doi: 10.1111/j.1742-7363.2008.00079.x

On large games with bounded essential coalition sizes

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We consider games in characteristic function form where the worth of a group of players depends on the numbers of players of each of a finite number of types in the group. The games have bounded essential coalition sizes: all gains to co-operation can be achieved by coalitions bounded in absolute size (although larger coalitions are permitted they cannot realize larger per-capita gains). We show that the utility function of the corresponding "limit" market, introduced in Wooders (1988, 1994a), is piecewise linear. The piecewise linearity is used to show that for almost all limiting ratios of percentages of player-types, as the games increase in size (numbers of players), asymptotically the games have cores containing only one payoff, and this payoff is symmetric (treats players of the same type identically). We use this result to show that for almost all limiting ratios of percentages of player-types, Shapley values of sequences of growing games converge to the unique limiting payoff.

Key words strong core, weak core, approximate cores, market games

JEL classification C71, C78, D71

Accepted 10 September 2007

1 Introduction

Cooperative games with bounded essential coalitions have the property that all gains to coalition formation can be realized by coalitions bounded in size; that is, any large coalition can achieve its worth by cooperation restricted to coalitions (in a partition of the coalition) containing fewer members than the bound. This sort of game is frequently used in the published literature. The most well-known of these examples is perhaps the "glove game", in which some players are each endowed with a right-hand glove, and others a left-hand glove, and a pair of gloves is worth $1. This example is analyzed in detail in Shapley and Shubik (1969a), where it is shown that the limit of the values

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This paper originally appeared as "On large games with bounded coalition sizes," (with E. Winter), University of Bonn, Sonderforschungsbereich 303 Discussion Paper A-317 (1990, revised 1991). We are grateful to an anonymous referee for helpful comments and to Tonmoy Islam for creating a figure for us. For financial support we are also grateful to the Social Sciences and Humanities Research Council of Canada and to the Deutsche Forschungsgemeinschaft.

of a particular growing sequence of games is in the limit core. In general, every two-sided market with n sellers and m buyers has bounded essential coalitions, and even situations with more than two types of players. A nice property of such games is that we can describe the payoff achievable by any coalition in terms of payoffs achievable by "small" coalitions and compare payoffs in games with differing numbers of players of each type. There are other underlying reasons why games of this sort frequently appear; their special properties serve well to illustrate a variety of phenomena of markets and games with many players.

It has been shown that games satisfying the property of "small group effectiveness", where *almost* all gains to collective activities can be realized by relatively small groups of players, are approximately market games: games derived from exchange economies where all agents have concave, quasi-linear payoff functions. When arbitrarily small percentages of players are ruled out, small group effectiveness is equivalent to the property that average or per capita payoffs are bounded over all games considered (see Wooders 1994a for both these results).[1] Moreover, under the same conditions, approximate cores are nonempty and converge to equal treatment cores (Wooders 1977, Shubik and Wooders 1982, Wooders 1994b, 2007). Besides being of interest themselves, games with bounded essential coalition sizes approximate games satisfying the property that the supremum of average payoffs is bounded sizes (or, in other words, games satisfying strict small group effectiveness).[2]

In this paper, for games with bounded essential coalition sizes (or, in other words, games satisfying *strict* small group effectiveness), we demonstrate a stronger convergence of games to market games than in Wooders (1988, 1994a) More precisely, we show that games derived from situations with a finite number of types of players and bounded essential coalition sizes are asymptotically equivalent to games derived from markets where all players have the same, piecewise-linear utility function. In the more general setting, Wooders (1994) shows that large games are asymptotically equivalent to games derived from markets where all players have the same concave, continuous, and 1-homogeneous utility function;[3] our main contribution here is showing that, with the further restrictions of a finite number of player types and bounded coalition sizes, we obtain the piecewise linearity. The results are obtained by sharpening, for our special case, some arguments in Wooders (1988). Using the properties of piecewise linearity of the utility function, we describe how our result implies that the limit core is a singleton for almost all limiting distributions of player types and also implies an asymptotic equivalence of cores and Shapley values.

Before leaving this introduction, we remark on what sorts of economic models generate games with bounded essential coalition sizes. First, we note that if there are gains to trade of private goods then typically increasing the numbers of traders leads to increasing

[1] Market games were introduced into the literature in Shapley and Shubik (1969b), who define a market game as a game derived from an economy where players have concave, quasi-linear preferences.

[2] The condition that per capita payoffs are bounded is very mild and, for TU games, simply rules out average or per capita payoff becoming infinite as the number of players goes to infinity. For much more general results concerning cores of games with many players, see, for example, Kovalenkov and Wooders (2003) and Wooders (2008).

[3] Wooders (1988) allows a compact metric space of attributes and a weaker condition bounding returns to group size.

opportunities to trade in larger coalitions. Thus, while such models might satisfy conditions of boundedness of average utilities (per capita boundedness) they are not likely to satisfy the condition of bounded essential coalition sizes. If the primary interest, however, is group or coalition formation, one may assume that there is only one private good (money or some composite commodity) and, therefore, gains to trade of commodities disappear. There are many examples of club economies with bounded essential coalition sizes.[4] Also, matching models typically satisfy bounded essential coalition sizes (see e.g. Roth and Sotomayor 1990).

2 An example

In this section, we present an example of our result.[5] The piecewise linearity of the utility function is illustrated by an indifference curve with linear segments.

We consider a situation where there are two types of players: cooks and helpers. There are only three sorts of "profitable" work-groups:

(a) 1 cook and 2 helpers can make a banquet;
(b) 4 cooks alone can make a banquet (cooks are not very efficient as helpers);
(c) A helper can find his way to the unemployment insurance office and collect unemployment benefits.

Suppose a banquet is worth \$10 and unemployment insurance is worth \$1. Because players of the same type are identical and exact substitutes, we can describe a coalition by a pair (x, y) of nonnegative integers, representing the number of cooks and helpers, respectively. We denote the payoff to a pair (x, y) by $\Psi(x, y)$. The function $\Psi(x, y)$, the maximum payoff that a coalition (x, y) can realize by forming groups, is the optimal value of the objective function of the following integer programming problem:

$$\begin{aligned} \text{Maximize} \quad & 10x_1 + 10\left[\tfrac{x_2}{4}\right] + y_1 \\ \text{subject to:} \quad & x_1 + x_2 \leq x, \\ & 2x_1 + y_1 \leq y, \quad \text{and} \\ & x_1, x_1 \text{ and } y_1 \text{ are nonnegative integers,} \end{aligned}$$

where $[\frac{x_2}{4}]$ is the largest integer less than or equal to $\frac{x_2}{4}$.

[4] If restricted to situations with only one private good, multiple papers in the literature on club economies (or on the "Tiebout Hypothesis") studying properties of price-taking equilibrium satisfy bounded essential coalition sizes. See Allouch and Wooders (2007), an exception to this, for further discussion and references.

[5] For the reader familiar with that paper we remark that the example is taken directly from Wooders (1988) and also appears in Wooders (1994b).

Some calculation reveals that, for $y > 3$, $\Psi(x, y)$ is given by

$$\Psi(x, y) = \left[\begin{array}{ll} 5x_2 + 10\left[\dfrac{x - \left(\frac{y-2}{2}\right)}{4}\right] - 8 & \text{if } y \text{ is even, } 2x - y = 8r + 6 \\ & \text{for some nonnegative integer, } r \\ 5y + 10\left[\dfrac{x - \left(\frac{y}{2}\right)}{4}\right] & \text{if } y \text{ is even, } 2x \geqslant y, \text{ and} \\ & 2x - y \neq 8r + 6 \\ & \text{for any nonnegative integer } r \\ & \text{if } y \text{ is odd and} \\ 5y + 10\left[\dfrac{x - \left(\frac{y-3}{4}\right)}{4}\right] - 12 & 2x - y = 8r + 5 \\ & \text{for some nonnegative integer } r \\ & \text{if } y \text{ is odd, } 2x \geqslant y, \text{ and} \\ 5y + 10\left[\dfrac{x - \left(\frac{y-1}{4}\right)}{4}\right] - 4 & 2x - y \neq 8r + 5 \\ & \text{for any nonnegative integer } r \\ 8x + y & \text{if } 2x \leq y \end{array}\right].$$

(For our purposes, we do not consider $y \leq 3$.)

When $2x \geq y$, roughly, the maximum value of the objective function is achieved by forming as many (a) groups as possible, and then putting the remainder of the cooks in (b) groups. However, this might leave 3 cooks “left-over” (if y is even, this is the condition that $2x - y = 8r + 6$). It then pays to break an (a) group, send 2 helpers to the unemployment insurance office, and form an additional (b) group. If only 1 or 2 cooks are “left-over”, they are (optimally) left unemployed. An additional complication arises when y is odd then one more helper is left unemployed (and gets unemployment insurance) than when y is even.

If $2x < y$, the maximum value of Ψ is achieved by simply forming as many (a) groups as possible, and then putting the remaining helpers in (c) groups.

For large groups of cooks and helpers, the per-capita payoff depends, approximately, only on the size of the group (the total number of cooks and helpers), and the distribution or composition of the group (the percentage of cooks and of helpers). To illustrate, let (x, y) be in $\mathbb{R}^2_+$, $(x, y) \neq (0, 0)$, and let $\{(x^n, y^n)\}$ be a sequence of pairs of nonnegative integers where $x^n + y^n \to \infty$ and $\frac{x^n}{x^n + y^n} \to \frac{x}{x+y}$ and $\frac{y^n}{x^n + y^n} \to \frac{y}{x+y}$ as $n \to \infty$. We leave it to the reader to verify that for any (x, y) with $2x > y$,

$$(x + y) \lim_{n\to\infty} \frac{\Psi(x^n, y^n)}{x^n + y^n} = \frac{5x}{2} + \frac{15y}{4}$$

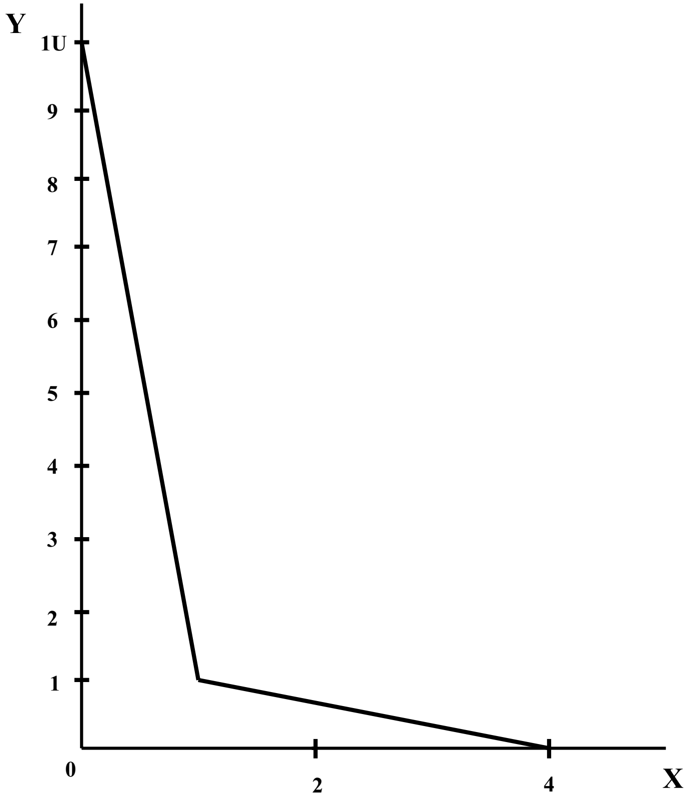

Figure 1 An indifference curve for the limiting utility function.

and with $2x \leq y$,

$$(x+y)\lim_{n\to\infty}\frac{\Psi(x^n,\ y^n)}{x^n+y^n}=8x+y.$$

To describe a market game generated by the game-theoretic information, we endow each agent with one unit of his or her type, and describe the utility function of an agent by the indifference map depicted in Figure 1. The utility function for an agent is given by:

$$u(x,\ y)=\begin{bmatrix}\dfrac{5x}{2}+\dfrac{15y}{4} & \text{if } 2x\geqslant y\\ 8x+y & \text{if } 2x<y\end{bmatrix}.$$

Observe that on the cones (x, y) with $2x \geq y$ and (x, y) with $2x \leq y$, the utility function is linear; our main result is that the utility function has the sort of property whenever essential coalition sizes are bounded. (In contrast, in Wooders (1988, 1994) it is demonstrated that under milder conditions, we still obtain convexity of the utility function.)

If we specify a total population of players in terms of the numbers of cooks and helpers in the population, if the ratio of helpers to cooks is greater than 2 then the data above determines a cooperative game with side payments. Suppose, for example, that a population N consists of 5 helpers and 2 cooks. Then the core of the game is nonempty and assigns a payoff of 4 to each cook and a payoff of 1 to each helper. In fact, in any population, if the ratio of helpers to cooks is greater than 2, then the core is nonempty and assigns each cook 4 and each helper 1; helpers are in relative abundant supply and can only achieve what they can do on their own. Because at least 1 helper must be unemployed, in a core outcome all helpers must receive only 1 each.

If the ratio of helpers to cooks is less than 2, cooks are in relatively excess supply. If the core is nonempty, it must be possible to assign each helper to a 2-helper, 1-cook coalition and each cook must be either with 2 helpers or in a 4-cook coalition (and there must be at least 1 such coalition). The existence of a 4-cook coalition implies that the core payoff must assign each cook $\frac{10}{4}$. The existence of a 2-helper, 1-cook coalition then implies that the core must assign each helper $\frac{15}{4}$. We can see from this case that if the population includes both cooks and helpers and if cooks are in relatively excess supply, then the nonemptiness of the core implies that the number of helpers, say y, must be even and the number of cooks, say x, must be equal to $4z + \frac{y}{2}$ for some non-negative integer z. It is also apparent that if the core is empty, but there are many players, we can form as many 2-helper, 1-cook coalitions and 4-cook coalitions as possible and have "few leftovers": at most 1 helper and at most 3 cooks.

The properties of this example are standard for games with bounded essential coalition sizes.

3 Games

In this section, we recall some definitions.

A *game* (with sidepayments) is a pair (N, ν) where N is a finite set (the set of players) and ν is a function (the *worth function* of the game) from the set 2^N of subsets of N to the set $\mathbb{R}_+$ of nonnegative real numbers, with the property that $\nu(\phi) = 0$. A nonempty subset S of N is called a *coalition* and the number $\nu(S)$ is the *worth* of the coalition S. If the player set N is understood, we frequently refer to ν itself as the game. The game (or ν) is *superadditive* if for all disjoint subsets S and S' of N we have

$$\nu\left(S \cup S'\right) \geq \nu(S) + \nu\left(S'\right).$$

Let (N, ν) be a game. Two players i and j, $i \neq j$ are *substitutes* if for all coalitions S with $i \notin S$ and $j \notin S$, it holds that $\nu(S \cup \{i\}) = \nu(S \cup \{j\})$.

A *payoff* for the game (N, ν) is a vector $x \in \mathbb{R}^N$; for convenience we use the notation $x(i)$ for the i^{th} component of x. The payoff is *feasible* if there is a partition of N into (disjoint) coalitions, say $(S_1 \ldots, S_L)$, such that

$$x(N) \leq \sum_{\ell=1}^{L} \nu(S_\ell), \tag{1}$$

where $x(S) = \sum_{i \in S} x(i)$, for any $S \subset N$.

For $\varepsilon \geq 0$, the payoff x is in the *weak ε-core* of (N, ν) if x is feasible and

$$x(S) \geq \nu(S) - \varepsilon\,|S|$$

for all coalitions S where $|S|$ denotes the number of elements in the set S. The payoff x is in the *strong ε-core* if it is feasible and if

$$x(S) \geq \nu(S) - \varepsilon$$

for all coalitions $S \subset N$. When $\varepsilon = 0$ the (weak or strong) ε-core is simply the *core*. The weak ε-core consists of those feasible payoffs with the property that no group of players could each be better off by ε, whereas the strong ε-core has the property that no group of players could collectively be better off by ε in total. (These concepts are introduced in Shapley and Shubik (1966).)

4 Pregames

To formalize the notion of a large game we use the notion of a pregame with a finite number, say T, of types of players. All players of the same type are substitutes.

Let $s \in \mathbb{Z}^T_+$ be the T-fold Cartesian product of the nonnegative integers. We call s a *profile* of a group of players, and interpret s_t as the number of players of type t in the group, $t = 1, \ldots, T$. We write $\underline{0}$ for the profile which is identically zero. We write $s' \leq s$ if $s'_t \leq s_t$ for each t and write χ^t for the profile given by

$$\chi^t_{t'} = \begin{bmatrix} 0 & \text{if } t' \neq t \\ 1 & \text{if } t' = t. \end{bmatrix}.$$

By the *norm* or *size* $\|s\|$ *of a profile* $s = (s_1, \ldots, s_T)$ we mean

$$\|s\| = \sum_{t=1}^{T} s_t.$$

A *partition of a profile* s is a collection of profiles $\{s^\ell\}$ satisfying the condition that $\sum_l s^\ell = s$. (This is a natural analogue to the notion of a partition of a set.)

A *pregame with types* is an ordered pair (T, Ψ) where T is a number of types and $\Psi : \mathbb{Z}^T_+ \to \mathbb{R}_+$, called the *worth function* (of the pregame), satisfies the conditions:

$$\Psi(\underline{0}) = 0 \quad \text{and} \tag{2}$$

$$\Psi(s) + \Psi(w) \geq \Psi(s + w) \text{ and for all profiles } s \text{ and } w \text{ (superadditivity).} \tag{3}$$

A pregame consists simply of a set of player types or attributes, and a worth function, specifying a payoff achievable by any group of players depending on the composition of the group. Pregames have now appeared in several places in the published literature.[6] In view of our definition of feasibility, the assumption of superadditivity is purely for notational convenience; we discuss this further after our next definition.

We say the pregame has *bounded essential coalition sizes* if there a constant B such that, for all profiles $s \in \mathbb{Z}_+^T$, there is a partition of s, say $\{s^\ell\}$, with the properties that

$$\begin{aligned} \Psi(s) &= \sum_\ell \Psi(s^\ell) \quad \text{and} \\ \|s^\ell\| &\leq B \quad \text{for each } \ell. \end{aligned}$$

In other words, a pregame has *B-bounded essential coalition sizes* if the worth of any coalition is equal to the sum of the worths of some collection of disjoint sub-coalitions, each containing no more than B members.

Remark 1 *On superadditivity*. An alternative approach would be to omit the requirement of superadditivity and instead require that for any profile s there is a partition$\{s^\ell\}$ of s with the property that $\| s^\ell \| \leq B$ for each s^ℓ in the collection and $\Psi(s) \leq \sum_\ell \Psi(s^\ell)$. This would not affect the set of feasible outcomes in any game derived from a pregame nor affect the core or ε-cores of the game. What is important, however, is that a group of players can split into smaller groups and each smaller group can realize its own worth, a property known as *essential superadditivity*, which is built into our definition of feasibility.[7]

To derive a game from a pregame (T, Ψ) we specify a finite set N and a function

$$\alpha : N \to \{1, \ldots, T\},$$

called a *type function.* With any subset $S \subset N$, we associate a profile, $prof(\alpha|S) \in \mathbb{Z}_+^T$, given by

$$prof(\alpha|S)(t) = \left|\alpha^{-1}(t) \cap S\right|;$$

$prof(\alpha|S)(t)$is simply the number of players of type t in S while $prof(\alpha\ |S)$ is a list of the numbers of players of each type in S. Given N and α, the worth function of the game (N, ν_α) is derived from the worth function of the pregame:

$$\nu_\alpha(S) = \Psi\left(prof(\alpha|S)\right).$$

The pair (N, ν_α) is called a *derived game*.

5 Premarkets with piecewise linear utility functions

A *premarket with a piecewise linear utility function* is a pair (G, U) where G is a number of types of goods, and U, called a *utility function*, is a function from R_+^G to $\mathbb{R}_+$ with the properties that:

[6] See, for example, Wooders (1983) for NTU pregames and Wooders and Zame (1987) for TU pregames.
[7] This issue is discussed for cooperative games more generally in Wooders (2008).

(5.1) For some collection of cones, say $C_1, \ldots, C_K$, all containing the origin, with the property that $\mathbb{R}_+^G = \bigcup_{k=1}^{K} C_K$, we have U a linear function on C_k for each $k = 1, \ldots, K$; that is, for any k, and any x, $y \in C_k$, we have $U(x) + U(y) = U(x+y)$

(5.2) $U(x) \geq 0$ for all $x \in \mathbb{R}_+^G$.

We note that these 2 conditions imply that U is superadditive and 1-homogeneous (i.e. for any positive λ and any $x \in C_k$, it holds that $U(\lambda\, x) = \lambda\, U(x)$).

Let (G, U) be a premarket. A vector $x \in \mathbb{R}_+^G$ is called a *commodity bundle*. Given a finite agent set N and an *endowment function* $e : N \to \mathbb{R}_+^G$, a *derived market* is determined. Given the agent set N, the function e assigns an endowment of commodities to each agent. The ordered triple (N, e, U) is called a *market*.

We will next derive a utility function from a pregame.[8] Let (T, Ψ) be a pregame. The *derived utility function U* is given by

$$U(x) = \|x\| \lim_{n\to\infty} \frac{\Psi(s^n)}{\|s^n\|}, \tag{2}$$

where $\{s^n\}$ is a sequence of profiles with the properties that

(a) $\| s^n \| \to \infty$ and

(b) $\|x - \frac{\|x\|}{\|s^n\|} s^n\| \to 0$ as $n \to \infty$.

(We note that the limit in the definition of U exists and is independent of the choice of the sequence $\{s^n\}$ in the sense that if another sequence of profiles, say $\{\widehat{s}^n\}$, satisfies properties (a) and (b) then (2) also holds for that sequence. This is easily demonstrated and is also immediate from Wooders (1988)).

Our first Proposition relates pregames with bounded essential coalition sizes to premarkets with piecewise linear utility functions.

Proposition 1 *Let (T, Ψ) be a pregame with T types and bounded essential coalition sizes. Then the derived utility function U is a concave and piecewise linear function; thus the pair (T, U) is a premarket, called the derived premarket.*

We first state and prove a lemma and introduce some auxiliary concepts. Then the proof of the proposition is almost immediate.

Let $\mathbb{E} = \{s \in \mathbb{R}_+^T : ||s|| \leq B\}$ denote the set of all profiles s bounded in size by B. We call the members of $\mathbb{E}$ *essential profiles*. From our assumption of bounded coalition sizes, note that for every profile s' there are nonnegative integers ℓ_s for each $s \in \mathbb{E}$ satisfying $\sum_{s\in\mathbb{E}} \ell_s s = s'$ and $\Psi(s') = \sum_{s\in\mathbb{E}} \ell_s \Psi(s)$.

[8] This is much in the spirit of Shapley and Shubik (1969b) where the authors relate markets (economies in which all agents have concave utility functions) and totally balanced games. Here, however, we relate pregames and premarkets. As our example in the preceding section illustrates, we will show that games with many players derived from pregames with bounded essential group sizes are market games where players have piecewise linear utility functions.

It holds that for each commodity bundle x there are weights $\lambda_s \in \mathbb{R}_+$ satisfying the condition that

$$U(x) = ||x|| \sum_{s \in \mathbb{E}} \lambda_s \Psi(s).$$

A collection $\mathbb{E}(x) \subset \mathbb{E}$ of essential profiles *supports the utility function U at a commodity bundle x* if and only if $\lambda_s > 0$ for each $s \in \mathbb{E}(x)$.

For every commodity bundle x there is a collection of essential profiles $\mathbb{E}(x)$ supporting the utility function U at x. This is because, for profiles $\{s^n\}$ with $\frac{\|x\|}{\|s^n\|} s^n \to x$ and $\|s^n\| \to \infty$ we have:

$$\begin{aligned} U(x) &= \|x\| \lim_{n \to \infty} \frac{\Psi(s^n)}{\|s^n\|} \\ &= \|x\| \lim_{n \to \infty} \sum_{s \in \mathbb{E}} \frac{\ell_s^n \Psi(s)}{\|s^n\|} \end{aligned}$$

(where each ℓ_s^n is a nonnegative integer for each $s \in \mathbb{E}$ and $\sum_{s \in S} \ell^n s = s^n$), and

$$U(x) = \|x\| \sum_{s \in \mathbb{E}} \Psi(s) \left(\lim_{n \to \infty} \frac{\ell_s^n}{\|s^n\|} \right),$$

where, if necessary, to ensure that $\lim_{n \to \infty} \frac{1}{\|s^n\|} \ell_s^n$ exists for each $s \in \mathbb{E}$, we pass to a subsequence of $\{s^n\}$. A candidate for the collection $\mathbb{E}(x)$ is simply $\{s \in \mathbb{E} : \lim_{n \to \infty} \frac{\ell_s^n}{\|s^n\|} > 0\}$. Note also that we can choose the profiles s^n so that the set of essential profiles supporting U at x supports the worth function Ψ at s^n. To see this, taking "$\approx$" to mean "approximately equal", observe that

$$U(x) = \|x\| \sum_{s \in \mathbb{E}(x)} \lambda_s \Psi(s) \text{ implies } U(x) \approx \|x\| \sum_{s \in \mathbb{E}(x)} r_s \Psi(s)$$

for some set of rational weights $r_s > 0$, so we have

$$U(x) \approx \|x\| \sum_{s \in \mathbb{E}} \frac{\ell_s^n}{\|s^n\|} \Psi(s),$$

where $s^n = \sum_{s^k \in \mathbb{E}} \ell_s^n \Psi(s)$ and ℓ_s^n satisfies $\frac{\ell_s^n}{\|s^n\|} = r_s$.

Lemma 1 *Let x and y be commodity bundles such that, for some collection $\mathbb{E}(x)$ of profiles supporting U at x, and for some collection $\mathbb{E}(y)$ of profiles supporting U at y, we have $\mathbb{E}(y) = \mathbb{E}(x)$. Then $U(x) + U(y) = U(x + y)$.*

PROOF: Let $\{s^n\}$ and $\{w^n\}$ be sequences of profiles where $\frac{\|x\|}{\|s^n\|} s^n \to x$, $\frac{\|y\|}{\|w^n\|} w^n \to y$, $\|s^n\| \to \infty$, $\|w^n\| \to \infty$, $U(x) = \|x\| \lim \sum \frac{\Psi(s^n)}{\|s^n\|}$, and $U(y) = \|y\| \lim \sum \frac{\Psi}{(w^n)} \|w^n\|$, $\mathbb{E}(s^n) = \mathbb{E}(x)$, and $\mathbb{E}(w^n) = \mathbb{E}(y)$ for each n, there are sets of positive integers

$\{\lambda_s^n : s \in \mathbb{E}(x)\}$ and $\{\rho_s^n : s \in \mathbb{E}(y)\}$ such that $\Psi(s^n) = \sum_{s \in \mathbb{E}(n)} \rho_s^n \Psi(s)$ and $\Psi(w^n) = \sum_{s \in \mathbb{E}(y)} \rho_s^n \Psi(s)$. Because $\mathbb{E}(x) = \mathbb{E}(y)$ it follows that

$$\Psi(s^n + w^n)$$
$$= \sum_{s \in \mathbb{E}(x)} \lambda_s^n \Psi(s) + \sum_{s \in \mathbb{E}(y)} \rho_s^n \Psi(s)$$
$$\sum\nolimits_{s \in \mathbb{E}(x)} (\lambda_s^n + \rho_s^n) \Psi(s).$$

Using some simple algebra, it follows that

$$\lim \frac{\|x+y\|}{\|s^n + w^n\|}(s^n + w^n) = x + y$$

and, therefore,

$$\|x+y\| \lim \frac{\Psi(s^n + w^n)}{\|s^n + w^n\|} = U(x+y). \qquad \square$$

Proof of Proposition 1: First, observe that because $\mathbb{E}$ is a finite set, there is only a finite number of (distinct) collections of nonempty subsets $\mathbb{E}$, say $\mathbb{E}^1, \ldots, \mathbb{E}^J$. Let y be a commodity bundle and let $\mathbb{E}(y)$ be a collection of essential profiles supporting U at y. It must hold that $\mathbb{E}(y) = \mathbb{E}^j$ for some j. From Lemma 1, for all commodity bundles x with $\mathbb{E}(y) = \mathbb{E}(x)$ it must hold that $U(x) + U(y) = U(x+y)$. Let C^j be the set of all commodity bundles x supported by $\mathbb{E}^j$. Because $\mathbb{E}(x) = \mathbb{E}(rx)$ for any non-negative real number x it follows that C^j is a cone. The one-homogeneity of U, its concavity, and the non-negativity of $U(x)$ for all commodity bundles x are easily demonstrated (and also already in the literature for the more general case in Wooders 1988, 1994a,b). $\square$

6 Limiting behavior of derived games

In this section, we investigate the limiting behavior of growing sequences of games derived from a pregame and the corresponding sequences of games derived from the associated premarket. First, for completeness, we state a result on nonemptiness of strong ε-cores, which follows from a result on strong ε-cores in Wooders (1994b).

Proposition 2 *Let (T, Ψ) be a pregame with bounded essential coalition sizes. Then there is an integer $\eta(\varepsilon)$ such that for every derived game (N, v_α) with $|N| \geq \eta(\varepsilon)$, the strong ε-core of (N, v_α) is nonempty.*

As we will show, if the distribution of players converges to some point in the interior of a cone on which the utility function is linear, then the limit of equal treatment strong ε-cores is given by the derivative of the utility function. Because, for games with many players, Shapley values are in weak ε-cores and Shapley values treat identical players identically, under the same conditions the limit of the Shapley values exists and equals the limit core.

As we will next demonstrate, these results follow easily from the piecewise linearity of the utility function U and results in the literature on values of large games, specifically those of Aumann and Shapley (1974) and Wooders and Zame (1987).

Before proceeding, we remark that we will describe a continuum limit game of the sort considered in Aumann and Shapley (1974) with coalitions of positive measure. Because we have in mind situations with finite, in fact bounded, coalition sizes, the interpretation of "coalitions" of positive measure may be unclear. Our interpretation is that a coalition of positive measure is an aggregate of finite coalitions. Under our conditions the f-core, the core of a continuum game with finite coalitions, is equivalent to the A-core, the core of a continuum game with coalitions of positive measure (see Kaneko and Wooders (1986), especially lemma 3.1 on the equivalence of feasibility conditions, and Hammond et al. (1989), on equivalence of cores).[9]

We now proceed by defining a limiting nonatomic game and stating some results from Wooders and Zame (1987).

We fix strictly positive real numbers $\Theta_1, \ldots, \Theta_T$ with $\sum \Theta_1 = 1$, and disjoint intervals $I_1, \ldots, I_T$ on the real line for which length $(I_t) = \Theta_T$. Set $I = \cup I_t$, let $\mathcal{B}$ be the family of Borel subsets of I, and let μ be the restriction to I of Lebesgue measure. (In interpretation, I represents a continuum of players of which the fraction $\Theta_T = \mu\ (I_T)$ are of type t. The family $\mathcal{B}$ of all Borel subsets of I is the family of admissible coalitions.)

To define a nonatomic game on I, in the sense of Aumann and Shapley [1974], we must define a set function λ of bounded variation. To this end, let β be an element of $\mathcal{B}$. If $\mu\ (\beta) = 0$, we define $\lambda\ (\beta) = 0$. Otherwise, we write $\beta_t = \mu\ (\beta \cap I_t)/\mu\ (\beta)$ for each t; note that $\beta_t \geq 0$ and that $\sum \beta_t = 1$ Choose a sequence $\{f^k\}$ of profiles on Ω such that $\| f^k \| \to \infty$ and $f_t^k/\| f^k \| \to \beta_t$ for each t. Define

$$\lambda(\beta) = \left(\lim_{k \to \infty} \frac{\Psi(f^k)}{\| f^k \|} \right) \mu(\beta),$$

which is interpreted as the per capita payoff to a group with composition given by β multiplied by the measure of players in the set β. (This limit exists and is independent of the particular sequence $\{f^k\}$ of profiles we choose.) To see that λ is of bounded variation, note that $0 \leq \frac{\Psi(f)}{\| f \|} \leq m$ for any profile f, where m is the maximum per-capita payoff of the pregame

$$m = \max_f \frac{\Psi(f)}{\| f \|} = \max_{f \in \mathbb{E}} \frac{\Psi(f)}{\| f \|}$$

(and this maximum exists). Therefore, $0 \leq \lambda\ (B) \leq m\mu\ (B)$. (This construction corresponds to the "fractionating process" for constructing nonatomic economies. We interpret the limit $\lim_{k \to \infty} \frac{\Psi(f^k)}{\| f^k \|}$ as the limiting per-capita payoff to a coalition with a given distribution of types, so $\lambda\ (B)$ is the limiting payoff, normalized relative to the measure of players.)

Recall that the *core of* λ consists of all nonnegative, finitely-additive set functions $\sigma : \beta \to \mathbb{R}$ such that $\sigma(I) = \lambda(I)$ and $\sigma(B) \geq \lambda(\beta)$ for each $\beta \in B$. (When σ is in the core,

[9] Roughly, the condition required for equivalence of the f-core, the A-core, is that the game can be represented as an economy without externalities. Examples of nonequivalence appear in Kaneko and Wooders (1986).

$\sigma(I_t)$ is interpreted as the total (normalized) payoff to the set of players of type t.) It is a useful fact that every element of the core treats players of the same type equally. The following lemma is from Wooders and Zame (1987).

Lemma 2 *Let σ belong to the core of λ. Then, for each t and each Borel subset A of I_t, $\sigma(A) = \sigma(I_t)\frac{\mu(A)}{\mu(I_t)}$.*

For each σ in the core of λ, define $\bar{\sigma} \in \mathbb{R}^T$ by $\bar{\sigma}(t) = \frac{\sigma(I_t)}{\mu(I_t)}$. In view of Lemma 2, we can interpret $\bar{\sigma}$ as the per-capita payoff to players of type t. Let $\bar{C}(\lambda) = \{\bar{\sigma} : \sigma$ is the core of $\lambda\}$, so that $\bar{C}(\lambda)$ is a subset of $\mathbb{R}^T$.

We now fix a sequence $\{(N_k, \nu_{\alpha_k})\}$ of games derived from the pregame (Ω, Ψ). We assume that $|N_k| \to \infty$ and that $\frac{\alpha_k(\varpi_t)}{|N_k|} \to \theta_t$ for each t; there is also no loss of generality in assuming that $\alpha_k^{-1}(\varpi_t) > 0$ for each k and t. We view the nonatomic game λ as a (normalized) limit of the games (N_k, ν_{α_k}). For $\varepsilon > 0$ a payoff x in the weak ε-core of $(N_k, \nu_{\alpha k})$ is an *equal-treatment payoff* if $x(i) = x(j)$ whenever $\alpha_k(i) = \alpha_k(j)$ (so that players of the same type receive the same payoff). For such an x, we define $\bar{x} \in \mathbb{R}^T$ by $\bar{x}(i)$ for any $i \in N_k$ with $\alpha(i) = t$; we write $\bar{C}_\varepsilon(N_k, \nu_{\alpha_k})$ for the set of such vectors, so that $\bar{C}_\varepsilon(N_k, \nu_{\alpha_k})$ is a subset of $\mathbb{R}^T$. Of course, for $x \in \bar{C}_\varepsilon(N_k, \nu_k)$, we may interpret $\bar{x}(t)$ as the per-capita payoff to players of type t. Evidently, then, to show that the sets $\bar{C}_\varepsilon(N_k, \nu_{\alpha k})$ and $\bar{C}(\lambda)$ are "close" is to show that, in a natural sense, the weak ε-core of $(N_k, \nu_{\alpha k})$ is close to the core of λ. This result first appeared in Wooders (1979) for sequences of games with a fixed distribution of types of players but the weaker condition of boundedness of average payoffs. Wooders and Zame (1987) continue by showing that, under stronger conditions (boundedness of individual marginal contributions) the Shapley value is in the limiting core.[10]

Theorem 1 *Given $\sigma_0 > 0$ and $\varepsilon_0 > 0$ there is an ε_1, with $0 < \varepsilon_1 < \varepsilon_0$, and an integer k_0 such that $dist(\bar{C}_{\varepsilon_1}(N_k, \nu_{\alpha_k}), \bar{C}(\lambda)) \leq \sigma_0$ for every $k > k_0$. Equivalently,*

$$\bar{C}(\lambda) = \bigcap_{\varepsilon>0} \limsup_{k\to\infty} \bar{C}_\varepsilon(N_k, \nu_{\alpha_k}) = \bigcap_{\varepsilon>0} \liminf_{k\to\infty} \bar{C}_\varepsilon(N_k, \nu_{\alpha_k}).$$

If $\zeta_K = Sh(N_k, \nu_{\alpha k})$, then $\lim_{k\to\infty} dist(\zeta_K, \bar{C}(\lambda)) = 0$ where $Sh(N_k, \nu_{\alpha_k})$ is the Shapley value of the game (N_k, ν_{α_k}) represented by a point in $\mathbb{R}^T_+$.

Given $\varepsilon > 0$, all sufficiently large games derived from a pregame with bounded essential coalition sizes have nonempty strong ε-cores. Since, for any game, the strong ε-core is contained in the weak ε-core, under our conditions the above theorem also holds when the weak ε-core is replaced by the strong ε-core. We now define a limiting continuum game using the utility function U derived from the pregame. Let $\{f^k\}$ be a sequence of profiles on Ω such that $\|f^k\| \to \infty$ and $\frac{f_t^k}{\|f^k\|} \to \beta_t$ for each t. Define

$$\lambda_M(\beta) = \lim_{k\to\infty} \left(\frac{U(f_t^k)}{\|f^k\|} \right) \mu(\beta).$$

[10] It can also be shown that for small ε, "most" payoffs, in ε-core of (N_k, ν_{α_k}) are "nearly" equal-treatment payoffs. This was first demonstrated in Wooders (1977); see also Wooders (1994b, 2007) for this result under substantially milder conditions.

It is immediate from the construction of U that $\lambda_M(\beta) = \lambda(\beta)$ for every Borel set β. We can now apply results for differentiable market games to the game λ, when θ, the limiting ratio of player types, is in the interior of one of the cones on which U is linear.

The following proposition is not surprising in view of the results of Aumann and Shapley (1974). In our context, it is very clear and easily proven.

Proposition 3 *When U is linear on a neighborhood of θ the core of λ_M consists of a singleton; that is, $|C(\lambda_M)| = 1$. Moreover, the core of the market game $C(\lambda_M)$ coincides with the core of the game $C(\lambda)$. At points where U is differentiable, the equal treatment payoff in the core of $C(\lambda)$ and $C(\lambda_M)$ is given by the gradient of the utility function U.*

PROOF: First, recall that every core payoff treats players of the same type identically, so we have $C(\lambda_M) = \bar{C}(\lambda_M)$. Let $x \in \mathbb{R}^T_+$ represent an equal-treatment payoff in the core of λ_M. Suppose $U(s)$ is given by $U(s) = a_1 s_1 + a_2 s_2 + \cdots + a_T s_T$ for all $s \in \mathbb{R}^T_+$ in a neighborhood of θ. Also suppose, for the purpose of obtaining a contradiction, that $x_{t'} < a_{t'}$ for some t'. Because x is the core of λ_M,

$$\sum_{j=1}^{T} x_j \theta_j = \sum_{j=1}^{T} a_j \theta_j .$$

Therefore, for some $t'' \neq t'$, we must have $x_{t''} > a_{t''}$. Consider the coalition given by $I \backslash I_{t'}$. For this coalition, we have

$$\sum_{\substack{t=1 \\ t \neq t'',}}^{T} x_j \theta_j < \sum_{t=1}^{T} a_j \theta_j .$$

From the definition of λ_M and of U it follows that $I \backslash I_{t'}$ can improve upon x, a contradiction. Thus, for x in $\bar{C}(\lambda_M)$ we must have $x_t = a_t$ for all t. Therefore, $\bar{C}(\lambda_M)$ consists of a singleton, given by the gradient of the utility function U. □

We can conclude that whenever θ is in the interior of one of the cones on which U is linear, the core of the original game λ is also a singleton. With Lebesgue measure on the simplex, we can say that for almost all θ in the simplex, sequences of games derived from the pregame with distribution of player types converging to θ have only one element in the limit of the ε-cores.

We also have the following result.

Proposition 4 *Assume θ is in the interior of one of the cones on which U is linear. Let $Sh(N_k, \nu_{\alpha k})$ represent the Shapley value of the game $(N_k, \nu_{\alpha k})$ where the t^{th} component of $Sh(N_k, \nu_{\alpha k})$ is the Shapley value of a player of type t. Then the sequence converges to a limit, say z^*, and $\{z^*\} = \bar{C}(\lambda)$.*

PROOF: This follows from the facts that, given any $\varepsilon > 0$, the Shapley values are in the ε-cores for all sufficiently large terms in the sequence, and the equal-treatment ε-cores converge to the limit core $\bar{C}(\lambda)$. □

Because of the remarkable simplicity of the market representation of a pregame with bounded essential coalition sizes, the feature that, for almost all ratios of measures of player types $\theta_1, \ldots, \theta_T$ the core consists of a singleton, emerges clearly and sharply.[11] Therefore, we have included Proposition 4, in this paper. We note that its implications for equivalence of the limit of values and the limit core hold more broadly.

Because inessentiality of large coalitions and finiteness of the set of player types ensures that, given θ, the game λ is representable as a market game where players all have the same concave utility function (Wooders 1994), the utility function is differentiable almost everywhere on the simplex and it follows that the core $\bar{C}(\lambda)$ is a singleton almost everywhere. Moreover, for any sequence of finite games $(N_k, v_{\alpha k})$ as above, with limiting proportions of player types given by $\theta_1, \ldots, \theta_T$, the limit of the values exists and equals $\bar{C}(\lambda)$ for almost every $(\theta_1, \ldots, \theta_T)$ in the simplex; this follows from the differentiability of the utility function.

7 Conclusions

In contrast to Wooders (1988) and more recent research (e. g. Wooders 2007, 2008), the present paper limits the set of player types to be finite rather than a compact metric space. With a compact metric space of player types (or, in other words, attributes or characteristics), boundedness of essential coalition sizes ensures that, given $\varepsilon > 0$ all sufficiently large derived games have nonempty strong ε-cores. We can also conclude, from Wooders (1988, 2008), that a pregame with a compact metric space of player types can be represented by a limiting premarket where all players have concave utility functions and 1-homogeneity continues to hold. If essential coalition sizes are bounded, with a compact metric space of attributes the piecewise-linear nature of the limiting utility function continues to hold as does nonemptiness of strong ε-core ($\varepsilon > 0$) cores of all sufficiently large games. Convergence of strong ε-cores also continues to hold but becomes somewhat more subtle.

References

Allouch, N., and M. Wooders (2007), "Price taking equilibrium in club economies with multiple memberships and unbounded club sizes," *Journal of Economic Theory*, doi: 10.1016/j.jet.2007.07.06.

Aumann, R. J., and L. S. Shapley (1974), *Values of Nonatomic Games*, New Jersey: Princeton University Press.

Hammond, P., M. Kaneko, and M. H. Wooders (1989), "Continuum economies with finite coalitions: Core, equilibria, and widespread externalities," *Journal of Economic Theory* **49**, 113–34.

Kaneko, M., and M. H. Wooders (1986), "The core of a game with a continuum of players and finite coalitions: the model and some results," *Mathematical Social Sciences* **12**, 105–37.

Kavalenkov, A., and M. Wooders (2003), "Approximate cores of games and economies with clubs," *Journal of Economic Theory* **110**, 87–120.

Roth, A., and M. Sotomayer (1990), *Two-sided Matching; A Study in Game-theoretic Modeling and Analysis*, Cambridge: Cambridge University Press.

Shapley, L. S., and M. Shubik (1966), "Quasi-cores in a monetary economy with nonconvex preferences," *Econometrica* **34**, 805–27.

[11] More precisely, for all $\theta_1, \ldots, \theta_T$ in the simplex except possibly for a set of Lebesque measure zero.

Shapley, L. S., and M. Shubik (1969a), "Pure competition, coalitional power, and fair division," *International Economic Review* **10**, 337–62.

Shapley, L. S., and M. Shubik (1969a), "On market games," *Journal of Economic Theory* **1**, 9–25.

Shubik, M., and M. H. Wooders (1982), "Near markets and market games," Cowles Foundation Discussion Paper No. 657. Published in part in *Economic Studies Quarterly* **37**, 289–299 and in full as "Clubs, markets, and near-market games" In: Wooders, M. H., ed., *Topics in Game Theory and Mathematical Economics: Essays in Honor of Robert J. Aumann.* Field Institute Communication Volume, American Mathematical Society.

Wooders, M. (1977), "Properties of quasi-cores and quasi-equilibria of coalition economies," Working Paper No. 184, State University of New York at Stony Brook.

Wooders, M. (1983), "The epsilon core of a large replica game," *Journal of Mathematical Economics* **11**, 277–300.

Wooders, M. (1988), "Large games are market games; 1. Large finite games," Discussion Paper No. 8842, CORE, Louvain-La-Neuve, Belgium.

Wooders, M. (1994a), "Equivalence of games and markets," *Econometrica* **62**, 1141–60.

Wooders, M. (1994b), "Large games and economies with effective small groups," J-F. Mertens, and S. Sorin, *Game-Theoretic Methods in General Equilibrum Analysis*, 145–266, eds, Boston and London: Kluwer Academic Publishers Dordrecht.

Wooders, M. (2007), "Competitive markets and games," *Review of Economic Design* (in press).

Wooders, M. (2008), "Small group effectiveness, per capita boundedness and nonemptiness of approximate cores," *Journal of Mathematical Economics*, doi: 10.1016/j.jmateco.2007.06.006.

Wooders, M. H., and W. R. Zame (1987), "Large games; fair and stable outcomes," *Journal of Economic Theory* **42**, 59–63.