properties driving the conclusion that economies with many agents pass the cooperative-game-theoretic tests for competitiveness—cores are nonempty, equilibria exist and equivalence of outcomes of in the core and price-taking equilibrium outcomes holds.

For equivalence of the outcomes of cooperation and competition to hold, since price taking equilibria have the equal treatment property (symmetry), it must be the case that approximate cores treat similar players similarly or nearly equally. The equal treatment property of cores for private goods economies has been studied in several papers, for example, [Shubik \(1959\)](#), [Green \(1972\)](#) and [Hildenbrand and Kirman \(1973\)](#). The

equal treatment property of the core has also attracted much interest in economies with clubs and/or local public goods; see for example [Wooders \(1980\)](#) and, for recent results, [Allouch and Wooders \(2009\)](#).

This paper explores conditions for many-player games with side payments to have nonempty approximate cores and for cores to converge to symmetric outcomes. The convergence is in the sense that for large numbers of players, any distribution of total payoff that is in the core treats similar players nearly equally, except for possibly a small exceptional set of players. To obtain our results, we use the framework of a pregame. A pregame consists of a compact metric space of player attributes (sometimes called “player types”) and a function assigning a worth to any finite list of attributes (repetitions allowed). A list of attributes is interpreted as a description of a possible group of players in terms of the attributes of the group members. The framework is sufficiently broad to accommodate games derived from economies with indivisibilities, non-monotonicities, non-convexities, local public goods and clubs.

The main conditions underlying our model and results are essential superadditivity and small group effectiveness. Essential superadditivity ensures that an option open to a group of players is to partition into smaller groups and realize the total payoffs attainable in these groups. Small group effectiveness, (SGE), dictates that almost all gains to collective activities can be realized by cooperation only within members of some partition of players into relatively small groups. The relationship of SGE to two other conditions, discussed below, and its usefulness in obtaining results for large economies motivate the emphasis that we place on the concept.

The concept of small group negligibility (SGN), is introduced and shown to be equivalent to SGE. SGN dictates that relatively small groups of players can have only (vanishingly) small effects on the payoffs of large groups. SGN is an analogue, for arbitrarily large (but finite) games of the condition implicit in continuum models following [Aumann \(1964\)](#) that sets of players of measure zero can be ignored.

SGE implies per capita boundedness (PCB), the condition that the supremum of average or per capita payoffs is uniformly bounded above. PCB is a very reasonable condition; if PCB were not satisfied then, as the size of the total player population is allowed to go to infinity, per capita payoffs could also go to infinity. With some further restrictions on the model, PCB is also necessary and sufficient for our results; in particular, if the set of player types is finite and there are “many” players of each type, then PCB (along with essential superadditivity) implies both nonemptiness of approximate cores and their convergence to symmetric outcomes. Examples are developed to show that with only PCB, however, if there are players of “scarce types” in the total player set (in other works, players with scarce attributes), then the equal treatment property of approximate cores may be lost, even if there is only a finite number of types of players; moreover, *even players of abundant types may be treated asymmetrically* by the core.

It is well known that if a coalition structure (a partition of players into groups) associated with a payoff vector in the core has the property that there are two identical players in two disjoint coalitions then the core must treat these two players equally; a version of this result is shown in [Wooders \(1983\)](#). However, for the class of games we consider it may be the case that there does not exist a core payoff with identical

players in disjoint coalitions. Moreover, it may hold that the ε -core, for $\varepsilon = 0$, is empty. In this paper we allow situations where per capita payoff may strictly continually increase as the number of players becomes large; in such situations, while it is impossible for all outcomes in the exact core to have the equal treatment property, we show that nevertheless most similar players are treated nearly equally by outcomes in approximate cores.¹

1.1 Some background for the model and results

The results in this paper grew out of research focusing on a special case—games with a fixed distribution of a finite number of player types, or in other words, replica games.² Moreover, the first results required that effective group sizes be uniformly bounded, say by an integer B (*strict* small group effectiveness, SSGE, a special case of SGE). Two examples are marriage models and soccer teams. Here we sketch the main ideas of this work for two special cases with the hope to provide some insight into what follows.

Let us first consider a very special example. Suppose all possible players are identical and any two-player group can cooperate and earn \$1. Groups of other sizes are worthless, but a larger group has open the possibility to form multiple two-player groups. Any specification of the total number of players now determines a game—a total player set and the worth of any subset of players. When as many two-person groups as possible are formed from the total player set, there will be at most one player left-over. For any even number of players greater than two, the core will be nonempty and symmetric (easy to show), assigning each player \$.50. If the total player set is ‘large’ then each player can be assigned nearly \$.50 and this assignment will be in an approximate core. There are many payoffs in the approximate core which treat some players worse than average and other players better than average; but any approximate core payoff vector (for close approximations) must assign most players nearly \$.50 (which is, hopefully, intuitive, and also not especially difficult to show for this example).

Let us next discuss games with T types of players, for some integer T . Let \mathbb{Z}_+^T denote the T -fold Cartesian product of the non-negative integers. Let $s = (s_1, \dots, s_T) \in \mathbb{Z}_+^T$ represent a group of players, where $s_t \in \mathbb{Z}_+$ denotes the number of players of type t in the group. Let $\Psi(s) \in \mathbb{R}_+$ be the total payoff that the group can realize, the worth of the group. The pair (T, Ψ) , where $\Psi : \mathbb{Z}_+^T \rightarrow \mathbb{R}_+$, is called a *pregame* (with a finite number of types). For convenience, we will assume that the pregame is essentially superadditive—that is, the worth of a group s is at least as great as the worth of any “partition” of the group into subgroups s^1, \dots, s^K ($s = \sum s^k$); thus,

¹ The total worth of a group p of n identical players, could be, for example, $n - \frac{1}{n}$, so the per capita worth would be $1 - \frac{1}{n^2}$. No (exact) core payoff vector would admit a partition of the total player set into two disjoint coalitions.

² The first working paper version of Wooders (1979) already provided statements and proofs of the results discussed in this subsection. For the convenience of the reader, the results described in this subsection are also proven in an Appendix.

$\Psi(s) \geq \sum \Psi(s^k)$.³ Note that a pregame is not a game since no total player set is specified.

Let $rn = r(n_1, \dots, n_T) \in \mathbb{Z}_+^T$ be a vector listing an integer number of players of each type $t = 1, \dots, T$, taken as a description of the total player set of a game. The pair, Ψ and rn , determines a game where, for each $s \in \mathbb{Z}_+^T, s \leq n$, the worth of s is given by $\Psi(s)$. From SSGE it follows that

$$\max_r \frac{\Psi(rn)}{r}$$

exists. Let r^* satisfy $\max_r \frac{\Psi(rn)}{r} = \frac{\Psi(r^*n)}{r^*}$. It now holds that any game derived from the pregame (T, Ψ) with total player set ℓr^*n has a nonempty core for any positive integer ℓ . (For the convenience of the reader, a proof is provided in the Appendix.⁴)

Any positive integer r can be written as $r = \ell r^* + j$ where $0 \leq j < r^*$. Thus, any game with the total player population described by $r(n_1, \dots, n_T), r \geq r^*$, contains a largest subgame with player set described by $\ell r^*(n_1, \dots, n_T)$ for some integer ℓ ; this subgame has a nonempty core. If $r \neq \ell r^*$, then “left-over” players, described by the vector $j(n_1, \dots, n_T)$ cause the core to be empty. But the number of left-overs is bounded above by $r^* \sum n_t$. Thus, given $\varepsilon > 0$, for large r , starting with a payoff vector in the core for a player set described by $\ell r^*(n_1, \dots, n_T)$, members of this player set can each be “taxed” ε per capita and transfers can be made to left-over players to create an outcome in the ε -core.

Now consider an outcome in the exact core of a game with player set described by $\ell r^*(n_1, \dots, n_T)$ for some $\ell r^* > B$, the bound on effective group sizes. An outcome in the core will have the property that if one player is assigned a smaller payoff than another player of the same type, then there will be a coalition, excluding the better-off player, that can do better for all its members, which is a contradiction.⁵ It follows easily that all players of type t must be assigned the same payoffs by an outcome in the core. It also follows that ε -cores must treat most players of the same type nearly equally. Here the argument is more complex but the basic ideas are intuitive. Given an outcome in the ε -core, let the “poor” be those players treated significantly worse than average and let the “rich” be those players treated significantly better than average. The ε -core outcome can treat some of the poor very badly but the number of poor players cannot be too large; otherwise, these players could join with some subset of the middle class (neither rich nor poor) and improve upon the ε -core payoff vector. The number of rich players is bounded by how much the poor and the middle class can be discriminated against.

The results discussed above all rely on *strict* small group effectiveness, SSGE. A beautiful feature of small group effectiveness, SGE, is that it allows us to approximate

³ One way that the group s may achieve its worth $\Psi(s)$ is by partitioning into subgroups, as in a club or local public good economy, for example.

⁴ The arguments for this special case are already in Wooders (1979) and, for NTU games, in Wooders (1983). See also Kovalenkov and Wooders (2003) for recent extensions for parameterized collections of games.

⁵ See Proposition 1 in Section 5. SSGE, with more players of each type than the bound B on group sizes implies that any outcome in the core will satisfy the conditions of that Proposition.

games with a compact metric space of player types by replication games with a finite number of player types satisfying SSGE. For our convergence results especially, the approximations can become quite complex but nevertheless, the finite type replication results for games satisfying SSGE underpin the general results. Another compelling aspect of SGE is its close relationship to small group negligibility, SGN, and PCB. It is hard to imagine an interesting economic model for which PCB would not be satisfied.

2 Games

We begin with some standard definitions from the theory of cooperative games with side payments.

Let (N, v) be a pair consisting of a finite set $N = \{1, \dots, n\}$, called the *player set*, and a function v , called the *worth function*, from subsets of N to the non-negative real numbers with $v(\emptyset) = 0$. The pair (N, v) is a *game* (with side payments). Nonempty subsets of N are called *groups*.⁶

Let (N, v) be a game. Let $\delta \geq 0$ be a non-negative real number. Two players i and j are δ -*substitutes* if for every group S with $i \notin S$ and $j \notin S$, it holds that

$$|v(S \cup \{i\}) - v(S \cup \{j\})| \leq \delta.$$

A *payoff vector* for a game (N, v) is a vector $x \in \mathbb{R}^N$. The payoff vector x is *feasible* if

$$x(N) \stackrel{\text{def}}{=} \sum_{i=1}^n x^i \leq \sum_{k=1}^K v(S^k) \quad (2.1)$$

for some partition $\{S^1, \dots, S^K\}$ of N .

Given $\varepsilon \geq 0$, a payoff vector x is in the ε -*core* of the game if it is feasible and if, for all groups $S \subset N$,

$$x(S) \geq v(S) - \varepsilon|S|.$$

Remark Our definition of feasibility ensures essential superadditivity, that is, an option open to a group of players is to partition itself into subgroups and realize, in total, the sum of the payoffs to the subgroups. As discussed at length in Wooders (2008), for the study of ε -cores, we can replace condition (2.1) by the condition that

$$x(N) \leq v^s(N), \quad (2.2)$$

where $v^s(N) \stackrel{\text{def}}{=} \max_{\mathbb{P}} \sum_{k=1}^K v(S^k)$ and \mathbb{P} is the set of all partitions $\{S^k\}$ of N . This is without loss of generality because, with the definition of feasibility given by (2.1),

⁶ To state our assumptions on the model we use the term “groups” instead of “coalitions” as we interpret the model as pertaining to socio-economic structures rather than to the cooperative behavior suggested by the word “coalition”.

for any $\varepsilon \geq 0$ the ε -core of a game (N, v) is equal to the ε -core of the superadditive cover game derived from (N, v) .⁷

3 Preambles

Let (Ω, d) be a compact metric space of player attributes (or types) equipped with a metric denoted by d . An element ω of Ω is interpreted as a description of a player. Let f be a function from Ω to \mathbb{Z}_+ . The support of f , denoted by $\text{support}(f)$, is defined by

$$\text{support}(f) = \{\omega \in \Omega : f(\omega) \neq 0\}.$$

A *profile on Ω* is a function f from Ω to \mathbb{Z}_+ with finite support, that is, $|\text{support}(f)|$, the number of elements in the set $\text{support}(f)$, is finite. In interpretation, a profile f is a description of a finite group of players in terms of the numbers of players of each type in the group. Let \mathcal{F} denote the set of all profiles on Ω . By the *norm of a profile f* we shall mean the L_1 vector norm:

$$\|f\|_1 \stackrel{\text{def}}{=} \sum_{\omega \in \text{support}(f)} f(\omega).$$

A *partition of a profile f* is a collection of profiles $\{f^k\}$, called *subprofiles of f* , satisfying the property that $\sum f^k = f$.

Definition 3.1 (A pregame) Let Ψ be a function from the set of profiles \mathcal{F} on Ω to \mathbb{R}_+ with $\Psi(\mathbf{0}) = 0$, where $\mathbf{0}$ denotes the profile that is identically zero. The pair (Ω, Ψ) is called a *pregame with worth function Ψ* .

In the definition of a pregame, the worth $\Psi(f)$ shall be interpreted as the total payoff a group of players, described by the profile f , can achieve by collective activities of the group membership.

We require an assumption ensuring that players whose attributes are close in attribute space are approximate substitutes for each other. To this end, we first define a metric on the set of profiles \mathcal{F} as follows: For any two profiles f, g , if $\|f\|_1 \neq \|g\|_1$ define

$$\text{dist}(f, g) \stackrel{\text{def}}{=} \max_{\omega_1, \omega_2 \in \Omega} d(\omega_1, \omega_2) + 1.$$

If $\|f\|_1 = \|g\|_1$, let $a = (a_1, \dots, a_{\|f\|_1})$ and $b = (b_1, \dots, b_{\|g\|_1})$ be lists of the elements in $\text{support}(f)$ and $\text{support}(g)$ with each element appearing as many times as its

⁷ The *superadditive cover* of a game (N, v) is the game (N, v^*) with, for each nonempty subset $S \subset N$, $v^*(S) \stackrel{\text{def}}{=} \max_{P \in \mathbb{P}} \sum_{S^k \in P} v(S^k)$ where $P = \{S^1, \dots, S^K\}$ is a partition of S and \mathbb{P} is the set of all partitions of S . See Wooders (2008) for further discussion.

multiplicity. Let θ be a permutation of the components of the index set $(1, \dots, \|g\|_1)$. Define

$$\text{dist}(f, g) \stackrel{\text{def}}{=} \min \max d(a_k, b_{\theta(k)})$$

where the maximum is over the index set $(1, \dots, \|g\|_1)$ and the minimum is over all permutations θ of the index set. It is easy to verify that dist is a metric on the set \mathcal{F} .

Definition 3.2 (Substitution, STN) Given $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $\text{dist}(f, g) < \delta(\varepsilon)$, then

$$\left| \frac{\Psi(f)}{\|f\|_1} - \frac{\Psi(g)}{\|g\|_1} \right| < \varepsilon. \tag{3.1}$$

Substitution ensures that similar profiles have similar worths.

For ease in notation and without any loss of generality (we could instead require only essential superadditivity), we will consider only superadditive pregames. A pregame (Ω, Ψ) is *superadditive* if

$$\Psi(f) = \max_{P \in \mathbb{P}} \sum_{g \in P} \Psi(g),$$

where \mathbb{P} is the set of all partitions P of the profile f . We will also require, throughout the following, that pregames satisfy STN (3.1).

Let us provide a simple example of a pregame based on the well known Shapley–Shubik glove game.

Example 1 A glove pregame. Suppose there are two types of players, players who each own a RH (right-hand) glove and players who each own a LH (left-hand) glove. A (RH, LH) pair of gloves is worth \$1.00. Formally, in the notation used above, let $\Omega = \{\omega_1, \omega_2\}$ denote a set of attributes, where ω_1 denotes the attribute “is endowed with a RH glove” and ω_2 denotes the attribute “is endowed with a LH glove”. In interpretation, a profile f describes a group consisting of $f(\omega_1)$ players with attribute ω_1 and $f(\omega_2)$ players with attribute ω_2 . Define $\Psi(f) = \min\{f(\omega_1), f(\omega_2)\}$. The pair (Ω, Ψ) is a pregame. Note that a pregame is not a game since we do not yet have a set of players.

3.1 Induced games

Let N be a finite set and let α be a map from N into Ω , called an *attribute function*. For any group $S \subset N$ let $\text{prof}(S, \alpha)$ be the function with domain Ω defined by

$$\text{prof}(S, \alpha)(\omega) \stackrel{\text{def}}{=} |\alpha^{-1}(\omega) \cap S|;$$

thus, $\text{prof}(S, \alpha)$ is a function stating the number of players assigned each attribute in the group S . For each $S \subset N$ define

$$v(S) \stackrel{\text{def}}{=} \Psi(\text{prof}(S, \alpha)).$$

Then the pair (N, v) is a game induced by the pregame (Ω, Ψ) and the attribute function α or simply an induced game.

It is sometimes convenient, especially for cases where Ω is a finite set, to describe a game induced by a pregame simply by a pair $[n, \Psi]$ where n is a profile (i.e., $n \in \mathcal{F}$). Let $\{\omega_1, \dots, \omega_T\}$ denote the elements in support(n). Denote a total player set by $N = \{(t, q) : t = 1, \dots, T \text{ and, for each } t, q = 1, \dots, n(\omega_t)\}$. As above, the profile of a subset $S \subset N$ can be defined by its components,

$$\text{prof}(S)_t = |\{(t, q) : q = 1, \dots, n(\omega_t)\} \cap S|$$

and the worth function v can also be defined as above.

Example 1 continued. Take as given the glove pregame described above in Example 1. Let N denote a finite set, called the set of *players*, and let α be an attribute function from N to Ω . In this example, the function α tells us whether player $i \in N$ owns a RH glove or a LH glove; if $\alpha(i) = \omega_1$ then player i owns a RH glove and if $\alpha(i) = \omega_2$ then player i is owns a LH glove. Given α , the worth of a group of players S is determined by the number of RH-LH glove pairs owned by the members of the group. Given α and $S \subset N$, define $\text{prof}(S, \alpha)$ by its components $\text{prof}(S, \alpha)(\omega_i) = |S \cap \alpha^{-1}(\omega_i)|$, $i = 1, 2$, simply a listing of the numbers of players with each attribute in the set S . For each group $S \subset N$, define $v(S) = \Psi(\text{prof}(S, \alpha))$. The pair (N, v) is then a game induced by the pregame.

3.2 Small group effectiveness, per capita boundedness, and small group negligibility

We first introduce the notion of per-capita boundedness, which simply bounds the supremum of average (or, in other words, per-capita) payoffs of games derived from a pregame.

Definition 3.3 (Per capita boundedness, PCB) A pregame (Ω, Ψ) satisfies *per capita boundedness*, PCB, if there is a constant C such that for all profiles f it holds that

$$\frac{\Psi(f)}{\|f\|_1} \leq C, \tag{3.2}$$

that is, per capita payoffs are bounded over all profiles f .

PCB is itself too weak to ensure nonemptiness of approximate cores of many-player games and core convergence.⁸ Our next condition ensures these results.⁹

Definition 3.4 (Small group effectiveness, SGE) A pregame (Ω, Ψ) satisfies *small group effectiveness*, SGE, if it is superadditive and if, given any real number $\varepsilon > 0$, there is an integer $\eta_0(\varepsilon)$ such that for each profile $f \in \mathcal{F}$, for some partition $\{f^k\}_k$ of f into subprofiles with

$$\|f^k\|_1 \leq \eta_0(\varepsilon) \quad \text{for each subprofile } f^k \text{ in the partition}$$

it holds that

$$\Psi(f) - \sum_k \Psi(f^k) \leq \varepsilon \|f\|_1. \tag{3.3}$$

Thus, for every profile f , almost all (within ε per capita) gains to collective activities can be realized by aggregating collective activities within groups of participants bounded in size (by $\eta_0(\varepsilon)$). Small group effectiveness is a natural relaxation of the condition that *all* gains to collective activities can be realized by groups of players uniformly bounded in size, now commonly called *strict small group effectiveness*. Example 1 satisfies strict small group effectiveness while Example 2 below does not.

Example 2 A pregame satisfying SGE but not strict SGE. Let $\Omega = \{\omega\}$, so there is only one attribute. For each profile f on Ω , let $\Psi(f) = f(\omega) - \frac{1}{\|f\|_1}$. Clearly the pregame (Ω, Ψ) satisfies PCB and also SGE but not strict small group effectiveness.

To further characterize SGE, we introduce another assumption limiting increasing returns to group formation. Roughly, this condition dictates that relatively small groups of players have only negligible effects on per-capita payoffs of large groups. Although there may be hints at such a condition in the literature, its formulation, at least for cooperative games, appears to be new to this paper and earlier working papers due to this author.

Definition 3.5 (Small group negligibility, SGN) A pregame (Ω, Ψ) satisfies *small group negligibility* if it satisfies PCB and if, for any sequence of profiles $\{f^v\}$ where

$$\begin{aligned} & \|f^v\|_1 \rightarrow \infty \text{ as } v \rightarrow \infty, \\ & \text{support}(f^v) = \text{support}(f^{v'}) \text{ for all } v \text{ and } v' \text{ and} \\ & \lim_{v \rightarrow \infty} \frac{1}{\|f^v\|_1} f^v \text{ and } \lim_{v \rightarrow \infty} \frac{\Psi(f^v)}{\|f^v\|_1} \text{ both exist,} \end{aligned} \tag{3.4}$$

⁸ For simple examples, suppose that there are only two types of players and all players of type 2 are dummies—a player of type 2 adds nothing to the worth of any group of players. Suppose any two players of type 1 can earn \$1.00 but a third player of type 1 adds nothing. To demonstrate possible emptiness of the core, suppose there are three players of type 1; then the core is empty. To demonstrate non-equal treatment, suppose that there are only two players of type 1. Then any division of \$1.00 is in the core.

⁹ This condition was introduced in Wooders (1992a, 1994). In the condition superadditivity could be relaxed to essential superadditivity.

then, for any sequence of profiles $\{\ell^v\}$ with

$$\lim_{v \rightarrow \infty} \frac{\|\ell^v\|_1}{\|f^v\|_1} = 0, \tag{3.5}$$

it holds that

$$\begin{aligned} \lim_{v \rightarrow \infty} \frac{\Psi(f^v + \ell^v)}{\|f^v + \ell^v\|_1} \text{ exists, and} \\ \lim_{v \rightarrow \infty} \frac{\Psi(f^v + \ell^v)}{\|f^v + \ell^v\|_1} = \lim_{v \rightarrow \infty} \frac{\Psi(f^v)}{\|f^v\|_1}. \end{aligned} \tag{3.6}$$

The property of small group negligibility appears quite mild. It simply ensures that a small group of possibly distinct player types cannot significantly affect per capita payoffs of large player sets.

Theorem 1 *Equivalence of small group effectiveness, SGE, and small group negligibility, SGN: Let (Ω, Ψ) be a pregame. Then (Ω, Ψ) satisfies SGE if and only if (Ω, Ψ) satisfies SGN.*

Informally, Theorem 1 states that small groups are effective for the realization of almost all gains to collective activities if and only if small groups become negligible in many-player games.¹⁰ Small group negligibility is a natural condition for games that can be approximated by games with an atomless continuum of players, since in such games (for example, those in [Aumann and Shapley 1974](#)), sets of measure zero are taken as unable to affect aggregates. Note that Theorem 1 does not require that $|\Omega|$ be finite.

If we require that there are many substitutes for each player and only a finite number of player attributes, then, as shown in [Wooders \(1994, Theorem 4\)](#), there is an equivalence between SGE and PCB. The following Theorem is an extension in that it states that SGE implies PCB, even with a compact metric space of player attributes. We first require a further definition.

Definition 3.6 (Thickness) Let (Ω, Ψ) be a pregame with $|\Omega| = T$ for some finite number T . Then, given a real number $\rho \in (0, 1)$, the ρ -thick restriction of (Ω, Ψ) is the pregame (Ω, Ψ_ρ) with admissible profiles f required to satisfy the condition that for each $t = 1, \dots, T$, either $\frac{f_t}{\|f\|} > \rho$ or $f_t = 0$.

Note that a sequence of profiles derived from the ρ -thick restriction of (Ω, Ψ) does not allow vanishingly small but positive percentages of players of any type.

Theorem 2 *Relating SGE and PCB. Let (Ω, Ψ) be a pregame.*

(a) *Suppose that (Ω, Ψ) satisfies SGE. Then (Ω, Ψ) satisfies PCB.*

¹⁰ For some intuition behind this, consider the special case of a marriage model. Two person groups can achieve all gains to collective activities but, if there are many players, no one or two player group can have a significant effect on per capita worths of large player sets.

- (b) Suppose that Ω is a finite set and that (Ω, Ψ) satisfies PCB. Then given any $\rho \in (0, 1)$, the ρ -thick restriction (Ω, Ψ_ρ) of (Ω, Ψ) satisfies SGE.

If SGE is not satisfied, then small groups of players of scarce types can have major impacts on per capita payoffs, which prevents the full equivalence of PCB and SGE in the finite-type case. The partial equivalence of Theorem 2 (b) demonstrates that if there are many substitutes for each player in a finite set of types, then the two conditions are equivalent. Theorem 2 (b) could be relaxed to hold for a compact metric space of player types, but then the statement of the Theorem would be more complex. In particular, thickness would need to be redefined to require that there be many near-substitutes (players with similar attributes) for each player in each admissible profile and the argument would use substitution, STN. SGE strengthens PCB to allow nonemptiness and convergence results for many-player games in which some types of players appear in vanishingly small percentages.

4 Nonemptiness and equal treatment properties of cores of games with many players

4.1 Nonemptiness

The following Theorem is an extension of the nonemptiness of approximate cores of many-player games of Wooders (1992a,b). The framework of that paper required PCB as part of the definition of a pregame. Since SGE implies PCB, when SGE is assumed the assumption of PCB is not required.

Theorem 3 (Nonemptiness of approximate cores of many player games.) *Let (Ω, Ψ) be a pregame satisfying SGE. Then:*

Given any positive real number $\varepsilon > 0$ there is a positive real number $v(\varepsilon)$ such that, for any induced game (N, v) , if $|N| > v(\varepsilon)$ then the game has a nonempty ε -core.

4.2 Equal treatment properties

Since the equal treatment properties of approximate cores of games with many players are easiest to state and understand for the case of a finite number of types, we first state a result for this case and then proceed to the case of a compact metric space of player types. The first theorem below states that, given a sufficiently small non-negative real number ε , for any game with a finite set of player attributes (or types) any payoff vector x in the ε -core of the game has the property that, for each type of player that appears in sufficient abundance in the population, most players of that type are treated approximately equally. Note that in interpretation of the theorem the numbers γ and λ are to be thought of as ‘small’.

Theorem 4 (Near equal treatment of most players of the same type.) *Let (Ω, Ψ) be a pregame where $\Omega = \{\omega_1, \dots, \omega_T\}$ is a finite set and assume that (Ω, Ψ) satisfies SGE. Then given any real numbers $\gamma > 0$, $\lambda > 0$ and $\delta > 0$ there is a positive real*

number ε^* and an integer η such that for each $\varepsilon \in [0, \varepsilon^*]$ and for every profile $n \in \mathcal{F}$ with $\|n\|_1 > \eta$, if $x \in \mathbb{R}^N$ is in the ε -core of the game $[n, \Psi]$ with player set

$$N = \{(t, q) : t = 1, \dots, T \text{ and, for each } t, q = 1, \dots, n(\omega_t)\}$$

then, for each $t \in \{1, \dots, T\}$ with $\frac{n(\omega_t)}{\|n\|_1} \geq \delta$, it holds that

$$|\{(t, q) : |x^{tq} - z_t| > \gamma\}| < \lambda n(\omega_t),$$

where

$$z_t = \frac{1}{n(\omega_t)} \sum_{q=1}^{n(\omega_t)} x^{tq},$$

the average payoff received by players of type t .

Note that the above result allows ‘scarce’ types; it is not required that *all* players have many close substitutes. Some players could be quite exceptional—extremely talented, handsome, and charismatic, or completely unable to dance the salsa, for example. With SGE the following Corollary, which admits scarce types in its conclusion, is a consequence of the total payoff to scarce types becoming small relative to the total numbers of players.

Corollary 1 *Let (Ω, Ψ) be a pregame where $\Omega = \{\omega_1, \dots, \omega_T\}$ is a finite set and assume that (Ω, Ψ) satisfies SGE. Then given any real numbers $\gamma > 0$ and $\lambda > 0$ there is a positive real number ε^* and an integer ρ such that for each $\varepsilon \in [0, \varepsilon^*]$ and for every profile $n \in \mathcal{F}$ with $\|n\|_1 > \rho$, if x is in the ε -core of the game $[n, \Psi]$ with player set*

$$N = \{(t, q) : t = 1, \dots, T \text{ and, for each } t, q = 1, \dots, n(\omega_t)\}$$

then, for each $t \in \{1, \dots, T\}$, it holds that

$$|\{(t, q) : |x^{tq} - z_t| > \gamma\}| < \lambda \|n\|_1, \tag{4.1}$$

where

$$z_t = \frac{1}{n(\omega_t)} \sum_{q=1}^{n(\omega_t)} x^{tq},$$

the average payoff received by players of type t .

Notice that in Corollary 1, the conclusion has an upper bound that depends on the size $\|n\|_1$ of the total player set. If some type, say t' , appears in only a small proportion in the population, then it may be the case that for *all* players of this type $|x^{t'q} - z_{t'}| > \gamma$.

We stress that the Corollary, and Theorem 4 need not hold under the assumption of only PCB; we refer the reader to Example 4 of the following section.

Our next result allows a compact metric space of player types. For ease of statement, Theorem 5 extends Corollary 1. We leave the extension of Theorem 4 to the interested reader.

Theorem 5 (Near equal-treatment of similar players.) *Let Ω be a pregame satisfying SGE. Then given any real numbers $\gamma > 0$ and $\lambda > 0$ there are real numbers $\varepsilon^* > 0$ and $\delta > 0$, integers T and ρ , and a partition of Ω into no more than T subsets, say $\Omega_1, \dots, \Omega_T$, each contained in a ball of diameter less than δ , such that for each $\varepsilon \in [0, \varepsilon^*]$ and for every game (N, v_α) induced by the pregame, if $x \in \mathbb{R}^N$ is in the ε -core of the game (N, v_α) and if $|N| \geq \rho$, it holds that*

$$|\{i \in N : \alpha(i) \in \Omega_t, |x^i - z_t| > \gamma\}| < \lambda|N|,$$

where

$$z_t = \frac{1}{|\{i \in N : \alpha(i) \in \Omega_t\}|} \sum_{i \in N : \alpha(i) \in \Omega_t} x^i,$$

the average payoff received by players with attributes in the set Ω_t .

5 Inequality and the importance of alternative opportunities

The next example demonstrates that under the assumption of SGE players of scarce types need not be treated approximately equally while the following example demonstrates that, in the absence of thickness of the player set (ensuring many substitutes for each player), even players of abundant types may be treated unequally.

Example 3 (Unequal treatment of scarce types.) Let (Ω, Ψ) be a pregame where $|\Omega| = 2$ and the payoff $\Psi(n)$ to any profile $n = (n_1, n_2)$ is given by:

$$\Psi(n) = \begin{cases} n(\omega_1) + n(\omega_2) & \text{if } n(\omega_1) \geq 2 \\ n(\omega_2) & \text{if } n(\omega_1) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the pregame satisfies SGE. Now consider a sequence of games (N^ν, v^ν) where the profile of N^ν is given by $n(\omega_1) = 2$, $n(\omega_2) = \nu$. Then for any ν , the payoff vector assigning 0 to one player of type 1, 2 to the other player of type 1, and 1 to each of the ν players of type 2 is in the core of the game (N^ν, v^ν) .

Given the equivalence, with thickness, of PCB and SGE, one might wonder whether PCB would suffice to obtain the results of Theorem 4 or 5. The following example illustrates that it will not. In the presence of small percentages of players of some types, that is, without thickness of the total player set (when PCB is equivalent to SGE) the result that players of *abundant* types are treated nearly equally may not hold.

Example 4 (Without thickness, PCB does not imply equal treatment, even of players of abundant types.) Let (Ω, Ψ) be a pregame where $\Omega = \{\omega_1, \omega_2\}$ and the worth $\Psi(n)$ to any profile n is given by:

$$\Psi(n) = \begin{cases} n(\omega_1) + n(\omega_2) & \text{if } n(\omega_1) > 0, n(\omega_2) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now consider a sequence of games (N^ν, v^ν) where the profile of N^ν is denoted by n^ν and satisfies $n^\nu(\omega_1) = 1, n^\nu(\omega_2) = \nu$. Then given ν , consider a payoff vector $x^\nu \in \mathbb{R}^{N^\nu}$ assigning $x_{2q}^\nu = \frac{q}{\nu}$ to the q^{th} player of type 2, $q = 1, \dots, \nu$, and assigning $1 + \nu - \sum_q x_{2q}^\nu$ to the one player of type 1. Then, for any ν , x^ν is in the core of the game (N^ν, v^ν) . With some additional work, the same conclusion can be obtained for approximate cores.

The following Proposition illustrates the importance of alternative opportunities for equal treatment;

Proposition 1 *Let (N, v) be a game and let $x \in \mathbb{R}^N$ be in the core of the game. Suppose that there are two players i and j who are δ -substitutes for each other, for some $\delta > 0$, and also suppose that there are two disjoint groups $S, S' \subset N$ satisfying $i \in S, j \in S'$ and $x(S) = v(S), x(S') = v(S')$. Then it follows that $|x^i - x^j| \leq \delta$.*

Notice that in the above Proposition there is no need for any topological structure on the set of player types. The key feature enabling the result is that there exist two disjoint coalitions containing players i and j which can both achieve the core payoff vector x for their members. The following Proposition also illustrates the effectiveness of disjoint coalitions containing similar players – that is, there are *alternative opportunities* – without any topological structure on the set of player types.¹¹

Proposition 2 *Let (N, v) be a game and let $x \in \mathbb{R}^N$ satisfy $x(N) \leq v(N)$ (so x is a feasible payoff vector). Suppose that two players i and j are 0-substitutes for each other, and also suppose that there are two disjoint groups $S, S' \subset N$ satisfying $i \in S, j \in S'$ and $x(S) = v(S), x(S') = v(S')$. Suppose $x_i > x_j$ and define $\gamma = x_i - x_j$. Set $\varepsilon_\gamma = \frac{\gamma}{2|N|}$. Then for all $\varepsilon \in [0, \varepsilon_\gamma]$ the payoff vector x cannot be in the ε -core of the game.*

As one can see from Propositions 1 and 2, if there are alternative opportunities for a player that do not require the participation of some substitute for that player, then the player and his substitute must be treated equally or nearly equally. It is not always the case, however, that such opportunities exist. In fact, while such opportunities arise in models of local public good/club economies with many players, congestion and one private good (cf., Wooders, 1980) they are not required for convergence of the core to equal treatment (cf., Allouch and Wooders, 2008). Indeed, assumptions

¹¹ Kovalenkov and Wooders (2001) provides related results for situations in which, instead of having an underlying space of player types, the concept of δ -substitutes is used to treat similar players. For their model, with “limited side payments,” approximate cores treat any two similar players nearly equally.

commonly made on private goods exchange economies do not ensure the existence of such opportunities. Thus, requiring the existence of alternative opportunities is a severe restriction.

6 Relationships to prior literature on cooperative games with many players

Approximate cores of economies with quasi-linear utility functions were introduced in [Shapley and Shubik \(1966\)](#), which showed that when the player set is replicated, then, for all sufficiently large replications, approximate cores are nonempty.¹² A contribution by [Owen \(1975\)](#) is also relevant. In this paper, Owen demonstrates that economies with linear production also generate totally balanced games. [Hurwicz and Uzawa \(1977\)](#) demonstrate that aggregation over large numbers of production sets yields approximate convexity. While approximate cores were studied in the context of economies (for example, [Kannai 1970, 1972](#)), there were few results treating approximate cores in the game-theoretic literature. Exceptions are [Weber \(1979, 1981\)](#) for games with a continuum of players. To obtain his results, Weber introduced concepts of balancedness for games with a continuum of players and demonstrated that, for every $\varepsilon > 0$, the ε -core was nonempty—the continuum of players did not suffice to obtain nonemptiness of the core.

Nonemptiness of approximate cores of TU games with many players, without balancedness assumptions, was initiated in [Wooders \(1979\)](#) under a condition of strict small group effectiveness and first results on convergence of cores to equal treatment cores were demonstrated for games with a fixed distribution of player types. Variations of the condition have appeared in a number of papers of this author and her co-authors. The nonemptiness results were extended to hold for NTU (and TU) games in [Wooders \(1983\)](#). [Shubik and Wooders \(1982\)](#) applied [Wooders \(1979\)](#) results for TU games satisfying PCB to demonstrate nonemptiness of approximate cores of games derived from economies with production and with possibly multiple-membership clubs. For the TU case, [Wooders and Zame \(1984\)](#) extended [Wooders'](#) earlier results to hold with a compact metric space of player types but, as it turns out, under the unnecessarily restrictive assumption of boundedness of individual marginal contributions to coalitions. Numerous other papers have since considered nonemptiness of approximate cores; see [Kovalenkov and Wooders \(2003\)](#) for a survey and [Wooders \(2008\)](#) for most recent results for NTU games satisfying a condition of small group effectiveness.

The condition of small group effectiveness in this paper appeared in [Wooders \(1992a,b\)](#). Both small group effectiveness and small group negligibility were introduced in earlier working papers due to this author (cf. [Wooders 1992a](#) and references therein).

As noted in the introduction, the study of equal treatment outcomes in cores of private goods economies has a long history with some papers demonstrating that under their conditions, cores have the equal treatment property ([Debreu and Scarf, 1963](#), for

¹² There has been vast literature on convergence of cores in models of economies, or ones where the worth of a coalition is achieved by joining together commodities or attributes owned by its members. Except for a few classic references here and in the introduction, we do not address this literature.

example) and other papers demonstrating that under other conditions, cores do not have the equal treatment property (Green, 1972, for example). Another paper, Khan and Polemarchakis (1978) shows, roughly, that in some sense an arbitrary outcome in the core of an economy is not likely to treat all individuals who are the same equally. They, as much of this literature, however, address the issue of whether individuals who have the same preferences and endowments will be assigned the same commodity bundles by an outcome in the core. Our concern has been with the issue of whether similar individuals will be assigned similar utilities or payoffs by payoff vectors in the core. Our results relate most, in spirit, to those of Hildenbrand and Kirman (1973) who show, as we do (but for a different model and different formulations), “size removes inequality.”

7 Conclusion

This paper contributes to a line of research investigating the competitive-economy-like properties of games with many players. The foundational papers of the early literature noted in the introduction take as given specific economic models and study their properties in depth. This paper contributes to a literature seeking to understand competitive properties of games. These properties may be satisfied by a diversity of economic models (subject to the constraint that utilities are linear in one commodity), including economies with public good, clubs, coalition production, and so on. Our results apply then to games derived from such economies, independently of further specification of economic structures. A companion paper relates the results of this paper to market-game equivalence, as in Shapley and Shubik (1969), but, as in Wooders (1994), for many-player games.

Besides our approximate core convergence results, an important part of the paper is demonstrating the equivalence of small group effectiveness and small group negligibility. SGN is an appealing condition since it relates well to models with a continuum of players. In such a model, it is implicit that sets of measure zero can be ignored. SGN dictates that relatively small groups of players can have only small effects on aggregate outcomes. Thus, SGN (or alternatively, SGE) arguably is a necessary condition underlying the use of the continuum as an approximation to models with a large but finite set of players.

Another important part of the paper is to reveal the importance of scarce player types. With only the assumption of PCB, finiteness of the supremum of average payoff, as in Wooders (1979) and Shubik and Wooders (1982), if some players have “few” substitutes then even players who have many close substitutes may be almost all treated far from the average for their types. Example 4, making this point within the context of cooperative games with many players, is, to the best of our knowledge, unique to this paper.

8 Appendix: Proofs

First, it is convenient to introduce some notational conventions. Given a profile n with support $\{\omega_1, \dots, \omega_T\}$, as previously, to discuss payoff vectors in the core we must have given a set of players rather than a list of numbers of players of each type. Thus,

we let $N = \{(t, q) : t = 1, \dots, T \text{ and } q = 1, \dots, n(\omega_T)\}$ denote an associated total player set. Similarly, for any positive integer r we denote the total player set of the game $[rn, \Psi]$ by $N_r = \{(t, q) : t = 1, \dots, T \text{ and } q = 1, \dots, rn(\omega_T)\}$. When we consider a sequence of games, $\{[n^v, \Psi]\}_v$ we denote the corresponding player sets by N^v .

8.1 A sketch of the claims in subsection 1.1

We will now sketch the proof of core nonemptiness from Sect. 1.1. First, we introduce balanced covers of games and pregames.

The *balanced cover game* generated by a game $[n, \Psi]$ is a game $[n, \Psi^b]$ where

1. $\Psi^b(s) = \Psi(s)$ for all $s \neq n$ and
2. $\Psi^b(n) \geq \Psi(n)$ and $\Psi^b(n)$ is as small as possible consistent with the nonemptiness of the core of $[n, \Psi^b]$.

From the Bondareva–Shapley Theorem it follows that $\Psi^b(n) = \Psi(n)$ if and only if the game $[n, \Psi]$ is balanced (ε -balanced, with $\varepsilon = 0$).

Let (Ω, Ψ) be a pregame where $\Omega = \{\omega_1, \dots, \omega_T\}$. For each profile f , define a *balanced collection of subprofiles of f* as a collection of subprofiles $\{g^k\}_k$ with corresponding weights $\{\gamma_k : \gamma_k \geq 0\}$ satisfying $\sum_k \gamma_k g^k = f$. The *balanced cover pregame*, denoted by (Ω, Ψ^b) is the pregame with

$$\Psi^b(f) \stackrel{\text{def}}{=} \max_{\beta} \sum_{g \in \beta} \gamma_k \Psi(g^k), \tag{8.1}$$

where the maximum is taken over all balanced collections β of subprofiles of f . Since a partition of a profile is a balanced collection of subprofiles it is immediately clear that $\Psi^b(f) \geq \Psi(f)$ for every profile f .

With the above definitions in hand, we consider a pregame satisfying strict small group effectiveness, SSGE, with bound B on effective group sizes. Formally, this means that for any profile f there is a partition of f into subprofiles $\{f^k\}$ (repetitions allowed) such that $\Psi(f) = \sum_k \Psi(f^k)$ and $\|f^k\| \leq B$ for each f^k in the partition.

Let a profile n be interpreted as the profile of an initially given player set and let r be an integer sufficiently large so that $rn(\omega_t) > B$ for each $t = 1, \dots, T$. From SSGE, there is a balanced collection of subprofiles of n , say $\{g^k\}_k$ such that for some corresponding weights $\{\gamma_k^*\}_k$ it holds that $\|g^k\| \leq B$ for each k and

$$\Psi^b(n) = \sum_k \gamma_k^* \Psi(g^k). \tag{8.2}$$

Without loss of generality, we can suppose that all the weights γ_k^* are rational numbers (Shapley 1967). Therefore there is an integer m_0 such that $m_0\gamma_k^*$ is an integer for each γ_k^* . Thus, there is a partition of n into subprofiles with $m_0\gamma_k^*$ elements g^k

in the partition for each k ; that is, $\sum_k (m_0 \gamma_k^*) g^k = n$. From superadditivity and (8.2) it follows that the games $[rm_0 n, \Psi]$ have nonempty cores for all positive integers r (Wooders 1979; Wooders 1983).

For the convenience of the reader we state the results before presenting their proofs. We first review the concept of balanced games, which will be used in the proofs.

8.2 Proof of Theorems 1 and 2

Given a pregame (Ω, Ψ) let Ψ^b denote the balanced cover of Ψ where, for each profile n , $\Psi^b(n)$ is defined as the smallest real number such that (N, v^b) is the balanced cover of (N, v) and (N, v) is the game induced by the pregame (Ω, Ψ) and the profile n .

Theorem 1 (Equivalence of small group effectiveness, SGE, and small group negligibility, SGN): *Let (Ω, Ψ) be a pregame. Then (Ω, Ψ) satisfies SGE if and only if (Ω, Ψ) satisfies SGN.*

Proof Part 1: SGE implies SGN. We proceed by supposing the assertion is false. Then there are sequences of profiles $\{f^v\}$ and $\{\ell^v\}$, and, for some integer T , a subset $\{\omega_1, \dots, \omega_T\} \subset \Omega$ and a function $f : \{\omega_1, \dots, \omega_T\} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} & \text{support}(f^v) = \{\omega_1, \dots, \omega_T\} \text{ for each } v, \\ & \frac{f^v(\omega_t)}{\|f^v\|_1} \text{ converges to } f(\omega_t) \text{ for each } \omega_t \text{ in } \{\omega_1, \dots, \omega_T\}, \\ & \lim_{v \rightarrow \infty} \frac{\|\ell^v\|_1}{\|f^v\|_1} \rightarrow 0 \text{ as } v \text{ becomes large, and} \\ & \lim_{v \rightarrow \infty} \frac{\Psi(f^v)}{\|f^v\|_1} \text{ exists,} \end{aligned}$$

but either

$$\lim_{v \rightarrow \infty} \frac{\Psi(f^v + \ell^v)}{\|f^v + \ell^v\|_1} \text{ does not exist,}$$

or for some $\varepsilon_0 > 0$ it holds that

$$\left| \lim_{v \rightarrow \infty} \frac{\Psi(f^v + \ell^v)}{\|f^v + \ell^v\|_1} - \lim_{v \rightarrow \infty} \frac{\Psi(f^v)}{\|f^v\|_1} \right| > 3\varepsilon_0. \tag{8.3}$$

For ease in notation for each v define

$$g^v = f^v + \ell^v.$$

Since (Ω, Ψ) satisfies SGE there is an integer $\eta(\varepsilon_0)$ such that for each profile g^v there is a partition $\{g^{vk} : k = 1, \dots, K\}$ of g^v satisfying

$$\begin{aligned} & \|g^{vk}\|_1 \leq \eta(\varepsilon_0) \quad \text{for each } k \quad \text{and} \\ & \Psi(g^v) - \sum_{k=1}^K \Psi(g^{vk}) \leq \varepsilon_0 \|g^v\|_1. \end{aligned}$$

From the definition of g^ν there are at most $\|\ell^\nu\|$ members of the partition $\{g^{\nu k}\}$ with the property that $\text{support}(g^{\nu k}) \cap \text{support}(\ell^\nu) \neq \emptyset$. Thus, by renumbering profiles if necessary we can suppose that for some integer K' ,

$$\begin{aligned} K' &\geq K - \|\ell^\nu\|_1, \\ \text{support}(g^{\nu k}) \cap \text{support}(\ell^\nu) &= \emptyset \text{ for all } g^{\nu k} \text{ with } k \leq K' \text{ and} \\ \sum_{k=1}^{K'} g^{\nu k} &\leq f^\nu. \end{aligned}$$

From PCB (implied by SGE, from Theorem 2 below) and since, for each $g^{\nu k}$, $\eta(\varepsilon_0) \geq \|g^{\nu k}\|_1$, it follows that there is a per capita bound C satisfying $\frac{\Psi \|g^{\nu k}\|_1}{\eta(\varepsilon_0)} < C$ for all subprofiles $\{g^{\nu k}\}_k$ and we have

$$\Psi(g^\nu) - \sum_{k=1}^{K'} \Psi(g^{\nu k}) - C \|\ell^\nu\|_1 \eta(\varepsilon_0) < \varepsilon_0 \|g^\nu\|_1.$$

Since $\frac{\|\ell^\nu\|_1}{\|f^\nu\|_1} \rightarrow 0$ as $\nu \rightarrow \infty$ it follows that, for all sufficiently large ν , $C \frac{\|\ell^\nu\|_1}{\|f^\nu\|_1} \eta(\varepsilon_0) < \varepsilon_0 \|g^\nu\|_1$ and

$$0 \leq \Psi(g^\nu) - \sum_{k=1}^{K'} \Psi(g^{\nu k}) < 2 \varepsilon_0 \|g^\nu\|_1.$$

Since $\sum_{k=1}^{K'} g^{\nu k} \leq f^\nu$ and from superadditivity it holds that $\sum_{k=1}^{K'} \Psi(g^{\nu k}) \leq \Psi(f^\nu)$ and $\Psi(g^\nu) \geq \Psi(f^\nu)$. Therefore,

$$0 \leq \Psi(g^\nu) - \Psi(f^\nu) < 2\varepsilon_0 \|g^\nu\|_1.$$

Since $\lim_{\nu \rightarrow \infty} \frac{\|\ell^\nu\|_1}{\|f^\nu\|_1} \rightarrow 0$, it follows that for all ν sufficiently large that

$$\left| \frac{\Psi(g^\nu)}{\|g^\nu\|_1} - \frac{\Psi(f^\nu)}{\|f^\nu\|_1} \right| = \left| \frac{\Psi(f^\nu + \ell^\nu)}{\|f^\nu + \ell^\nu\|_1} - \frac{\Psi(f^\nu)}{\|f^\nu\|_1} \right| < 3\varepsilon_0,$$

a contradiction to (8.3).

Part 2: SGN implies SGE. Given a positive integer ν , a partition $\{g^k\}_k$ of a profile g will be ν -bounded if $\|g^k\|_1 < \nu$ for each k . Suppose (Ω, Ψ) satisfies SGN but does not satisfy SGE. Then there is a positive real number $\varepsilon_0 > 0$ and a sequence of profiles $\{f^\nu\}_\nu$ such that for each integer ν , for every ν -bounded partition $\{f^{\nu k}\}$ of f^ν it holds that

$$\Psi(f^\nu) - \sum \Psi(f^{\nu k}) > 4\varepsilon_0 \|f^\nu\|_1. \tag{8.4}$$

Let δ be a positive real number satisfying the property that whenever two profiles f and g have $\text{dist}(f, g) < \delta$, then $\|f\|_1 = \|g\|_1$ and $|\Psi(f) - \Psi(g)| < \varepsilon_0 \|f\|_1$; this

is possible from STN (3.1). Let Ω, \dots, Ω_T be a partition of Ω into nonempty subsets, each contained in a ball of diameter less than δ , and for each $t \in \{1, \dots, T\}$ let ω_t be a point in Ω_t . For each profile f^ν define the profile g^ν by

$$g^\nu(\omega_t) = \sum_{\omega \in \text{support}(f^\nu) \cap \Omega_t} f^\nu(\omega) \text{ and} \\ g^\nu(\omega) = 0 \text{ for } \omega \notin \{\omega_1, \dots, \omega_T\}.$$

Note that $\text{dist}(f^\nu, g^\nu) < \delta$ so, from STN (3.1),

$$|\Psi(f^\nu) - \Psi(g^\nu)| \leq \varepsilon_0 \|f^\nu\|_1.$$

We can suppose without loss of generality that the sequence $\left\{ \frac{1}{\|g^\nu\|_1} g^\nu \right\}_\nu$ converges, say to g .

Observe that for some attributes ω_t it may be the case that $g(\omega_t) = 0$. We now define another sequence $\{\widehat{g}^\nu\}$ as follows:

$$\widehat{g}^\nu(\omega_t) = \begin{cases} g^\nu(\omega_t) & \text{if } g(\omega_t) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that from SGN, for all ν sufficiently large,

$$|\Psi(g^\nu) - \Psi(\widehat{g}^\nu)| < \varepsilon_0 \|g^\nu\|_1$$

Since the pregame (Ω, Ψ) satisfies PCB and since the sequence $\{\widehat{g}^\nu\}$ satisfies the property that the percentages of players of each type that appears in positive proportions in the game is bounded away from zero, we can now apply Wooders (1994, Theorem 4) to the sequence $\{\widehat{g}^\nu\}$ and conclude that there is an integer η such that each profile \widehat{g}^ν has a partition into subprofiles, say $\{\widehat{g}^{\nu k}\}_{k=1}^K$, with $\|\widehat{g}^{\nu k}\|_1 < \eta$ for each $k = 1, \dots, K$ and

$$\left| \Psi(\widehat{g}^\nu) - \sum_{k=1}^K \Psi(\widehat{g}^{\nu k}) \right| < \varepsilon_0 \|\widehat{g}^\nu\|_1.$$

For each ν define a profile ℓ^ν on $\{\omega_1, \dots, \omega_T\}$ by

$$\ell^\nu(\omega_t) = g^\nu(\omega_t) - \sum_{k=1}^K \widehat{g}^{\nu k}(\omega_t).$$

Observe that for each t , $\frac{\ell^\nu(\omega_t)}{\|g^\nu\|_1} \rightarrow 0$ as $\nu \rightarrow \infty$. Thus, from SGN, for all ν sufficiently large,

$$\Psi(\ell^\nu) < \varepsilon_0 \|\widehat{g}^\nu\|_1.$$

Let $\{f^{vk}\}$ be a partition of f^v into subprofiles where for $k = 1, \dots, K$, $f^{vk} = \widehat{g}^{vk}$ and for $k > K$, f^{vk} satisfies $\|f^{vk}\| \leq \eta$ and for each t ,

$$\sum_{k \geq K+1} f^{vk}(\omega_t) = \ell^v(\omega_t).$$

We now have

$$\begin{aligned} & \left| \Psi(f^v) - \sum_k \Psi(f^{vk}) \right| \\ & < \left| \Psi(f^v) - \Psi(g^v) \right| + \left| \Psi(g^v) - \Psi(\widehat{g}^v) \right| + \left| \Psi(\widehat{g}^v) - \sum_{k=1}^K \Psi(\widehat{g}^{vk}) \right| \\ & \quad + \left| \sum_{k=1}^K \Psi(\widehat{g}^{vk}) - \sum_k \Psi(f^{vk}) \right| \\ & < \varepsilon_0 \|f^v\|_1 + \varepsilon_0 \|f^v\|_1 + \varepsilon_0 \|f^v\|_1 + \left| \sum_{k>K} \Psi(f^{vk}) \right| \\ & < 3\varepsilon_0 \|f^v\|_1 + \Psi(\ell^v) \\ & < 4\varepsilon_0 \|f^v\|_1, \end{aligned}$$

which is a contradiction to (8.4). □

Theorem 2 *Relating SGE and PCB. Let (Ω, Ψ) be a pregame.*

- (a) *Suppose that (Ω, Ψ) satisfies SGE. Then (Ω, Ψ) satisfies PCB.*
- (b) *Suppose that Ω is a finite set and that (Ω, Ψ) satisfies PCB. Then given any $\rho \in (0, 1)$, the ρ -thick restriction (Ω, Ψ_ρ) of (Ω, Ψ) satisfies SGE.*

Proof In view of Wooders (1994, Theorem 4) we need only prove part (a) of the theorem. To prove (a), given a pregame (Ω, Ψ) satisfying SGE and $\varepsilon_0 > 0$, let $\eta(\varepsilon_0)$ satisfy the condition given in the definition of SGE.

Define

$$C' = \max_{\{g \in \mathcal{F}: \|g\|_1 \leq \eta(\varepsilon_0)\}} \frac{\Psi(g)}{\|g\|_1}.$$

It from some simple algebra that

$$\sup_f \frac{\Psi(f)}{\|f\|_1} \leq \varepsilon_0 + C'.$$

which implies that $\varepsilon_0 + C'$ is a per capita bound. □

8.3 Proofs of equal treatment Theorems

The following Theorem, which appeared in [Wooders \(1979\)](#) and in [Shubik and Wooders \(1982\)](#), will be used in the proof of our main results. For the purposes of the next proof, we remark that for each type t the bound λr could be replaced by $\lambda r n(\omega_t)$, the number of players of type t in the r^{th} game, since this would increase the size of an upper bound and thus constitute a relaxation of the bound. Notice that this theorem differs from [Theorem 4](#) in that the theorem requires a fixed distribution of player types.

Theorem 6 ([Wooders 1979](#); [Shubik and Wooders 1982](#)). *Let Ω be a finite set with $|\Omega| = T$ and let (Ω, Ψ) be a pregame satisfying PCB. Let n be a given profile on Ω . For any positive integer r , let*

$$N_r \stackrel{\text{def}}{=} \{(t, q) : t = 1, \dots, T \text{ and } q = 1, \dots, rn(\omega_t)\}.$$

Given any real numbers $\gamma > 0$ and $\lambda > 0$ there is a positive real number ε^ and an integer r^* such that for each $\varepsilon \in [0, \varepsilon^*]$ and for any $r \geq r^*$, if $x \in \mathbb{R}^{N_r}$ is in the ε -core of $[rn, \Psi]$, then for each t in $\text{support}(n)$,*

$$|\{(t, q) : |x^{tq} - z_t| > \gamma\}| < \lambda r,$$

where $\bar{z}_t \stackrel{\text{def}}{=} \frac{1}{rn(\omega_t)} \sum_{q=1}^{rn(\omega_t)} x^{tq}$, the average payoff received by players of type t .

Proof Let n be a profile over Ω . Given real numbers λ and γ greater than zero, select ε^* , r^* and $r_0 < r^*$ so that:

- (i) $\varepsilon^* > 0$ and $\varepsilon^* < \min_t \{ \frac{\lambda \gamma}{4\|n\|_1}, \frac{\gamma n(\omega_t)}{2\|n\|_1} \}$ where the minimum is over all t with $n(\omega_t) \neq 0$;
- (ii) for all $r \geq r_0$, $\left| \frac{\Psi(rn)}{r\|n\|_1} - \frac{\Psi^{b}(r_0n)}{r_0\|n\|_1} \right| < \varepsilon^*$;
- (iii) $\frac{\varepsilon^* r_0}{r^*} < \frac{\lambda}{2\|n\|_1}$.

Since $\lambda > 0$ and $\gamma > 0$, and $\left| \frac{\Psi(rn)}{r\|n\|_1} - \frac{\Psi^{b}(r_0n)}{r_0\|n\|_1} \right| \rightarrow 0$ as $r_0 \rightarrow \infty$ and $r \rightarrow \infty$, such a selection is possible. (This follows from the nonemptiness results of [Wooders 1979](#), or [1983](#) restricted to TU games, and also from the market-game equivalence of [Wooders 1994](#).)

Select $r \geq r^*$, let $\varepsilon \in [0, \varepsilon^*]$, and let x be in the ε -core of $[rn, \Psi]$. For each t , define z_t as in the statement of the Theorem. Since, for any $\varepsilon \geq 0$, the ε -core is convex, we have that the vector $z = (z_1, \dots, z_T) \in \mathbb{R}^T$, defined as in the statement of the Theorem, represents a payoff vector in the equal-treatment ε -core of the game $[rn, \Psi]$. (Note that to obtain the vector z as a convex combination, for each type form new payoff vectors by permuting the payoffs of players of that type and then take the average of all the payoff vectors thus constructed.) It follows that for all profiles $s \leq rn$,

$$z \cdot s \geq \Psi(s) - \varepsilon \|s\|_1$$

and

$$z \cdot rn \leq \Psi(rn).$$

Thus, z represents a payoff vector in the equal-treatment ε -core of the game $[rn, \Psi]$.

Let $N_r = \{(t, q) : t = 1, \dots, T, q = 1, \dots, rn(\omega_t)\}$. It is convenient to establish the convention that for each coalition $S \subset N_r$, S_t denotes the subset of players in S of type t , i.e., $S_t = \{(\hat{t}, q) : (\hat{t}, q) \in S \text{ and } \hat{t} = t\}$ for each $t = 1, \dots, T$. We define the profile of a coalition S by $s \in \mathbb{Z}_+^T$ with t^{th} component given by $|S_t|$ for each t .

Select a subset P of N_r so that the profile of P is r_0n and P contains the “worst-off” players of each type (the “poor”); thus, if $(t, q) \notin P$ then $x^{tq} \geq x^{tq'}$ for all q' with $(t, q') \in P$. Define $P_t = \{(t', q) \in P : t' = t\}$. Suppose that, for some type t^* ,

$$|P \cap \{(t^*, q) \in N_r : x^{t^*q} < z_{t^*} - \gamma\}| = r_0n(\omega_{t^*});$$

i.e. all players of type t^* in P receive less than the average payoff for players of that type minus γ . We then have

$$\Psi^b(r_0n) - \varepsilon r_0 \|n\|_1 \leq x(P) < r_0(z \cdot n) - \gamma r_0n(\omega_{t^*}) \leq r_0 \frac{\Psi(rn)}{r} - \gamma r_0n(\omega_{t^*}).$$

The first inequality follows from the fact that x in the ε -core of N_r . The second follows from the facts that $z_{t^*} \geq x^{t^*q} + \gamma$ for each q with (t^*, q) in P_{t^*} and $x(P_t) \leq r_0z_tn(\omega_t)$ for each t . The final inequality is from the feasibility of z , that is, $z \cdot rn \leq \Psi(rn)$. From the above inequality, the following inequality is immediate:

$$\Psi^b(r_0n) - \varepsilon r_0 \|n\|_1 < r_0 \frac{\Psi(rn)}{r} - \gamma r_0n(\omega_{t^*}).$$

Subtracting $\Psi^b(r_0n)$ from both sides of the expression, adding $\gamma r_0n(\omega_{t^*})$ to both sides, and dividing by $r_0 \|n\|_1$ we obtain

$$\frac{\gamma n(\omega_{t^*})}{\|n\|_1} - \varepsilon < \frac{\Psi(rn)}{\|rn\|_1} - \frac{\Psi^b(r_0n)}{\|r_0n\|_1}.$$

From (i) above and the fact that $\varepsilon \leq \varepsilon^*$, it holds that $\frac{\gamma n(\omega_{t^*})}{\|n\|_1} - \varepsilon^* > \varepsilon^*$ which, along with the preceding expression, implies that $\varepsilon^* < \frac{\Psi(rn)}{\|rn\|_1} - \frac{\Psi^b(r_0n)}{\|r_0n\|_1}$, a contradiction to (ii). Therefore, for each $t^* = 1, \dots, T$ it holds that

$$|P \cap \{(t^*, q) \in N_r : x^{t^*q} < z_{t^*} - \gamma\}| < r_0n(\omega_{t^*});$$

of the worst off players of type t^* , fewer than $r_0n(\omega_{t^*})$ can be treated worse than the average payoff for that type minus γ . This means that $\{(t, q) : x^{tq} - z^t < -\gamma\} \subset P$.

From the facts that:

$$\begin{aligned}
 r_0 \frac{\Psi(rn)}{r} - 2\varepsilon^* r_0 \|n\|_1 &\leq \Psi^b(r_0 n) - \varepsilon^* r_0 \|n\|_1 \quad (\text{from (ii)}), \\
 &\leq x(P) \quad (\text{since } x \text{ is in the } \varepsilon^* \text{-core}), \\
 &\leq r_0 z \cdot n \quad (\text{from the definition of } P), \\
 &\leq r_0 \frac{\Psi(rn)}{r} \quad (\text{from feasibility of } x),
 \end{aligned}$$

it follows that

$$0 \leq r_0 z \cdot n - x(P) \leq 2\varepsilon^* r_0 \|n\|_1. \tag{8.5}$$

Informally, the above expression says that, for each t , on average players of type t in P are receiving payoffs within $2\varepsilon^*$ of z_t .

We now turn to those players who are receiving payoffs significantly more (that is, more than γ) than the average for their types and put an upper bound on the number of such players. Define the set of “best off” players (the “rich”) R by

$$R = \{(t, q) \in N_r : x^{tq} > z_t + \gamma\}.$$

Define the set of “middle class” players M by

$$M = N_r \setminus (R \cup P).$$

Observe that, since $\sum_{(t,q) \in N_r} (x^{tq} - z_t) = 0$, it follows that

$$\gamma |R| \leq \sum_{(t,q) \in R} (x^{tq} - z_t) = \sum_{(t,q) \in P \cup M} (z_t - x^{tq}).$$

From (8.5) and the above expression,

$$\gamma |R| < 2\varepsilon^* r_0 \|n\|_1 + \sum_{(t,q) \in M} (z_t - x^{tq}).$$

Obviously, the larger the value of $\sum_{(t,q) \in M} (z_t - x^{tq})$, the larger it is possible for $|R|$ to be. We claim that $\sum_{(t,q) \in M} (z_t - x^{tq}) \leq 2\varepsilon^* |M|$. This follows from the fact that the players in P are the worst off, and they are, on average, each within $2\varepsilon^*$ of the average payoff for their types. Since those players in M are at least as well off, they must receive on average no less than the average for their types minus $2\varepsilon^*$. Therefore, $\sum_{(t,q) \in M} (z_t - x^{tq}) \leq 2\varepsilon^* |M|$. It now follows that

$$\gamma |R| \leq 2\varepsilon^* r_0 \|n\|_1 + 2\varepsilon^* |M|.$$

From $|M| + |R| = r \|n\|_1 - r_0 \|n\|_1$, $|M| \leq r \|n\|_1 - r_0 \|n\|_1$, and

$$\gamma |R| \leq 2\varepsilon^* r_0 \|n\|_1 + 2\varepsilon^* (r \|n\|_1 - r_0 \|n\|_1) \leq 2\varepsilon^* r \|n\|_1,$$

it follows that $\frac{|R|}{r \|n\|_1} \leq 2\frac{\varepsilon^*}{\gamma}$ and from (i), that $\frac{|R|}{r \|n\|_1} \leq \frac{\lambda}{2\|n\|_1}$.

Counting the number of players who may be treated significantly differently than the average we see that:

$$\frac{|P|}{\|rn\|_1} + \frac{|R|}{\|rn\|_1} \leq \frac{\varepsilon^* r_0}{r} + \frac{\lambda}{2\|n\|} < \frac{\lambda}{\|n\|} \text{ from (i) and (iii) above.}$$

The conclusion of the Theorem is immediate from the observation that if x is in the ε -core of rn for $r \geq r^*$ and $0 \leq \varepsilon \leq \varepsilon^*$, then x is in the ε^* -core of rn . □

Theorem 4 (Near equal treatment of players of the same type.) *Let (Ω, Ψ) be a pregame where $\Omega = \{\omega_1, \dots, \omega_T\}$ is a finite set and assume that (Ω, Ψ) satisfies SGE. Then given any real numbers $\gamma > 0$, $\lambda > 0$ and $\delta > 0$ there is a positive real number ε^* and an integer η such that for each $\varepsilon \in [0, \varepsilon^*]$ and for every profile $n \in \mathcal{F}$ with $\|n\|_1 > \eta$, if $x \in \mathbb{R}^N$ is in the ε -core of the game $[n, \Psi]$ with player set*

$$N = \{(t, q) : t = 1, \dots, T \text{ and, for each } t, q = 1, \dots, n_t\}$$

then for each $t \in \{1, \dots, T\}$ with $\frac{n(\omega_t)}{\|n\|_1} \geq \delta$ it holds that

$$|\{(t, q) : |x^{tq} - z_t| > \gamma\}| < \lambda n(\omega_t),$$

where

$$z_t = \frac{1}{n(\omega_t)} \sum_{q=1}^{n(\omega_t)} x^{tq},$$

the average payoff received by players of type t .

Proof of Theorem 4 Suppose the statement of the Theorem is false. Then there is a pregame (Ω, Ψ) , where $\Omega = \{\omega_1, \dots, \omega_T\}$ for some T , satisfying the condition of SGE, and real numbers $\gamma > 0$ and $\lambda > 0$ such that: for every integer ν and every positive real number ε_ν there is a real number $\varepsilon \in [0, \varepsilon_\nu]$, a game $[n^\nu, \Psi]$ and a payoff vector $x_\nu \in \mathbb{R}^{N^\nu}$ in the ε -core of the game with the property that, for some t with $\frac{n^\nu(\omega_t)}{\|n^\nu\|_1} > \delta$,

$$|\{(t, q) : q = 1, \dots, n^\nu(\omega_t) \text{ and } |x_\nu^{tq} - z_\nu^t| > \gamma\}| > \lambda n^\nu(\omega_t),$$

where $z_\nu^t \stackrel{\text{def}}{=} \frac{1}{n^\nu(\omega_t)} \sum_{q=1}^{n^\nu(\omega_t)} x_\nu^{tq}$, the average payoff received by players of type t .

[As a guide to the reader, the basic strategy of the following is to first describe all sufficiently large games in the sequence $\{[n^\nu, \Psi]\}_\nu$ as replica games plus some

‘leftovers.’ The leftovers will constitute only a small fraction of the total player set. Moreover, by Theorem 1, their effects on per capita payoff of large groups they might join become small as the games grow in size. Thus, any approximate core payoff vector for a sufficiently large game must also be an approximate core payoff vector—for a slightly less close approximation—for the subgame consisting of a large replica of some player set.]

By passing, if necessary, to a subsequence of the sequence of games $\{[n^v, \Psi]\}$, we can without loss of generality assume that for each $\omega_t \in \Omega$ the sequence $\left\{ \frac{1}{\|n^v\|_1} n^v(\omega_t) \right\}$ converges. Define

$$\bar{n}_t = \lim_{v \rightarrow \infty} \frac{1}{\|n^v\|_1} n^v(\omega_t).$$

By relabelling points in Ω we can assume that for some $T' \leq T$ it holds that $\bar{n}_t > 0$ for $t = 1, \dots, T'$ and $\bar{n}_t = 0$ for $t = T' + 1, \dots, T$. We can similarly suppose (since SGE implies PCB) that the sequence $\left\{ \frac{\Psi(n^v)}{\|n^v\|_1} \right\}$ converges.

Let $\{h^v\}$ be a sequence of profiles on Ω with the properties that:

1. $\|h^v\|_1 \rightarrow \infty$.
2. For each $t \in \{1, \dots, T'\}$, $\lim_{v \rightarrow \infty} \frac{1}{\|h^v\|_1} h^v(\omega_t) = \bar{n}_t$ and for each $t \in \{T' + 1, \dots, T\}$ and each v , $h_t^v = 0$.
3. For each $t \in \{1, \dots, T'\}$, $h^v(\omega_t) \leq n^v(\omega_t)$ and $\lim_{v \rightarrow \infty} \frac{h^v(\omega_t)}{n^v(\omega_t)} = 0$.

Now for each v consider the sequence of induced games $\{[rh^v, \Psi]\}_{r=1}^\infty$. Note that this sequence satisfies the conditions of Theorem 6. Let $\hat{\varepsilon}_v > 0$ be a positive real number and let \hat{r}_v be an integer such that for each $\varepsilon \in [0, \hat{\varepsilon}_v]$ and for any $r \geq \hat{r}_v$, if $y \in \mathbb{R}^{\hat{N}^v}$ is in the ε -core of $[rh^v, \Psi]$, with total player set denoted by $\hat{N}_r^v = \{(t, q) : t = 1, \dots, T, q = 1, \dots, rh^v(\omega_t)\}$, then for each $t \in \{1, \dots, T'\}$

$$|\{(t, q) \in \hat{N}_r^v : q = 1, \dots, rh^v(\omega_t) \text{ and } |y^{tq} - \tilde{z}_v^t| > \frac{\varepsilon}{2}\}| < \frac{\varepsilon}{2} rh_t^v,$$

where $\tilde{z}_v^t \stackrel{\text{def}}{=} \frac{1}{rh^v(\omega_t)} \sum_{q=1}^{rh^v(\omega_t)} y^{tq}$, the average payoff assigned by y to players of type t in the player set \hat{N}^v .

Next, let m_v be the largest integer such that $m_v h^v(\omega_t) \leq n^v(\omega_t)$ for each $t = 1, \dots, T'$. Since for each t , $\frac{h^v(\omega_t)}{m_v} \rightarrow 0$ as $v \rightarrow \infty$ (from 3.) for all v sufficiently large, say $v \geq v^*$, it holds that $m_v \geq \hat{r}_v$. Thus, for all sufficiently large games in the sequence $\{[m_v h^v, \Psi]\}_v$, the conclusion of Theorem 6 holds. That is, there is an integer v^* and a positive real number $\tilde{\varepsilon}_v$ such that for each $\varepsilon \in [0, \tilde{\varepsilon}_v]$ and for any $v \geq v^*$, if $y \in \mathbb{R}^{\tilde{N}^v}$ is in the ε -core of $[m_v h^v, \Psi]$, (with total player set denoted by $\tilde{N}^v = \{(t, q) : t = 1, \dots, T, q = 1, \dots, m_v h^v(\omega_t)\}$), then for each $t \in \{1, \dots, T'\}$ $|\{(t, q) \in \tilde{N}^v : |y^{tq} - \tilde{z}_v^t| > \frac{\varepsilon}{2}\}| < \frac{\varepsilon}{2} m_v h_t^v$, where $\tilde{z}_v^t \stackrel{\text{def}}{=} \frac{1}{m_v h^v(\omega_t)} \sum_{q=1}^{m_v h^v(\omega_t)} y^{tq}$, the average payoff assigned by y to players of type t in the player set \tilde{N}^v .

For each v , let $\varepsilon_v^* = \min\{\tilde{\varepsilon}_v, \frac{\varepsilon}{2}\}$.

Let ℓ^ν be a profile satisfying the property that for each $t \in \{1, \dots, T\}$,

$$m_\nu h^\nu(\omega_t) + \ell^\nu(\omega_t) = n^\nu(\omega_t);$$

in terms of our informal discussion of the proof, the profile ℓ^ν represents the ‘leftovers.’ Observe that $\lim_\nu \frac{m_\nu h^\nu(\omega_t) + \ell^\nu(\omega_t)}{\|m_\nu h^\nu + \ell^\nu\|_1} = \bar{n}_t$ and that $\lim_\nu \frac{\ell^\nu(\omega_t)}{\|m_\nu h^\nu + \ell^\nu\|_1} = 0$. It follows from Theorem 1 that

$$\lim_{\nu \rightarrow \infty} \frac{\Psi(n^\nu)}{\|n^\nu\|_1} = \lim_{\nu \rightarrow \infty} \frac{\Psi(m_\nu h^\nu + \ell^\nu)}{\|m_\nu h^\nu + \ell^\nu\|_1} = \lim_{\nu \rightarrow \infty} \frac{\Psi(m_\nu h^\nu)}{\|m_\nu h^\nu\|_1}.$$

By passing to a subsequence and re-numbering if necessary, we can suppose that for each ν sufficiently large, say $\nu \geq \nu^*$,

$$\left| \frac{\Psi(n^\nu)}{\|m_\nu h^\nu\|_1} - \frac{\Psi(m_\nu h^\nu)}{\|m_\nu h^\nu\|_1} \right| \leq \frac{\varepsilon_\nu^*}{2}. \tag{8.6}$$

Let ν_0 be sufficiently large so that for all $\nu \geq \nu_0$ it holds that

$$\frac{\sum_t \ell^\nu(\omega_t)}{n^\nu(\omega_t)} \leq \frac{\lambda}{2}$$

Select $\nu \geq \max\{\nu^*, \nu_0\}$, an $\varepsilon \in \left[0, \frac{\varepsilon_\nu^*}{2}\right]$, and a payoff vector $x_\nu \in \mathbb{R}^{N^\nu}$ be in the ε -core of the induced game $[n^\nu, \Psi]$ with player set denoted by $N^\nu = \{(t, q) : t = 1, \dots, T \text{ and, for each } t, q = 1, \dots, n^\nu(\omega_t)\}$ so that

$$|\{(t, q) \in N^\nu : q = 1, \dots, n_t^\nu \text{ and } |x_\nu^{tq} - z_t^\nu| > \gamma\}| > \lambda n^\nu(\omega_t),$$

where $z_t^\nu = \frac{1}{n(\omega_t)} \sum_{q=1}^{n_t(\omega_t)} x_\nu^{tq}$. From our initial supposition, such a selection is possible.

Let N_0^ν be a subset of N^ν with profile equal to $m_\nu h^\nu$. We claim that, for each $t \in \{1, \dots, T\}$ it holds that

$$\left| \left\{ (t, q) \in N_0^\nu : |x_\nu^{tq} - z_t^\nu| > \frac{\gamma}{2} \right\} \right| < \frac{\lambda}{2} n^\nu(\omega_t) \tag{8.7}$$

where again $z_t^\nu = \frac{1}{n^\nu(\omega_t)} \sum_{q=1}^{n^\nu(\omega_t)} x_\nu^{tq}$. Since x is in the ε -core for $\varepsilon \in \left[0, \frac{\varepsilon_\nu^*}{2}\right]$, the payoff vector x_ν^* defined by $x_\nu^{*tq} = x_\nu^{tq} - \frac{\varepsilon_\nu^*}{2}$ ($\geq x_\nu^{tq} - \frac{\gamma}{2}$) for each $(t, q) \in N_0^\nu$ and $x_\nu^{*tq} = x_\nu^{tq}$ otherwise, is in the ε' -core for $\varepsilon' = \varepsilon + \frac{\varepsilon_\nu^*}{2} \leq \varepsilon_\nu^*$. This holds since x^* cannot be ε' improved upon by any coalition by at least ε' and because, from (8.6) and the fact that x is in the ε -core of the game $[n^\nu, \Psi]$, and thus feasible for the total player set N^ν ,

$$x_\nu^*(N_0^\nu) = x(N_0^\nu) - \frac{\varepsilon_\nu^*}{2} |N_0^\nu| \leq \Psi(n^\nu) - \frac{\varepsilon_\nu^*}{2} |N_0^\nu| \leq \Psi(m_\nu h^\nu),$$

which implies that $(x_v^{*tq} : (t, q) \in N_0^v)$ is feasible for N_0^v . Thus, for the subgame with player set N_0^v , (8.7) holds.

We now have, given $t' \in \{1, \dots, T'\}$,

- (a) $|\{(t, q) \in N^v : t = t', q = 1, \dots, n_t^v, |x_v^{tq} - z_v^t| > \gamma\}|$
- (b) $\leq |\{(t, q) \in N_0^v : t = t', q = 1, \dots, n^v(\omega_t), |x_v^{tq} - z_v^t| > \frac{\gamma}{2} + \frac{\varepsilon_v^*}{2}\}| + n^v(\omega_t) - m_v n^v(\omega_t)$
- (c) $= |\{(t, q) \in N_0^v : t = t', q = 1, \dots, n^v(\omega_t), |(x_v^{tq} - \frac{\varepsilon_v^*}{2}) - (z_v^t - \frac{\varepsilon_v^*}{2})| > \frac{\gamma}{2} + \frac{\varepsilon_v^*}{2}\}| + \frac{\lambda}{2}$
- (d) $\leq |\{(t, q) \in N_0^v : t = t', q = 1, \dots, n^v(\omega_t), |x_v^{*tq} - z_v^t + \frac{\varepsilon_v^*}{2}| > \frac{\gamma}{2} + \frac{\varepsilon_v^*}{2}\}| + \frac{\lambda}{2}$
- (e) $\leq |\{(t, q) \in N_0^v : |x_v^{*tq} - z_v^t| > \frac{\gamma}{2}\}| + \frac{\lambda}{2}$
- (f) $\leq (\frac{\lambda}{2} + \frac{\lambda}{2}) n^v(\omega_t) = \lambda n^v(\omega_t)$

where (a) follows from our supposition, (b) follows from the choice of $\varepsilon_v^* \geq \frac{\gamma}{2}$ and the fact that the constraining value $\frac{\gamma}{2} + \frac{\varepsilon_v^*}{2}$ is less than or equal to γ , (c) is simple algebra, (d) follows from the definition of $x_v^{*tq} = x_v^{tq} - \frac{\varepsilon_v^*}{2}$, (e) follows from the properties of the absolute value, and (f) follows from (8.7) and the fact that $|N^v| \geq |N_0^v|$. This gives us the desired contradiction and completes the proof. \square

Theorem 5 (Near equal treatment of similar players.) *Let Ω be a pregame satisfying SGE. Then given any real numbers $\gamma > 0$ and $\lambda > 0$ there are real numbers $\varepsilon^* > 0$ and $\delta > 0$, integers T and ρ , and a partition of Ω into no more than T subsets, say $\Omega_1, \dots, \Omega_T$, each contained in a ball of diameter less than δ , such that for each $\varepsilon \in [0, \varepsilon^*]$ and for every game (N, v_α) induced by the pregame, if $x \in \mathbb{R}^N$ is in the ε -core of the game (N, v_α) and if $|N| \geq \rho$, it holds that*

$$|\{i \in N : \alpha(i) \in \Omega_t, |x^i - z_t| > \gamma\}| < \lambda |N|,$$

where

$$z_t = \frac{1}{|\{i \in N : \alpha(i) \in \Omega_t\}|} \left(\sum_{i \in N : \alpha(i) \in \Omega_t} x^i \right),$$

the average payoff received by players with attributes in the set Ω_t .

Proof of Theorem 5 From (3.1), given any $\varepsilon > 0$ there is a positive real number $\delta(\varepsilon)$ such that whenever f and g are profiles satisfying $dist(f, g) < \delta(\varepsilon)$ then $\left| \frac{\Psi(f)}{\|f\|_1} - \frac{\Psi(g)}{\|g\|_1} \right| < \varepsilon$. It follows that given $\delta(\varepsilon)$ we can partition Ω into a finite number of subsets, say $\{\Omega^t\}_{t=1}^{T(\varepsilon)}$, each contained in a ball of diameter less than $\delta(\varepsilon)$, and in any game induced by the pregame, players with attributes in Ω^t are $\delta(\varepsilon)$ -substitutes for each other. In addition, we can select a finite number of points in Ω , say

$$\Omega(\varepsilon) := \{\omega_t\}_{t=1}^{T(\varepsilon)},$$

with $\omega_t \in \Omega_t$ for each $t = 1, \dots, T(\varepsilon)$, such that for every profile f there is a profile g with the support of g contained in $\Omega(\varepsilon)$ and with $\left| \frac{\Psi(f)}{\|f\|_1} - \frac{\Psi(g)}{\|g\|_1} \right| < \varepsilon$. Given $\varepsilon > 0$ let $(\Omega(\varepsilon), \Psi)$ denote the pregame determined by $\Omega(\varepsilon)$ and Ψ with the domain of Ψ restricted to profiles with support in $\Omega(\varepsilon)$.

Given λ and γ let ε^* be a positive real number, let $\delta(\varepsilon^*)$ and $\Omega(\varepsilon^*)$ satisfy the properties required in the preceding paragraph, and let ρ be an integer with the property that: For any game (N, v) induced from the pregame $(\Omega(\varepsilon^*), \Psi)$ with $|N| > \rho$, and for any $\varepsilon \in [0, 2\varepsilon^*]$, if x is in the ε -core of the game then it holds that

$$|\{i \in N : \alpha(i) \in \Omega_t \text{ and } |x^i - \widehat{z}_t| > \gamma\}| < \lambda |N|,$$

where $\widehat{z}_t = \frac{1}{|\{i \in N : \alpha(i) \in \Omega_t\}|} \sum_{i \in N : \alpha(i) \in \Omega_t} x^i$.

Now let (N, v) be a game induced by the pregame (Ω, Ψ) and an attribute function α . Let n denote the profile of N (given the attribute function α). Define a new attribute function α' as follows: For each Ω_t define

$$\alpha'(i) = \omega_t \quad \text{for all } i \in N \text{ with } \alpha(i) \in \Omega_t.$$

Let n' denote the profile of N under the attribute function α' , and let (N, v') be the game induced by the pregame and the attribute function α' . Given $\varepsilon \in [0, \varepsilon^*]$, let $x \in \mathbb{R}^N$ be in the ε -core of (N, v) . Define the payoff vector y by $y^i = x^i - \varepsilon^*$. Observe that y must be feasible for (N, v') , since $x(N) \leq v(N)$, $dist(n, n') < \delta(\varepsilon^*)$ and

$$\left| \frac{\Psi(n)}{\|n\|_1} - \frac{\Psi(n')}{\|n'\|_1} \right| < \varepsilon^*$$

implies $y(N) = x(N) - \varepsilon^* |N| \leq \Psi(n) - \varepsilon^* |N| \leq \Psi(n') = v'(N)$. Also, $x(S) \geq v(S) - \varepsilon |S|$ implies that $y(S) \geq v'(S) - (\varepsilon + \varepsilon^*) |S|$. This implies that y is in the $(\varepsilon + \varepsilon^*)$ -core of (N, v') . Since $\varepsilon + \varepsilon^* \leq 2\varepsilon^*$, for each t with $|\alpha^{-1}(N) \cap \Omega_t| \geq \rho$ it holds that

$$|\{i \in N : |y^i - z'_t| > \gamma\}| < \lambda |N|,$$

where $z'_t = \frac{1}{|\{i \in N : \alpha^{-1}(N) \cap \Omega_t\}|} (\sum_{i \in N : \alpha(i) \in \Omega_t} y^i)$, the average payoff received by players with attributes in the set Ω_t , $t = 1, \dots, T(\varepsilon^*)$. From the above inequality it follows that

$$|\{i \in N, \alpha(i) \in \Omega_t : |x^i - z_t| > \gamma\}| < \lambda |N|$$

where

$$\begin{aligned} z_t &= \frac{1}{|i \in N : \alpha^{-1}(N) \cap \Omega_t|} \left(\sum_{i \in N: \alpha(i) \in \Omega_t} x^i \right) \\ &= \frac{1}{|i \in N : \alpha^{-1}(N) \cap \Omega_t|} \left(\sum_{i \in N: \alpha(i) \in \Omega_t} y^i + \varepsilon \right) \\ &= \widehat{z}_t + \varepsilon. \end{aligned}$$

□

Open Access This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

References

- Allouch N, Wooders M (2008) Price taking equilibrium in economies with multiple memberships in clubs and unbounded club sizes. *J Econ Theory* 140:246–278
- Arrow K, Debreu G (1954) Existence of an equilibrium for a competitive economy. *Econometrica* 22:265–290
- Arrow KJ, Hurwicz L (1958) On the stability of the competitive equilibrium, I. *Econometrica* 26:522–552
- Aumann RJ (1964) Markets with a continuum of traders. *Econometrica* 32:39–50
- Aumann RJ (1987) Game theory. The new palgrave: a dictionary of economics. In: Eatwell J, Milgate M, Newman P (eds) *The Macmillan Press*
- Aumann RJ, Dreze J (1974) Cooperative games with coalition structures. *Int J Game Theory* 3:217–237
- Aumann RJ, Shapley LS (1974) Values of nonatomic economies. Princeton University Press, Princeton
- Bondareva ON (1963) Some applications of linear programming methods to the theory of cooperative games (in Russian). *Problemy Kibernetiki* 10:119–139
- Debreu G, Scarf H (1963) A limit Theorem on the core of an economy. *Int Econ Rev* 4:235–246
- Foley D (1970) Lindahl's solution and the core of an economy with public goods. *Econometrica* 38:66–72
- Green J (1972) On the inequitable nature of core allocations. *J Econ Theory* 4:121–129
- Hildenbrand W, Kirman AP (1973) Size removes inequality. *Rev Econ Stud* 40(3):305–319
- Hurwicz L (1995) Cores, almost competitive prices, and the approximate optimality of Walrasian allocations in discrete spaces. In: Maruyama T, Takahashi W (eds) *Nonlinear and convex analysis in economic theory*. Springer. Also (in Japanese translation): in Mita *J Econ* 88:86–105
- Hurwicz L (1999) Revisiting externalities. *J Public Econ Theory* 1(2):225–245
- Hurwicz L, Uzawa H (1977) Convexity of asymptotic average production possibility sets. In: Arrow KJ, Hurwicz L (eds) *Studies in resource allocation processes*. Cambridge University Press, Cambridge
- Kannai Y (1970) Continuity properties of the core of a market. *Econometrica* 38(6):791–815
- Kannai Y (1972) Continuity properties of the core of a market: a correction. *Econometrica* 40(5):955–958
- Khan MA, Polemarchakis H (1978) Unequal treatment in the core. *Econometrica* 46:1475–1481
- Kovalenkov A, Wooders M (2001) Epsilon cores of games with limited side payments; Nonemptiness and equal treatment. *Games Econ Behav* 36(2):193–218
- Kovalenkov A, Wooders M (2003) Approximate cores of games and economies with clubs. *J Econ Theory* 110:87–120
- Lindahl E (1958) Just taxation—a positive solution. In: Musgrave R, Peacock A (eds) *Classics in the Theory of Public Finance*. Macmillan, London
- Mas-Colell A (1980) Efficiency and decentralization in the pure theory of public goods. *Q J Econ* 94:625–641

- McKenzie MW (1959) On the existence of general equilibrium for a competitive market. *Econometrica* 27:54–71
- Owen G (1975) On the core of linear production games. *Math Program* 9:358–370
- Samuelson P (1954) The pure theory of public expenditures. *Rev Econ Stat* 36:387–389
- Shapley LS (1967) On balanced sets and cores. *Navel Res Logist Q* 14:453–460
- Shapley LS, Shubik M (1966) Quasi-cores in a monetary economy with nonconvex preferences. *Econometrica* 34:805–827
- Shapley LS, Shubik M (1969) On market games. *J Econ Theory* 1:9–25
- Shubik M (1959) Edgeworth market games. In: Luce FR, Tucker AW (eds) *Contributions to the theory of games IV*. *Annals of mathematical studies*, vol 40. Princeton University Press, Princeton, pp 267–278
- Shubik M, Wooders M (1982) Near markets and market games. Cowles Foundation Discussion Paper No. 657. Published as Clubs, near markets and market games. In: *Topics in mathematical economics and game theory; essays in honor of Aumann RJ, Wooders MH* (eds) American Mathematical Society Fields Communication Volume, on-line at <http://www.myrnawooders.com>
- Weber S (1979) On ϵ -cores of balanced games. *Int J Game Theory* 8:241–250
- Weber S (1981) Some results on the weak core of a non-sidepayment game with infinitely many players. *J Math Econ* 8:101–111
- Winter E, Wooders M (2008) On large games with bounded essential coalition sizes, University of Bonn Sonderforschungsbereich 303 Discussion Paper B-149, 1990, and *Int J Econ Theory* 4(2):191–206 June 2008
- Wooders M (1979) A characterization of approximate equilibria and cores in a class of coalition economies. (Stony Brook Department of Economics Working Paper No. 184, Revised 1979)
- Wooders M (1980) The Tiebout Hypothesis: near optimality in local public good economies. *Econometrica* 48:1467–1486
- Wooders M (1983) The epsilon core of a large replica game. *J Math Econ* 11:277–300
- Wooders M (1992a) Inessentiality of large groups and the approximate core property; an equivalence theorem. *Econ Theory* 2:129–147
- Wooders M (1992b) Large games and economies with effective small groups, Sonderforschungsbereich 303 University of Bonn Discussion Paper No. 215, Revised September 1992. In: Mertens J-F, Sorin S (eds) *Game-theoretic methods in general equilibrium analysis*. Kluwer, Dordrecht (1994), pp 145–206
- Wooders M (1994) Equivalence of games and markets. *Econometrica* 62:1141–1160
- Wooders M (2008) Small group effectiveness, per capita boundedness and nonemptiness of approximate cores. *J Math Econ* 44(7–8):888–906. doi:[10.1016/j.jmateco.2007.06.006](https://doi.org/10.1016/j.jmateco.2007.06.006)
- Wooders M, Zame WR (1984) Approximate cores of large games. *Econometrica* 52:1327–1350