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Price taking equilibrium in economies with multiple memberships in clubs and unbounded club sizes☆

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Abstract

We model an economy with clubs (or jurisdictions) where individuals may belong to multiple clubs and where clubs sizes are arbitrary—clubs may be restricted to consist of only one or two persons, or as large as the entire economy, or anything in-between. Notions of price-taking equilibrium and the core, both with communication costs, are introduced. These notions take into account that there is a small communication cost of deviating from a given outcome. We demonstrate that, given communication costs, for all sufficiently large economies the core is nonempty and the set of price-taking equilibrium outcomes is equivalent to the core.

JEL classification: C71; C62; D71; D46; H41; D41

Keywords: Competitive pricing; Clubs; Local public goods; Hedonic coalitions; Edgeworth; Tiebout hypothesis; Core; Decentralization

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1. Motivation

Gains to cooperation by large groups of individuals may be substantial. For example, in economies with public goods, coordination of activities and decreasing per capita costs of providing public goods may yield increasing benefits to ever larger organizations. Consider questions of global pollution, global harmonization of productive activities and memberships in networks. If we wish a model to describe organizations such as the World Trade Organization, the United Nations, the World Environmental Organization, or religions that wish to embrace all people, then a model with bounded club sizes, where clubs become infinitesimal in large economies, is not appropriate. However, much economic activity is carried out within small clubs—marriages, small firms, and swimming pool clubs, for example; thus small clubs should also be permitted. Moreover, a general model should also allow overlapping clubs so that a participant may belong, for example, to a two-person partnership, a dance club, and a world-wide social movement.

Recent literature suggests that whenever almost all gains to collective activities can be realized by relatively small groups of participants then, when there are many participants, diverse economies resemble markets. This includes economies with indivisibilities, nonconvexities, local public goods, and club economies with multiple memberships. In particular, under apparently mild conditions—essential superadditivity, boundedness of feasible average or per capita utilities, and thickness of the total set of consumers [1] —approximate cores are nonempty, approximate cores treat similar people similarly and economies modeled as games with side payments generate market games. In addition, analogues of the Laws of Demand and Supply hold. [2] Except for situations where the "commodities" to be priced are the consumers themselves, models of games with many consumers, however, cannot treat the properties of price-taking economic equilibrium. To obtain richer results on price-taking equilibrium, more detailed economic models are required. Our primary focus is the extent to which increasing returns to club formation in larger and larger economies is consistent with existence of price-taking equilibrium and equivalence of the outcomes of price-taking equilibrium with cooperative outcomes.

In this paper we explore the boundaries of price-taking equilibrium in club economies where clubs may overlap and also are unrestricted in size and composition. Providing most consumers have many close substitutes, if an economy is sufficiently large then an equilibrium with communication costs and possibly some frictions, captured by the presence of an exceptional set of consumers, exists and is in the core. Communication costs are parameterized by a non-negative real number ε and ε can be allowed to tend to zero as the economy becomes large. An interesting feature of our model is that, depending on the affordability of coalition formation costs for potentially improving coalitions, equilibrium may or may not have similar individuals paying similar costs to belong to a club. If communication costs are affordable, in a sense made precise in the paper, then most similar consumers must be treated approximately equally.

Our research grows out of the seminal works of Tiebout [34] and Buchanan [6]. Tiebout conjectured that, in economies with sufficient diversity of communities in terms of their local public good offerings, competitive forces would lead to a "market-like outcome." Buchanan stressed that

[1] Essential superadditivity captures the idea that an option open to a group of consumers is to realize the outcomes achievable by coalitions in a partition of the group; boundedness of feasible average or per capita utilities implies that there is a uniform upper bound on per capita utilities; and thickness of the total consumer set implies that there are many close substitutes for most consumers.

[2] See Wooders [40] for market games and Kovalenkov and Wooders [23,26] for the most recent results treating cores of games with many players and discussion of related literature.

there may be congestion so that optimal club sizes may exist; that is, there may exist some finite population at which all gains to membership size would be exhausted. There are now many models showing that large economies with small optimal groups (communities, firms, clubs, jurisdictions, and so on) generate markets; club membership is simply another commodity. For example, think of movie theaters. Movies can be provided by clubs or by profit maximizing entrepreneurs. It may be that, when the demand is small, they are provided by non-market organizations—foreign film clubs, for example—and price discrimination of some sort may be required to cover costs. Most models, however, rule out large clubs that are few in number, for example, the individual States in the United States. Allowing clubs to be unrestricted in size, with possibly increasing returns to club size, leads to a situation that appears, in essence, to be fundamentally different from a private goods economy or an economy where small groups of consumers can exhaust *all* gains to coalition formation.

Our paper is one of a few allowing the possibility of large clubs, perhaps as large as the entire population, and the first to study price-taking equilibrium in contexts permitting both overlapping clubs and large clubs. Moreover, we allow a compact metric space of consumer types so it does not necessarily hold that there are many exact substitutes for any consumer. Other than some standard conditions such as desirability of private goods, the main assumption of our research is that sufficient wealth, measured in terms of private goods, can compensate for ever-larger club sizes. This permits ever-increasing returns to club size while maintaining boundedness of average payoffs (per capita boundedness). A simple example demonstrating these points is provided.

In the following, Section 2 develops the model and Section 3 introduces the equilibrium concept and states the theorem that an equilibrium state of the economy is in the communication core. Section 4 introduces our main economic assumption, desirability of wealth, and states our existence theorem. Section 5 introduces the concept of Edgeworth equilibrium and states our convergence theorem. Section 6 relates our results to the literature and Section 7 concludes the main body of the paper. Appendix A, containing proofs, follows.

2. A club economy allowing large clubs

2.1. Consumers

Let Ω be a compact set with metric d. An element of Ω, typically denoted by ω, is interpreted as a description of a consumer. Given a finite set N, let α be a function from N to Ω. In interpretation, N will be a set of consumers and $\alpha(i)$ $(i \in N)$ will describe all relevant attributes (or characteristics) of consumer i, including his endowment, preferences, productive abilities, crowding attributes, and so on. An *economy* is a pair (N, α) where $N = \{1, \dots, n\}$ is a *set of consumers* and $\alpha\colon N \longrightarrow \Omega$ is an *attribute function*. For $\omega \in \Omega$, the set of consumers in N with attributes ω is $N \cap \alpha^{-1}(\omega)$ and $|N \cap \alpha^{-1}(\omega)|$ is their number. Given Ω, we denote the set of all economies (N, α) by $F(\Omega)$.

2.2. Clubs and club structures

Let (N, α) be an economy. A *coalition* is simply a nonempty subset of N. A *club* is also a nonempty subset of N but, in interpretation, engages in some club activities. These could be, for example, consumption of a local public good or some shared activities, such as listening to music or swimming in a pool belonging to the club. The members of a coalition may form multiple clubs. We will typically denote a club by S_k and a coalition simply by S.

Let S be a coalition and let $\{S_1, \ldots, S_k, \ldots, S_K\}$ denote a covering of S (with no repetitions) by clubs.[3] Such a covering is called a *club structure* of S. Let $\mathbf{C}(S)$ denote the set of all possible club structures of S. We denote a generic element of $\mathbf{C}(S)$ by $C(S)$.

We assume that there is an upper bound M on the number of clubs to which a consumer can belong. Such a bound is eminently reasonable since membership in clubs, even just to join, takes time and typically other resources. Observe that there are no *a priori* restrictions on heterogeneity of club membership or on club size; for any economy (N, α) the total consumer set N may constitute a club.

Given economy (N, α), coalition $S \subset N$, club structure $C(S) = \{S_1, \ldots, S_k, \ldots, S_K\} \in \mathbf{C}(S)$, and $i \in S$, let

$$C[i; S] = \{S_k | S_k \in C(S) \text{ and } i \in S_k\}$$

denote the set of all clubs in $C(S)$ that contain consumer i. Define

$$\mathbf{C}[i; S] = \bigcup_{\{C(S) \in \mathbf{C}(S)\}} C[i; S],$$

where the union is taken over all club structures $C(S)$ of S. We shall call $\mathbf{C}[i; S]$ the *club consumption set* relative to S for consumer $i \in N$.

Given economy (N, α) and coalition $S \subset N$, we denote the subeconomy with consumer set S, $(S, \alpha_{|S})$, simply by (S, α). Observe that, given S and $i \in S$, any club structure $C(S)$ of S can be embedded (not necessarily uniquely) in a club structure of N, say $C(N)$, so $C[i; S] = C[i; N]$.

2.3. Further specification of the model

Let (N, α) be an economy. Let $e^i = (e^i_1, \ldots, e^i_\ell, \ldots, e^i_L) \in \mathbb{R}^L_{++}$ be the *endowment of consumer* $i \in N$. We assume that there exists a real number $\tau > 0$ satisfying $e^i_\ell > \tau$ for all ℓ and for all $i \in N$. The utility function of consumer $i \in N$ is denoted by $u^i(\cdot, \cdot)$ and maps $\mathbb{R}^L_+ \times \mathbf{C}[i; N]$ into $\mathbb{R}$ where $\mathbb{R}^L_+$ is the private good consumption set for consumer i and $\mathbf{C}[i; N]$ is his club consumption set.

For any consumer $i \in N$ and any club structure $C[i; N] \in \mathbf{C}[i; N]$ of N, the utility function u^i satisfies the usual properties of monotonicity, continuity and convexity. Specifically, given $i \in N$ and club consumption $C[i; N]$, the utility function u^i satisfies:

(a) *Monotonicity*: $u^i(\cdot, C[i; N])$ is an increasing function; that is, if $x < x'$ then $u^i(x, C[i; N]) < u^i(x', C[i; N])$.

(b) *Continuity*: $u^i(\cdot, C[i; N])$ is a continuous function.

(c) *Convexity*: $u^i(\cdot, C[i; N])$ is a quasi-concave function.

(d) *Desirability of endowment*: If $u^i(e^i - \tau\overline{1}, \{i\}) < u^i(x^i, C[i; N])$, where $\overline{1} = (1, \ldots, 1) \in \mathbb{R}^L$ and $\tau > 0$ is the lower bound on endowments of each commodity, then $x_i \gg 0$.[4]

[3] In principle, our techniques allow there to be two (or more) clubs with identical membership offering different activities. To introduce this formally would significantly increase notational complexity. Also, in principle, some particular clubs may be inadmissible—for example, three-person marriages may be ruled out, at least legally. Inadmissible clubs can be accommodated within our framework by simply assigning to them negative utilities so that being a member of such a club would not be individually rational.

[4] This assumption could be weakened but at the cost of more notation and without significant gain in economic understanding.

The following assumption must hold uniformly over all economies. Its purpose is to ensure that, no matter how large the economy and thus, no matter how large the number of distinct clubs, private goods continue to have a strictly positive value to the consumer.

(e) *Private goods are valuable*: Given any attribute $\omega \in \Omega$ and a real number $\varepsilon > 0$ there is a real number $\rho_\varepsilon^\omega > 0$ such that, for any economy (N, α) and any $i \in N$ with $\alpha(i) = \omega$, it holds that

$$u^i(x^i, C[i; N]) + \rho_\varepsilon^\omega < u^i(x^i + \varepsilon\bar{1}, C[i; N]) \quad \text{for any } x^i \in \mathbb{R}_+^L.$$ [5]

(the marginal utility of each consumer for some goods or combinations of goods is uniformly bounded away from zero).

With the exception of (d) and (e), the conditions above are all standard. Condition (d) incorporates the Hammond–Kaneko–Wooders [16] and Kaneko–Wooders [19] condition that the endowment is preferred to any outcome which assigns the consumer zero of any of the indivisible (club) goods.[6] Condition (e) is a nonsatiation assumption, since it dictates that if the amounts of all private goods available to a consumer were increased, the utility of the consumer would increase by an amount bounded below, independently of the size of the economy.

In addition, we require that consumers who are similar in attribute space are near-substitutes in the economy. Note that, in the above, endowments, utilities, and club structures depend on the attribute function α. For ease in reading, we omitted making this explicit. In some cases, however, such as in the following assumptions, noting this dependence is important and we do so.

(f) *Continuity with respect to attributes* 1: Given $\varepsilon > 0$ there exists a real number $\theta > 0$ such that for any set of consumers N and any pair of economies (N, α) and (N, β), if $d(\alpha(i), \beta(i)) \leqslant \theta$ for all $i \in N$ then, for any $x^i \in \mathbb{R}_+^L$,

$$u^{\alpha(i)}(x^i, C[i; N^\alpha]) < u^{\beta(i)}(x^i + \varepsilon\bar{1}, C[i; N^\beta]).$$

Note that $C[i; N^\alpha]$ and $C[i; N^\beta]$ are the same collections of clubs in terms of the names of club members but only similar in terms of the attributes of club members. Condition (f) ensures that consumers who are similar in terms of their attributes have similar utility functions and also that consumers who are "close" in attribute space are "crowding substitutes" for each other. Informally, (f) ensures that if the attributes of consumers in a club were slightly perturbed then a small increase in their private goods allocations would compensate for any loss in utility due to the perturbation.

We also need the condition that those consumers who have similar attributes have similar endowments.

(g) *Continuity with respect to attributes* 2: Given $\varepsilon > 0$ there exists a real number $\theta > 0$ such that for any set of consumers N and any pair of economies (N, α) and (N, β), if $d(\alpha(i), \beta(i)) \leqslant \theta$ for any $i \in N$, then

$$e^{\alpha(i)} \leqslant e^{\beta(i)} + \varepsilon\bar{1}.$$

[5] The assumption plays a role only in the proof of Theorem 3 and is used to ensure that private goods do not become valueless as the economy grows large.

[6] In the literature of private goods exchange economies, related, more restrictive conditions go back to Broome [5] and Mas-Colell [28].

2.4. Production of club activities

Let (N, α) be an economy and let $S_k \subset N$ be a club. The production of the club activity for S_k requires $z_{S_k} \in -\mathbb{R}^L_+$ *inputs of private goods*. Note that in fact it is possible that zero inputs of private goods are required. For a purely "hedonic" club—a club where the membership of the club itself is the only benefit of the club—this may be especially natural.

In addition, we require that clubs of the same size whose members have similar attributes are similar. Informally, the inputs for production of the club good are continuous in the attributes of the club membership.

(h) *Continuity with respect to attributes* 3: For every $\varepsilon > 0$ there exists a real number $\theta > 0$ such that, for every club S_k, and for any attribute functions α and β, if $d(\alpha(i), \beta(i)) \leqslant \theta$ for every consumer $i \in S_k$ it holds that

$$z_{S_k^\alpha} \leqslant z_{S_k^\beta} + \varepsilon\bar{1}.$$

2.5. States of the economy and communication costs

Let (N, α) be an economy, let S be a coalition, and let $C(S)$ be a club structure of S.

Definition. A state *of the economy for S relative to* $C(S)$ is an ordered pair $(x^S, C(S))$, where $x^S = (x^i \in \mathbb{R}^L_+ : i \in S)$ is called an *allocation* (of private goods) for S. The state $(x^S, C(S))$ is *feasible* if

$$\sum_{i \in S} (x^i - e^i) \leqslant \sum_{S_k \in C(S)} z_{S_k}.$$

If a group of consumers is to form an alliance—a coalition or a club—then the consumers must communicate with each other and possibly reallocate goods among themselves. This motivates the introduction of communication costs required to form a coalition. Denote these communication cost for coalition S by $c(\varepsilon, S)$; we assume that

$$c(\varepsilon, S) \stackrel{\text{def}}{=} \varepsilon|S|\bar{1},$$

where ε is a non-negative real number. [7]

Definition. A state of the economy $(x^S, C(S))$ is $c(\varepsilon, S)$-*feasible* if

$$\sum_{i \in S} (x^i - e^i) \leqslant \sum_{S_k \in C(S)} z_{S_k} - \varepsilon|S|\bar{1}.$$

Note that implicitly all coalitions face the same per member communication costs. This can be justified by the assumption that there is a common communication technology. The commonality of communication costs, however, could be relaxed but at the cost of more notation and complexity. In particular, it could be more costly to communicate with some types of consumers than with

[7] Note also that some of each private good is required for communication. Less restrictive forms of this assumption would increase complexity without gain in economic insight. In particular, we could allow possibly different values of ε for every coalition, as long as these were uniformly bounded away from zero. The property that the same amount of each private good is required is without loss of generality since this can be obtained by a normalization.

others; what is important is that there is some lower bound on the per consumer communication costs.

2.6. The core with communication costs

The following concept of the core parameterized by $\varepsilon \in \mathbb{R}_+$, called the $c(\varepsilon)$-core, can be interpreted as either a notion of an approximate core arising from market frictions or as an exact core relative to communication costs. Let $(x^N, C(N))$ be a state of the economy relative to the club structure $C(N)$. A coalition S can $c(\varepsilon)$-*improve upon* the state $(x^N, C(N))$ if there is a club structure $C(S)$ of S and a $c(\varepsilon, S)$-feasible state of the economy $(y^S, C(S))$ for S such that for all consumers $i \in S$ it holds that

$$u^i(y^i, C[i; S]) > u^i(x^i, C[i; N]).$$

A state of the economy $(x^N, C(N))$ is in the $c(\varepsilon)$-*core* of the economy if it is feasible and cannot be $c(\varepsilon)$-improved upon by any coalition S. In spirit, one might think of a state of the economy in the $c(\varepsilon)$-core as "secession proof," as in the line of recent research by Haimanko, Le Breton and Weber [15] and Le Breton and Weber [27] for example; some state of the economy is taken as given and the question asked is whether that state is vulnerable to secession.

It is clear that when $\varepsilon = 0$ the notion of the $c(\varepsilon)$-core coincides with the standard notion of the core.

2.7. The communication core with remainders

Depending on the composition of a population N it may be that some consumers cannot be accommodated in their preferred clubs. If these consumers constitute only a small proportion of the total population, then a solution concept ignoring an exceptional set of consumers may provide reasonable approximations to outcomes of an exact solution. Thus, we weaken our notion of the $c(\varepsilon)$-core to take account of these observations.

Given $\varepsilon_1 \geqslant 0$ and $\varepsilon_0 \geqslant 0$, an ε_1-*remainder* $c(\varepsilon_0)$-*core state of the economy* (N, α) is a state of the economy $(x^N, C(N))$ satisfying the property that for some subset $N^0 \subset N$ with $\frac{|N \backslash N^0|}{|N|} \leqslant \varepsilon_1$, the state $(x^{N^0}, C(N^0))$ is a $c(\varepsilon_0)$-core state of the economy (N^0, α). An ε_1-remainder $c(\varepsilon_0)$-core state of the economy thus simply ignores an exceptional set of consumers.

When the economy is large, the proportion of left-over consumers is small. In this case, we regard the notion of an ε_1-remainder $c(\varepsilon_0)$-core state of the economy (N, α) as a reasonable approximation to a state of the economy in the $c(\varepsilon_0)$-core .

3. Equilibrium with communication costs

In this section we first define communication cost equilibrium, called $c(\varepsilon_0)$-equilibrium, and state our theorem that an equilibrium state of the economy is in the core.

Given $\varepsilon_0 \geqslant 0$, a *price system for private goods* is a vector $p \in \mathbb{R}^L_+$. A *participation price system* is a set

$$\Pi = \{\pi^i(S_k) \in \mathbb{R}\colon S_k \subset N \text{ and } i \in S_k\},$$

stating a participation price—positive, negative, or zero—for each consumer in each club S_k.

A $c(\varepsilon_0)$-*equilibrium* (for an economy with club goods) is an ordered triple $((x^N, C(N)), p, \Pi)$ consisting of a state of the economy $(x^N, C(N))$, a club structure $C(N) = \{J_1, \ldots, J_g, \ldots, J_G\}$, a price system $p \in \mathbb{R}^L_+ \setminus \{0\}$ for private goods, and a participation price system Π, and satisfying:

(i) $\sum_{i\in N}(x^i - e^i) \leqslant \sum_{J_g \in C(N)} z_{J_g}$ (feasibility);

(ii) for each possible club $S_k \subset N$,

$$p \cdot z_{S_k} + \sum_{i\in S_k} \pi^i(S_k) \leqslant 0$$

(no possible club profit);

(iii) for any consumer $i \in N$, any $S \subset N$ with $i \in S$, and any club structure $C(S)$ of S, if

$$u^i(y^i, C[i; S]) > u^i(x^i, C[i; N])$$

then

$$p \cdot y^i + \sum_{S_k \in C[i;S]} \pi^i(S_k) > p \cdot e^i - \varepsilon_0 p \cdot \overline{1}$$

(maximization of utility given costs of coalition formation and budget constraints);

(iv)

$$-\varepsilon_0 \sum_{i\in N} p \cdot \overline{1} \leqslant \sum_g p \cdot z_{J_g} + \sum_g \sum_{i \in J_g} \pi^i(J_g) \leqslant 0$$

(clubs cannot be significantly far, in aggregate, inside their budget sets), and

$$-\varepsilon_0 \sum_{i\in N} p \cdot \overline{1} \leqslant \sum_{i\in N} p \cdot (x^i - e^i) + \sum_{i\in N} \sum_g \pi^i(J_g) \leqslant 0$$

(consumers cannot be significantly far, in aggregate, outside their budget sets [8]).

Our notion of $c(\varepsilon_0)$-equilibrium does not require that the budgets of all consumers balance. This is motivated by communication costs, which affect not only opportunities to change club memberships but also opportunities to purchase different commodity bundles. In interpretation, given a state of the economy, prices, and communication costs, no consumer can improve upon his situation in that state of the economy.

An ε_1-*remainder* $c(\varepsilon_0)$-*equilibrium* is an ordered triple $((x^N, C(N)), p, \Pi)$ with the property that, for some subset of consumers $N^0 \subset N$ with $\frac{|N \setminus N^0|}{|N|} < \varepsilon_1$, $((x^{N^0}, C(N^0)), p, \Pi)$ is a $c(\varepsilon_0)$-equilibrium as defined above.

Our notion of ε_1-remainder $c(\varepsilon_0)$ -equilibrium dictates that all consumers in the economy are competitive or almost competitive except perhaps a small proportion of “left-over” consumers. Concepts of approximate equilibrium or cores involving left-over consumers are common in the literature of game theory and economics. The left-overs may have unsatisfied demands. Such situations may arise from imperfections in markets.

Our first result establishes that an ε_1-remainder $c(\varepsilon_0)$-equilibrium state of the economy is in the ε_1-remainder $c(\varepsilon_0)$-core.

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[8] This condition can be derived from conditions on the model and other parts of the definition of equilibrium. We include the condition, however, to convey the understanding of the concept and indicate that, in equilibrium, most consumers cannot be very far outside of their budget sets.

Theorem 1. *Let (N, α) be an economy. An ε_1-remainder $c(\varepsilon_0)$-equilibrium state of the economy is in the ε_1-remainder $c(\varepsilon_0)$-core.*

The proof of Theorem 1 extends standard techniques arising from Debreu and Scarf [10] and is relegated to Appendix A.

4. Existence of equilibrium

Before stating our existence theorem, we must introduce our main economic assumption that permits ever-increasing gains from larger and larger clubs while, at the same time, allows small clubs. This assumption ensures that there is a private goods bundle sufficiently large to compensate for the benefits of membership in large coalitions.

Desirability of wealth: There is a bundle of private goods $x^* \in \mathbb{R}^L_+$ and an integer η such that, for any economy (N, α) and any consumer $i \in N$, there is a coalition $S \subset N$ with $|S| \leqslant \eta$ and a club structure $C(S)$ satisfying the condition that, for any club structure $C(N)$ of N,

$$u^i(x^i + x^*, C[i; S]) \geqslant u^i(x^i, C[i; N])$$

for any $x^i \in \mathbb{R}^L_+$.[9]

Desirability of wealth ensures that wealth, in terms of private goods, can substitute for membership in clubs with arbitrarily many members. Informally, desirability of wealth dictates that, given any state of any economy, if a consumer were sufficiently wealthy he could provide club goods for himself and just a few others (no more than η) and achieve a preferred outcome. For simplicity x^* is independent of the attribute of the consumer. Also, note that x^* may not be feasible for the individual consumer or even for the coalition S. Desirability of wealth is considerably weaker than bounding club sizes.

Example 1. As a simple example, consider an economy where all consumers are identical and suppose consumers derive utility only from money and from sharing some common activity with other consumers. Suppose that each consumer can belong to at most two clubs ($M = 2$). For any economy (N, α) the utility function of a representative consumer $i \in N$ is given by

$$u^i(\xi, C[i; N]) = \xi - \frac{4}{g(C[i; N])},$$

where $g(C[i; N])$ is the total number of members of the clubs to which consumer i belongs and $\xi \geqslant 0$ is a private good. The endowment of each consumer is equal to some positive number, say $w \in \mathbb{R}_{++}$. Note that, in any feasible state of the economy, $\xi - \frac{4}{g(C[i;N])} \leqslant \xi - \frac{4}{2|N|-1}$. Possible values for η and x^*, in the definition of desirability of wealth, are $\eta = 3$ and $x^* = 1$. To verify this, let S be a subset of N containing consumer i with $|S| = 3$ and let $C(S)$ consist of S and another club S' where $i \in S'$, $\left|S'\right| = 2$. Observe that

$$u^i(\xi + x^*, C[i; S]) = \xi + 1 - \frac{4}{5} \geqslant \xi - \frac{4}{2\,|N| - 1}$$

[9] In a previous version of this paper, we required desirability of wealth only for replication sequences, defined below, but, for presentation of the paper, we strengthened the assumption to its current form.

for any consumer set N. We highlight that desirability of wealth is satisfied and the feasible per capita utility level as a function of the economy size does not achieve a maximum—desirability of wealth does not imply the existence of an optimal club size.

Theorem 2. Assume desirability of wealth. Then, given any $\varepsilon_1, \varepsilon_0 > 0$ there is an integer $n(\varepsilon_1, \varepsilon_0)$ such that, for any economy (N, α), if $|N| > n(\varepsilon_1, \varepsilon_0)$ then there exists a state of the economy $(x^N, C(N))$, a price system for private goods p and a participation price system Π with the property that $((x^N, C(N)), p, \Pi)$ is an ε_1-remainder $c(\varepsilon_0)$-equilibrium.

Theorem 2 is an immediate consequence of Theorems 3 and 4 in the next section. These theorems require the notion of "Edgeworth equilibrium," which we introduce in the next section.

5. Edgeworth equilibrium and convergence of cores

Our existence of equilibrium result (Theorem 2) depends on extending results for replica games (games with a fixed distribution of consumers on attribute space) to a compact metric space of attributes of consumers. Moreover, the definition of Edgeworth equilibrium—informally, a state of an economy whose replicas are in the cores of the corresponding replica economies—requires the notion of replicating a state of the economy. Thus, we next introduce replica economies. The following subsection provides our convergence theorems. We then provide a discussion of the motivation for Edgeworth equilibrium.

5.1. Replication economies

Given $(N, \alpha) \in F(\Omega)$, for each positive integer r we define the rth *replica economy*, denoted by $(N_r, \alpha_r) \in F(\Omega)$, as the economy with set of consumers

$$N_r = \{(i, q) : i = 1, \ldots, N \text{ and } q = 1, \ldots, r\},$$

and attribute function $\alpha_r : N_r \to \Omega$ where $\alpha_r(i, q) = \alpha(i)$, $q = 1, \ldots, r$ (i.e., all consumers (i, q), (i, q') are identical in terms of their attributes). The consumer (i, q) is called the qth *consumer of attribute* i. To replicate a state of the economy, in addition to replicating the set of consumers, we also replicate the club structure and private goods allocation so that a consumer and all his replicas are in clubs with identical profiles (that is, identical numbers of consumers with each attribute) and are allocated identical consumptions.

Let $C(N) = \{J_1, \ldots, J_g, \ldots, J_G\}$ be a club structure of N and let r be a positive integer. Let $C(N_r)$ be a club structure of N_r containing rG clubs and denoted by

$$C(N_r) = \{J_{gq} : q = 1, \ldots, r \text{ and } g = 1, \ldots, G\},$$

where for each $q = 1, \ldots, r$ and each $g = 1, \ldots, G$, if $i \in J_g$ then consumer $(i, q) \in J_{gq}$. (Note that the profile of J_{gq} equals the profile of J_g.) Then $C(N_r)$ is the rth *replication of* $C(N)$.

Let $(x^N, C(N))$ be a state of the economy (N, α). A state of the replicated economy (N_r, α_r), denoted by $(x^{N_r}, C(N_r))$, is the rth *replication of* $(x^N, C(N))$ if

(a) for each $g = 1, \ldots, G$ and each $q = 1, \ldots, r$,

$$z_{J_{gq}} = z_{J_g};$$

(b) for each consumer $i \in N$ all r consumers (i, q), $q = 1, \ldots, r$, in the replicated consumer set N_r are allocated the same private goods bundle as i.

Note that since consumer (i, q) is the qth replica of consumer $i \in N$ and, when $i \in J_g$, it holds that in the replicated state of the economy, (i, q) is in club J_{gq}, it then follows from (b) that replications (i, q) of consumer i are allocated consumption bundles that are identical to the consumption bundle of consumer i.

5.2. Equivalence of Edgeworth states of the economy and equilibrium states

First, we define the notion of $c(\varepsilon_0)$-*Edgeworth state of the economy* (N, α).

Definition. Let (N, α) be an economy and let $\varepsilon_0 \geqslant 0$ be given. A state of the economy $(x^N, C(N))$ is a $c(\varepsilon_0)$-Edgeworth state of the economy (N, α) if each replica $(x^{N_r}, C(N_r))$ of $(x^N, C(N))$ is in the $c(\varepsilon_0)$-core of the corresponding replica of the economy (N, α).

Clearly, if a state of the economy $(x^N, C(N))$ is in the $c(\varepsilon_0)$-core but is not a $c(\varepsilon_0)$-Edgeworth state then, for some replication r of the economy, the corresponding replication $(x^{N_r}, C(N_r))$ of that state is not in the $c(\varepsilon_0)$-core of the replicated economy. Thus, the $c(\varepsilon_0)$-core converges to the set of $c(\varepsilon_0)$-Edgeworth states as the economy is replicated.

We also define the notion of ε_1-*remainder* $c(\varepsilon_0)$-*Edgeworth state of the economy* (N, α). This notion simply admits an exceptional set of players.

Definition. Let (N, α) be an economy and let $\varepsilon_1, \varepsilon_0 \geqslant 0$ be given. A state of the economy $(x^N, C(N))$ is an ε_1-remainder $c(\varepsilon_0)$-Edgeworth state of (N, α) if for some $N^0 \subset N$ with $\left|N \setminus N^0\right| \leqslant \varepsilon_1 |N|$ it holds that $(x^{N^0}, C(N^0))$ is a $c(\varepsilon_0)$-Edgeworth state of the economy (N^0, α).

The essence of the above remark above continues to hold: If a state of the economy is in the ε_1-remainder $c(\varepsilon_0)$-core but is not an ε_1-remainder $c(\varepsilon_0)$-Edgeworth state then, for some replication r of the economy, the corresponding replication of that state is not in the ε_1-remainder $c(\varepsilon_0)$-core.

From Theorem 3 below, for all sufficiently large economies an Edgeworth state of the economy exists. From Theorem 4, such a state of the economy is an equilibrium state.

Theorem 3. *Assume desirability of wealth. Then, given any* $\varepsilon_1, \varepsilon_0 > 0$ *there is an integer* $n(\varepsilon_1, \varepsilon_0)$ *such that: For any economy* (N, α), *if* $|N| > n(\varepsilon_1, \varepsilon_0)$ *then there exists an* ε_1-*remainder* $c(\varepsilon_0)$-*Edgeworth state of the economy* (N, α).

Omitting epsilons for ease of statement: From the definition of Edgeworth states of the economy and Theorem 3, the core of an economy converges, as the economy is replicated, to the set of Edgeworth states. Our next result states that every Edgeworth state of the economy can be decentralized by price-taking equilibrium. Thus, from Theorems 3 and 4, the core converges to the set of price-taking equilibrium states.

Theorem 4. *Let* (N, α) *be an economy and let* $\varepsilon_1, \varepsilon_0 > 0$ *be given. If* $(x^N, C(N))$ *is an* ε_1-*remainder* $c(\varepsilon_0)$-*Edgeworth state of the economy then there exists a price system for private goods* p *and a participation price system* Π *with the property that* $((x^N, C(N)), p, \Pi)$ *is an* ε_1-*remainder* $c(\varepsilon_0)$-*equilibrium*.

Theorem 2 follows immediately from Theorems 3 and 4; for all sufficiently large economies, price-taking equilibrium exists.

5.3. *Equal treatment property of Edgeworth states of the economy*

It is clear from the definition of equilibrium that when communication costs are non-zero, identical consumers may not be treated identically. If communication costs were zero, then any state of the economy in the core for all replications of the economy must have the equal-treatment property; this can be demonstrated using standard arguments as in Debreu and Scarf ([10], Theorem 2) for economies or Wooders ([36], Theorem 3) for NTU games. In this section, we demonstrate that if communication costs are sufficiently small and thus, in a sense to be made precise, affordable, then any state of the economy in the $c(\varepsilon)$-core for all replications of that state must assign similar consumers nearly equal utilities.

Let (N, α) be an economy. A state of the economy $(x^N, C(N))$ has the *equal treatment property* if, for all $i, i' \in \{1, \ldots, N\}$, whenever $\alpha(i) = \alpha(i')$,

$$u^{i'}(x^{i'}, C[i'; N]) = u^i(x^i, C[i; N]).$$

To show that every state of the economy in the core (the core with zero communication costs) has the equal treatment property, it would be necessary to make some assumptions ensuring that core utility payoffs can be achieved by strict subsets of the population.[10] Our results below demonstrate that, if communication costs are not too high, then states of the economy in the $c(\varepsilon_0)$-core for all replications cannot differ substantially from equal treatment states. We first present the results and then a discussion.

Given an economy $(N, \alpha) \in F(\Omega)$ and $\varepsilon_0 \geqslant 0$, let $(x^N, C(N))$ be a state of the economy in the $c(\varepsilon_0)$-core for all replications of the economy. For each attribute $\omega \in \Omega$ that appears in the economy (i.e., each $\omega \in \Omega$ for which $N \cap \alpha^{-1}(\omega) \neq \emptyset$) select two consumers $\underline{i_\omega}$ and $\overline{i_\omega}$ as follows:

$$u^{\underline{i_\omega}}(x^{\underline{i_\omega}}, C[\underline{i_\omega}; N]) = \min_{i \in N \cap \alpha^{-1}(\omega)} u^i(x^i, C[i; N])$$

and

$$u^{\overline{i_\omega}}(x^{\overline{i_\omega}}, C[\overline{i_\omega}; N]) = \max_{i \in N \cap \alpha^{-1}(\omega)} u^i(x^i, C[i; N]).$$

That is, according to the state of the economy $(x^N, C(N))$, consumer $\underline{i_\omega}$ is one of the worst-off consumers with attribute ω and $\overline{i_\omega}$ is one of the best of consumers with attribute ω. We call $\underline{i_\omega}$ *the representative of the poor with attribute* ω and similarly we call $\overline{i_\omega}$ *the representative of the rich with attribute* ω. In such cases, we say that *communication is affordable for* $\overline{i_\omega}$ if

$$x^{\overline{i_\omega}} - \varepsilon_0 |N| \overline{1} \geqslant 0.$$

Proposition 1. *Given an economy* $(N, \alpha) \in F(\Omega)$ *and* $\varepsilon_0 \geqslant 0$, *let* $(x^N, C(N))$ *be a state of the economy in the* $c(\varepsilon_0)$*-core for all replications of the economy. For each attribute* $\omega \in \alpha(N)$, *if*

[10] It is well known that, under arguably mild conditions, if relatively small groups of consumers are nearly effective for the realization of utility levels of states of the economy in the core, then cores and approximate cores must assign most (or all) similar consumers similar utility levels (see, for example, Kovalenkov and Wooders [21]). Nevertheless it is desirable to obtain such results for the class of economies considered in this paper.

communication is affordable for $\overline{i_\omega}$ *then*

$$\begin{aligned}0 &\leqslant u^{\overline{i_\omega}}(x^{\overline{i_\omega}}, C[\overline{i_\omega}; N]) - u^{\underline{i_\omega}}(x^{\underline{i_\omega}}, C[\underline{i_\omega}; N]) \\ &\leqslant u^{\overline{i_\omega}}(x^{\overline{i_\omega}}, C[\overline{i_\omega}; N]) - u^{\overline{i_\omega}}(x^{\overline{i_\omega}} - \varepsilon_0|N|\overline{1}, C[\overline{i_\omega}; N]).\end{aligned}$$

That is, the difference in utilities between the representatives of the poor and of the rich is bounded by the utility loss of the representative of the rich if he had to pay communication costs.

Proof. The idea of the proof is as follows. Suppose the conclusion of the proposition does not hold for an economy (N, α) and a state of the economy $(x^N, C(N))$. For the second replication of $(x^N, C(N))$ we can select a subset of consumers that is identical, in terms of its size and the attributes of its members, to N. Moreover, we can select the subset to have the same consumers as N except that we replace $\overline{i_\omega}$ by the replica of $\underline{i_\omega}$. This subset of consumers can then all be better off than in the replicated state of the economy, which is a contradiction.

More formally, suppose the conclusion of the proposition is false. Then for some given $\varepsilon_0 \geqslant 0$ there exists $\omega \in \Omega$, such that $N \cap \alpha^{-1}(\omega) \neq \emptyset$ and

$$\begin{aligned}0 &\leqslant u^{\overline{i_\omega}}(x^{\overline{i_\omega}}, C[\overline{i_\omega}; N]) - u^{\overline{i_\omega}}(x^{\overline{i_\omega}} - \varepsilon_0|N|\overline{1}, C[\overline{i_\omega}; N]) \\ &< u^{\overline{i_\omega}}(x^{\overline{i_\omega}}, C[\overline{i_\omega}; N]) - u^{\underline{i_\omega}}(x^{\underline{i_\omega}}, C[\underline{i_\omega}; N]).\end{aligned} \tag{1}$$

Since $(x^N, C(N))$ is in the $c(\varepsilon_0)$-core for all replications of the economy, it follows that the twice replication of $(x^N, C(N))$ is in the $c(\varepsilon_0)$-core of the corresponding replica economy. Let S be a coalition consisting of $N \setminus \{\overline{i_\omega}\} \cup \{i'\}$ where i' is the replica of $\underline{i_\omega}$. From the fact that communication is affordable for $\overline{i_\omega}$, the state of the economy $((y^i : i \in S), C(S))$ is $c(\varepsilon_0, S)$-feasible state for S, where $y^{i'} = x^{\overline{i_\omega}} - \varepsilon_0|N|\overline{1}$, $y^i = x^i$ otherwise, and $C(S)$ is equal to $C(N)$ with $\underline{i_\omega}$ replaced by i'. Note that, from (1), this state of the economy makes consumer i' strictly better off. Therefore, from continuity and monotonicity of preferences with respect to private goods, coalition S can $c(\varepsilon_0)$-improve upon the second replication of $(x^N, C(N))$. This is a contradiction. □

From the above proposition, when $\varepsilon_0 = 0$, a state of the economy in $c(\varepsilon_0)$-core for all replications of the economy treats all consumers with the same attribute equally in terms of their utilities. The following proposition demonstrates that, whether or not communication is affordable, if there exists states of the economy in the $c(\varepsilon_0)$-core for all replications, then there exists such states with the equal-treatment property.

Proposition 2. *Given an economy* $(N, \alpha) \in F(\Omega)$ *and* $\varepsilon_0 \geqslant 0$, *let* $(x^N, C(N))$ *be a state of the economy in the* $c(\varepsilon_0)$*-core for all replications of the economy. Then there exists an equal treatment state of the economy* $(y^N, C(N))$ *in the* $c(\varepsilon_0)$*-core for all replications*; *that is, for any two consumers* $i, j \in N$ *with* $\alpha(i) = \alpha(j)$ *it holds that*

$$u^i(y^i, C[i; N]) = u^j(y^j, C[j; N]).$$

Moreover, if $u^i(x^i, C(N)) > u^i(y^i, C(N))$ the allocation y^i can be chosen as a fraction of the allocation x^i.

Proof. Suppose for some $\omega \in \Omega$ and some consumer $i \in N \cap \alpha^{-1}(\omega)$ it holds that $u^i(x^i, C[i; N]) > u^{\underline{i_\omega}}(x^{\underline{i_\omega}}, C[\underline{i_\omega}; N])$. Since $(x^N, C(N))$ is individually rational (that is, taking into account communication costs, no consumer can do better using his own resources in a club consisting of himself alone) it follows that

$$\begin{aligned} u^i(x^i, C[i; N]) &> u^{\underline{i_\omega}}(x^{\underline{i_\omega}}, C[\underline{i_\omega}; N]) \\ &\geqslant u^{\underline{i_\omega}}(e^{\underline{i_\omega}} - \varepsilon_0\overline{1}, \{\underline{i_\omega}\}) = u^i(e^i - \varepsilon_0\overline{1}, \{i\}), \end{aligned}$$

where the final equality results from the fact that $\underline{i_\omega}$ and i have the same attribute. From desirability of endowments it follows that $x^i > 0$. From continuity of utility functions and the Mean Value Theorem there exists $0 < \lambda_i \leqslant 1$ such that $u^i(\lambda_i x^i, C[i; N]) = u^{\underline{i_\omega}}(x^{\underline{i_\omega}}, C[\underline{i_\omega}; N])$. We consider a state of the economy $(y^N, C(N))$ satisfying $y^i = \lambda_i x^i$ if $u^i(x^i, C[i; N]) > u^{\underline{i_\omega}}(x^{\underline{i_\omega}}, C[\underline{i_\omega}; N])$ and $y^i = x^i$ otherwise. It is obvious that the state of the economy $(y^N, C(N))$ is feasible and satisfies the equal treatment property. Moreover, one can easily show that $(y^N, C(N))$ is in the $c(\varepsilon_0)$-core of the economy for all replications of the economy. (If not, eventually a coalition consisting of the worst-off consumers with each attribute and their replicas could improve.) □

Remark. We note that the aggregate deviation of the allocations received by the better-off consumers cannot exceed the communication costs. More formally, continuing from the above proof, for each consumer $i \in N$ with attribute ω and for whom $u^i(x^i, C[i; N]) > u^{\underline{i_\omega}}(x^{\underline{i_\omega}}, C[\underline{i_\omega}; N])$ define $\mathbf{\Delta}_i = (1 - \lambda_i)x^i$; otherwise define $\Delta_i = 0$. Define $\mathbf{\Delta} = \sum_i \Delta_i$. It cannot hold that $\mathbf{\Delta} > c(\varepsilon_0, N)$. If it were the case $\mathbf{\Delta} > c(\varepsilon_0, N)$ then, along similar lines as the proof of the first proposition, one can obtain a contradiction. In the case of one-private-good, it must hold that $\mathbf{\Delta} \leqslant c(\varepsilon_0, N)$.

Our first proposition demonstrates that, within the context of our model, given $\varepsilon_0 > 0$, if a state of the economy in the $c(\varepsilon_0)$-core for all replications and the best-off consumer with a given attribute could feasibly cover communication costs, then all consumers with the same attribute must be treated nearly equally. This allows us to place a bound on the differences between the utilities of the best-off and worst-off consumers with that attribute. Since ε_0 can be made arbitrarily small, the bound can be made "small" and approximate equal treatment holds for all consumers with the given attribute.

We also demonstrate that, if any consumers are treated better than the worst-off consumers of each attribute by a state of the economy in the $c(\varepsilon_0)$-core for all replications, then a state of the economy where the private goods allocations of the better-off consumers are reduced until equal treatment is satisfied is also in the $c(\varepsilon_0)$-core for all replications. This indicates that utilities cannot differ "substantially" from equal treatment.

From our assumptions on preferences, it follows that any state of the economy in the $c(\varepsilon_0)$-core must assign each consumer a strictly positive amount of each private good so, for sufficiently small ε_0, the communication costs would be affordable for every consumer i. Thus, given our assumptions, affordability of the communication costs is not restrictive. The main idea, however, which could result from a number of different assumptions on endowments, preferences, and communication costs, is that the cost of forming coalitions bounds the extent of inequality of states of the economy in the $c(\varepsilon_0)$-core for all replications.

Turning to an ε_1-remainder $c(\varepsilon_0)$-Edgeworth state of the economy for $\varepsilon_1, \varepsilon_0 > 0$, the propositions above will apply to a subset of consumers $N^0 \subset N$ for whom the restriction of the ε_1-remainder $c(\varepsilon_0)$-Edgeworth state to the members of that subset is in the $c(\varepsilon_0)$-core for all replications of the economy. The remainders, $N \setminus N^0$, may be treated vastly unequally.

5.4. Motivation for Edgeworth equilibrium

Edgeworth [11] conjectured that if the set of consumers of an economy were replicated, then the contract curve (the core in allocation space) would shrink to the competitive equilibrium. Debreu and Scarf [10] gave a rigorous formulation of Edgeworth's conjecture and demonstrated that if the set of consumers in an economy is replicated, the only allocations in the core of the replicated economy for all replications are competitive equilibrium allocations; that is, if an allocation remains in the core for all replications then there exists a price system that, together with the allocation, constitutes a competitive equilibrium. The first step in Debreu and Scarf's approach to proving Edgeworth's conjecture, their Theorem 2, was to demonstrate that (with strictly quasi-concave preferences) when an economy is replicated, all replications of a consumer must receive the same consumption bundle in any state of the economy in the core. This result enables Debreu and Scarf to consider the core in a space of fixed dimension, independent of the number of replications of the economy. The second step is to decentralize allocations or states of the economy in the core.

A new approach to showing convergence of cores of economies to competitive equilibrium outcomes is developed in Aliprantis, Brown and Burkinshaw [1], who introduced the concept of the "Edgeworth equilibrium" in the context of economies with infinite dimensional commodity spaces. An Edgeworth equilibrium is defined as a state of the economy with the property that each replication of that state is in the core of the corresponding replication economy. Aliprantis, Burkinshaw and Brown show that every Edgeworth equilibrium is an equilibrium. [11] Treating Edgeworth equilibrium avoids the necessity of a result such as Debreu and Scarf's Theorem 2. [12] Also, note that using the concept of Edgeworth equilibrium allows the separation of the question of existence of states of the economy in the core for all replications from the question of decentralizing prices for such states of the economy. In the infinite dimensional commodity space case and the case treated in this paper, it is advantageous to break down the proof into two steps. Since each step has its own set of difficulties, it is convenient to take one step at a time.

In this paper, we use the notion of an approximate Edgeworth equilibrium, defined as a state of the economy with the property that all replications of that state are in approximate cores of the corresponding replica economies. We encounter new problems, however, that require new approaches. First, we cannot embed the core in a fixed dimensional finite space. In our framework, new commodities—in particular, new possible clubs—emerge when the set of consumers grows large. Thus, the dimensionality of the space of possible club memberships goes to infinity as the population grows large. Second, although we require quasi-concavity of utility functions over private commodities, we do not make such an assumption for club memberships as we believe this would be too restrictive (even if we took account of the indivisibility of consumers). In addition, the usual problems of existence of equilibrium in an economy with clubs or local public goods are present; except under special assumptions, the core (without communication costs) may be empty.

[11] The notion of an Edgeworth equilibrium (for private goods economies) has become well known and appears in a number of papers, including, for example, Florenzano [13], Boyd and McKenzie [4], Predtetchinski [29], and Allouch and Predtetchinski [3], among others. A similar concept appears in Wooders [37] where, for a substantially more restrictive model than that of this paper, it is demonstrated that if all replications of a state of a local public goods economy are in approximate cores of corresponding replicated economies, then that state is a Tiebout equilibrium.

[12] Nevertheless, since Allliprantis, Burkinshaw, and Brown (and subsequent papers treating Edgeworth equilibrium in infinite dimensional exchange economies) assume quasi-concavity, it is apparent that every Edgeworth equilibrium state of the economy satisfies the equal treatment property in utility; that is, consumers with identical endowments and preferences achieve the same utilities from an Edgeworth equilibrium state.

Our assumptions, most notably desirability of wealth, and our approach allow us to both circumvent the dimensionality issue and the equal treatment issue and separate the question of states of the economy in the $c(\varepsilon_0)$-core for all replications of an economy from the question of existence of equilibrium prices. Also, we are able to demonstrate an approximate equal treatment property under the same assumptions and an additional assumption on the size of coalition formation costs. This assumption, which dictates that a potentially improving coalition can afford the communication costs, illustrates also that large costs of coalition formation hinder fairness or equal treatment of similar consumers. Desirability of wealth also allows us to place an upper bound on the possible increases in utilities due to the possibility of larger and larger clubs and enables us to demonstrate nonemptiness of cores with communication costs.

6. Further relationships to the literature

The literature on economies with clubs or local public goods, in particular the inspiring contributions of Tiebout [33] and Buchanan [6], relating to this paper is discussed at some length in survey papers by Kovalenkov and Wooders [25] and Conley and Smith [8] and in a condensed form in Wooders [42]. These papers also discuss the works of Wooders [34,35], which, although they treat very special cases relative to our model herein, initiated much research leading to the current paper. Thus, we discuss here only the most salient aspects of some related papers.

6.1. Multiple memberships in clubs

To the best of our knowledge, the first paper to allow consumers to be members of multiple jurisdictions or clubs is Shubik and Wooders [32], which demonstrated nonemptiness of approximate cores of large economies with quasi-linear preferences and convergence of approximate cores to equal treatment cores. Except for the feature of quasi-linear utility functions, the framework of Shubik and Wooders was very general, allowing coalition production with multiple private goods and other nonconvexities and indivisibilities. Most recently, Kovalenkov and Wooders [23] demonstrated conditions under which large finite games and economies with clubs and possibly multiple memberships in clubs have nonempty approximate cores. (The general game theoretic results of Wooders [36,40] and subsequent papers, including Kovalenkov and Wooders [22,23], all apply to games derived from economies with clubs, with or without multiple memberships. It is only necessary that games derived from the economies satisfy the conditions of the papers. Most notably, the economies must satisfy the condition that feasibility allows the set of consumers to partition itself into self-sufficient groups/clubs/firms and so on—that is, the economies are essentially superadditive.) Ellickson et al. [12] introduced a model of an economy with multiple memberships and obtained approximate versions of existence of equilibrium and equivalence of the core and the set of equilibrium outcomes. Their model is more restrictive than the prior model of Shubik and Wooders [32] and those of Kovalenkov and Wooders in the sense that Ellickson et al. [12] allow only a bounded number of distinct sorts of clubs; thus clubs become negligible as the economy grows large. One interpretation of the Ellickson et al.'s approach is that the space of clubs becomes analogous to a finite dimensional space of private goods. In the one-private-good case, the restrictions of Ellickson et al. [12] transform the economy into an essentially private goods economy with indivisibilities and a consistency condition on club memberships that yield an appropriate feasibility condition.

6.2. Unbounded club sizes

Since we allow unbounded club sizes and ever-increasing returns to club sizes, prior approaches to price-taking equilibrium in situations with multiple memberships in clubs will not suffice. The fact that the set of possible clubs grows without bound and there may be increasing returns to larger and larger clubs are the motivating features of our approach to equilibrium.

In view of the prior literature on large games and large economies demonstrating nonemptiness of approximate cores with possibly unbounded returns to club size, one might hope for approximate equivalence in large finite economies even with multiple memberships in clubs and with potentially ever-increasing returns to club size. The crucial restriction appears to be that *almost* all gains to collective activities are realized by groups bounded in size; that is, "small groups are effective." Our research demonstrates an asymptotic equivalence when arbitrarily large clubs and ever increasing returns to club size are allowed. Our desirability of wealth assumption ensures that groups bounded in size can realize almost all gains to coalition formation—that is, small groups are effective. To see this, given ε_1 and ε_0, there exists states of the economy in the ε_1-remainder $c(\varepsilon_0)$-core for all replications. If gains to economy size were not nearly exhausted then such a result could not hold; some large enough replication would allow a large coalition to significantly improve.

In the literature on approximate cores of games and economies with collective activities and clubs, there are a number of models permitting ever-increasing gains to coalition and club sizes (cf., Wooders [36]; Kovalenkov and Wooders [21–24]). Moreover, these models permit games derived from economies where consumers may belong to overlapping clubs. In addition, following Shubik and Wooders [32], Kovalenkov and Wooders [23] explicitly allow a consumer to belong to multiple clubs. None of these papers, however, treat price-taking equilibrium. Other papers treating price taking equilibrium require significantly stronger assumptions on gains to club size. [13]

While Wooders [37,38] allows club sizes to be unbounded, to demonstrate existence of states of the economy in $c(\varepsilon)$-core for all replications she requires that there be a "minimum efficient scale"—utility levels that can be realized in an arbitrarily large economy can be realized with clubs bounded in size (Wooders [37], Theorem 3). To demonstrate the existence of equilibrium similar restrictions are made in Wooders [41] (in subsequent research, Gilles and Scotchmer [43] also make similar assumptions). Moreover, the problem of multiple memberships is not treated in any of these papers and the papers are restricted to a finite number of types of players and replication.

6.3. The techniques of our decentralization result

To ensure that the games derived from the economies satisfy per capita boundedness [14] —simply boundedness of the set of equal treatment payoffs—we make an assumption of "desirability of wealth." Informally, this assumption dictates that there is some level of wealth, measured in terms of a bundle of private goods, such that a consumer would prefer that level of wealth and membership in some set of clubs, all bounded in size, to any feasible equal-treatment outcome in

[13] See, for example, Conley and Wooders [9], Ellickson et al. [12], and Wooders ([37,41]). Except for some results in the last two papers, all these papers bound club sizes and require that there is a fixed finite number of different "sorts" of clubs (such as two-person marriage clubs, and firms who can each hire no more than a bounded number of workers of a bounded number of productivities).

[14] See Lemma 2 in Appendix A.

any economy, no matter how large. Loosely, desirability of wealth implies that private goods can compensate for membership in large clubs. At the nub of our proof are results from Wooders [36] showing convergence of equal treatment utility vectors for replica games satisfying per capita boundedness.

A crucial innovation in the current paper is our construction of the commodity space. Part of this innovation is in extending and further developing the Foley [14]—Wooders [37] proof technique of defining "preferred sets of allocations of private goods" for individual consumers (Foley) and for coalitions/clubs (Wooders). Recall that, given a state of the economy that is in the core for all replications of the total consumer set, Debreu and Scarf [10] define the set of preferred net trades of each consumer in the economy and show that the convex hull of union of these sets can be separated from the origin. For an economy with pure public goods, Foley [14] extends the commodity space to make the public good a separate good for each consumer. Wooders [37] further extends the commodity space to make local public goods for each consumer in each possible jurisdiction separate commodities. In this paper, we build on these three approaches. More precisely, we extend the club good space so that each club and its membership is a different commodity for each consumer in the club. Having done so, extensions of the techniques of Debreu and Scarf [10] can be applied. We also introduce a virtual production set. Even though we have no production in the current paper, our virtual production set plays a similar role to the extended production sets in Foley [14] and Wooders [37]. In particular, the feasibility requirements ensuring the club choices are consistent are imposed on the virtual production set.

6.4. *Equal treatment of similar individuals*

We have demonstrated that when communication costs are affordable, then consumers with the same attributes are treated nearly equally by any Edgeworth state of the economy. Since the core converges to the set of Edgeworth states and since, as communication costs become small, communication costs become affordable, it follows that the core converges to the equal treatment core, treating identical consumers identically.

Our equal treatment result has a number of precursors in the literature. For private goods exchange economies we have already noted Debreu and Scarf ([10, Theorem 2]). Another relevant paper for economies with private goods is Hildenbrand and Kirman [18], which demonstrate equal treatment under a broader set of circumstances. For economies with local public goods, the equal treatment property of the core has been demonstrated in a number of papers already noted. None of these models, however, encompass the model of the current paper.

Since cooperative games with many players encompass games derived from economies, the equal treatment property of the core of cooperative games with many players and many close substitutes is also relevant. For such games with sidepayments, approximate cores treat most similar players nearly equally—see, for example, Shubik and Wooders [32]. More recently Kovalenkov and Wooders [21] demonstrate that with "limited side payments" approximate cores treat all similar players similarly. The results of the current paper demonstrate conditions under which economies with clubs, possibly with many members, generate games that satisfy per capita boundedness and essential superadditivity.

6.5. *Anonymous pricing*

As emphasized in Conley and Wooders [9] in the club context (and by others in other contexts) to be called "competitive," decentralizing prices that support allocations in the core should depend on observable information and, in particular, should not depend on tastes. For this reason, Conley and Wooders characterize consumers by their "crowding types," their observable attributes or characteristics that may directly affect other members of the same clubs, and define prices to depend only on observable characteristics of consumers, rather than their names. When preferences of consumers for club members depend only on their crowding types, then Conley and Wooders demonstrate that equilibrium outcomes are Pareto-optimal. Ellickson et al. [12], Cole and Prescott [7], and Allouch, Conley and Wooders [2] adopt related approaches to the problem of competitive pricing. In the current paper we have not separated observable crowding attributes of consumers from their unobservable taste attributes.

The crucial aspect of economies with clubs making it possible to define first best Pareto optimal prices depending only on crowding attributes of consumers is highlighted in Conley and Wooders [9]. The authors demonstrate that, in a core state of the economy with sufficiently many consumers, consumers in the same jurisdiction with the same crowding type must make the same implicit monetary contribution to the club good, regardless of their preferences. [15] Thus, a price system with only one price for consumers of the same crowding type, equal to the monetary contribution of a consumer of that type to the club good, can be defined and shown to support Pareto optimal outcomes. In the current paper, since we have asymptotic equal treatment (in utility terms) of identical consumers, a similar approach could be taken, possibly at the cost of more assumptions [16] and equivalence of the set of price taking equilibrium outcomes (with anonymous prices depending only on crowding attributes) and the core could be obtained.

7. Conclusions

The major economic importance of our research is that equilibrium clubs may be unbounded—they do not necessarily become infinitesimal as the economy grows large—and there may be increasing returns to larger and larger clubs. Although they consider an economy of a fixed size, in this respect, our model is similar to, for example, Konishi, Le Breton and Weber [20], where jurisdictions may be "large." [17] This aspect of our modeling is especially relevant for questions of political economy, for example, and to issues of regulation of large firms, such as multinationals. We hope to study these issues, as well as other issues relating to labor markets in economies with large firms/jurisdictions in future research. [18]

Finally, the research in this paper contributes to a body of work demonstrating that whenever almost all gains to collective activities can be realized by relatively small groups of consumers then large economies—with clubs, coalition production, exchange economies, indivisibilities, nonconvexities, and so on—are asymptotically competitive in the sense that if there are many consumers, price-taking equilibrium exists and generates Pareto-optimal outcomes and equivalence of the core and the set of equilibrium outcomes obtains.

[15] This result deepens the equal treatment result of Wooders [36], Theorem 3, which shows equal treatment only in utility terms.

[16] For example, with quasi-linearity of utility functions in one commodity.

[17] That is, the Konishi, Le Breton, and Weber model allows an arbitrary number of jurisdictions; there may be as many jurisdictions as individuals, for example, or there may be only one jurisdiction.

[18] Even in the absence of ever-increasing returns to larger and larger clubs, unbounded club sizes create special problems; see Allouch, Conley and Wooders [2].

Acknowledgments

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Appendix A

A.1. Theorem 1

Theorem 1. *Let (N, α) be an economy. An ε_1-remainder $c(\varepsilon_0)$-equilibrium state of the economy is in the ε_1-remainder $c(\varepsilon_0)$-core.*

Proof. Suppose the Theorem is false. Then there exists at least one ε_1-remainder $c(\varepsilon_0)$-equilibrium (otherwise the result would be vacuously true). Let $((x^{N^0}, C(N^0)), p, \Pi)$ be the associated $c(\varepsilon_0)$-equilibrium with $\frac{|N \backslash N^0|}{|N|} \leqslant \varepsilon_1$. If the state of the economy $(x^{N^0}, C(N^0))$ is not in the $c(\varepsilon_0)$-core, then there is a coalition $S \subset N^0$, a club structure $C(S)$ of S and a state of the economy $(y^S, C(S))$ such that

$$\sum_{i \in S}(y^i - e^i) \leqslant \sum_{S_k \in C(S)} z_{S_k} - \varepsilon_0 |S| \overline{1}$$

and

$$u^i(y^i, C[i; S]) > u^i(x^i, C[i; N^0]) \quad \text{for each } i \in S.$$

From (ii) of the definition of an $c(\varepsilon_0)$-equilibrium it holds that

$$p \cdot z_{S_k} + \sum_{i \in S_k} \pi^i(S_k) \leqslant 0$$

and from utility maximization, it holds that

$$p \cdot y^i + \sum_{S_k \in C[i; S]} \pi^i(S_k) > p \cdot e^i - \varepsilon_0 p \cdot \overline{1}.$$

Summing up these above inequalities, one obtains

$$\sum_{i \in S} p \cdot (y^i - e^i) > \sum_{S_k \in C(S)} p \cdot z_{S_k} - p \cdot \varepsilon_0 |S| \overline{1},$$

which is a contradiction. □

A.2. Theorem 3

Theorem 3. *Assume desirability of wealth. Then, given any $\varepsilon_1, \varepsilon_0 > 0$ there is an integer $n(\varepsilon_1, \varepsilon_0)$ such that for any economy (N, α), if $|N| > n(\varepsilon_1, \varepsilon_0)$ then there exists an ε_1-remainder $c(\varepsilon_0)$-Edgeworth state of the economy (N, α).*

Proof. Preliminaries. Recall that τ denotes a lower bound on consumption sets. Given an economy $(N, \alpha) \in F(\Omega)$ and $\varepsilon_0 \in (0, \tau]$ denote the game induced by the economy by (N, V^{ε_0}), where V^{ε_0} is a correspondence mapping subsets S of N into $\mathbb{R}^N$. For each subset S of N, define $V^{\varepsilon_0}(S)$ as the set of payoff vectors $v \in \mathbb{R}^N$ with the property that for some club structure $C(S)$ of S and some $c(\varepsilon_0, S)$-feasible state with associated state of the economy $(x^S, C(S))$ we have $v^i \leqslant u^i(x^i, C[i; S])$ for each $i \in S$. (Note that coordinates of consumers not in S are unconstrained.)

The proof is now divided into three steps.

Step 1: Suppose the claim of the Theorem is not true. Then there exists $\varepsilon_1, \varepsilon_0 > 0$ and a sequence of economies $(N^\nu, \alpha^\nu)_{\nu=1}^{\infty}$ such that, for every ν, $|N^\nu| > \nu$ and the set of ε_1-remainder $c(\varepsilon_0)$-Edgeworth states of (N^ν, α^ν) is empty.

We first approximate the sequence of economies $(N^\nu, \alpha^\nu)_{\nu=1}^{\infty}$ by a sequence of economies with a finite set of types. From continuity assumptions (f), (g) and (h), there exists a real number $\theta > 0$ such that, for any consumer set N and for any attribute functions $\alpha : N \to \Omega$ and $\beta : N \to \Omega$, if $d(\alpha(i), \beta(i)) \leqslant \theta$ for every $i \in N$ then

$$u^{\alpha(i)}(x^i, C[i; S^\alpha]) < u^{\beta(i)}\left(x^i + \frac{\varepsilon_0}{3}\overline{1}, C[i; S^\beta]\right) \quad \text{for every } x^i \in \mathbb{R}^L_+,$$

$$e^{\beta(i)} - \frac{\varepsilon_0}{18}\overline{1} \leqslant e^{\alpha(i)} \leqslant e^{\beta(i)} + \frac{\varepsilon_0}{18}\overline{1},$$

$$z_{S_k^\beta} - \frac{\varepsilon_0}{18M}\overline{1} \leqslant z_{S_k^\alpha} \leqslant z_{S_k^\beta} + \frac{\varepsilon_0}{18M}\overline{1} \quad \text{for every club } S_k \subset N.$$

Since Ω is a compact set, there is a partition $\Omega_1, \ldots, \Omega_T$ of Ω such that if $\omega, \omega' \in \Omega_i$ then $d(\omega, \omega') \leqslant \theta$. For each $t = 1, \ldots, T$ select an arbitrary $\omega_t \in \Omega_t$. For each (N^ν, α^ν) let (N^ν, γ^ν) be an economy where the attribute function γ^ν satisfies $\gamma^\nu(i) = \omega_t$ whenever $\alpha^\nu(i) \in \Omega_t$. Since the range of each γ^ν is $\{\omega_1, \ldots, \omega_T\}$ we may represent the sequence of sets of consumers in the economies (N^ν, γ^ν) by

$$N^\nu = \{(t, q) : t = 1, \ldots, T \text{ and } q = 1, \ldots, n_t^\nu\},$$

where all consumers (t, q) and (t', q') with $t = t'$ are substitutes for each other—that is, they have the same attribute ω_t. By passing to a subsequence if necessary we can assume, without any loss, that for each $t = 1, \ldots, T$,

$$\frac{|N_t^\nu|}{|N^\nu|} \quad \text{converges to a limit } n_t.$$

We next approximate the sequence of economies (N^ν, γ^ν) by a sequence of replica economies (economies with a fixed proportion of players of each type) using the following lemma from Wooders [39].

Lemma 1. *Let $\{N^\nu\}$ be a sequence of sets of consumers where*

$$N^\nu = \{(t, q) : t = 1, \ldots, T \text{ and } q = 1, \ldots, n_t^\nu\}$$

for some integers n_t^ν, $t = 1, \ldots, T$, Then, given $\varepsilon_1 > 0$ there exists a vector of integers, $\overline{n} = (\overline{n}_1, \ldots, \overline{n}_T)$, such that for all ν sufficiently large, for some $r^\nu \in \mathbb{Z}_+$ and $\ell^\nu \in \mathbb{Z}_+^T$ it holds that:

$$n^\nu = (n_1^\nu, \ldots, n_T^\nu) = r^\nu \overline{n} + \ell^\nu$$

and

$$\frac{\|\ell^v\|}{\|n^v\|} < \frac{\varepsilon_1}{2}$$

where, for any vector n, $\|n\| \stackrel{\text{def}}{=} \Sigma_t \; n_t$.[19]

Step 2: Let $\overline{N} \stackrel{\text{def}}{=} \{(t, q) : t = 1, \ldots, T \text{ and } q = 1, \ldots, \overline{n}_t\}$. Now we consider a new economy $(\overline{N}, \overline{\gamma})$ where the consumers have the same attributes as in the economies $(N^v, \gamma^v)_{v=1}^{\infty}$ except for their endowments; we posit that for every $i \in \overline{N}$,

$$e^{\overline{\gamma}(i)} = e^{\gamma^v(i)} - \tfrac{4}{9}\varepsilon_0 \overline{1}.$$

Our next lemma shows that desirability of wealth implies per capita boundedness for the sequence of replica economies $(\overline{N}_r, \overline{\gamma}_r)_{r=1}^{\infty}$.

Lemma 2. *Assume desirability of wealth. Then there is a positive real number K such that for any replication number r and for any feasible equal treatment state $(x^{\overline{N}_r}, C(\overline{N}_r))$ of the rth economy $(\overline{N}_r, \overline{\gamma}_r)$,*

$$\max_{i \in \overline{N}_r} u^i(x^i, C[i; \overline{N}_r]) < K$$

(*per capita boundedness*).

Proof. First, define $(\overline{N}_r, V_r^0)$ as the game induced by the rth replication of the economy $(\overline{N}, \overline{\gamma})$. To show per-capita boundedness of the derived sequence of games $(\overline{N}_r, V_r^0)_{r=1}^{\infty}$ (and thus of $(\overline{N}_r, V_r^{\varepsilon})_{r=1}^{\infty}$ for any $\varepsilon_0 \in [0, \tau]$), we construct an auxiliary sequence of replica "*-economies" and their induced games, denoted by $(\overline{N}_r, V_r^*)_{r=1}^{\infty}$. We demonstrate that $V_r^0(\overline{N}_r) \subset V_r^*(\overline{N}_r)$ and that $(\overline{N}_r, V_r^*)_{r=1}^{\infty}$ satisfies per-capita boundedness.

We choose an integer r^* such that $|N_{r^*}| \leqslant \eta$, where η satisfies the desirability of wealth condition. For each *-economy, let the utility function of consumer i be defined by

$$u^{*i}(x^i) = \max_{r \leqslant r^*} u^i(x^i + x^*, C[i; \overline{N}_r]).$$

The utility functions u^{*i} are well defined and are quasi-concave. Also, it is clear that given any $(x^{\overline{N}_r}, C[i; \overline{N}_r])$ for any integer r we have

$$u^{*i}(x^i) \geqslant u^i(x^i, C[i; \overline{N}_r])$$

from desirability of wealth.

For each r, the allocation $(x^{\overline{N}_r})$, *is *-feasible* if

$$\sum_{iq \in \overline{N}_r} (x^{iq} - e^{iq}) \leqslant 0.$$

[19] Observe that $|N^v| = \|n^v\|$.

The set of all *-feasible allocations is denoted by A_r^*. Let K be a real number such that

$$K > \max_{i \in \overline{N}} \left\{ \max_{x=(x^1,\ldots,x^{|N|}) \in A_1^*} u^{*i}(x^i) \right\}.$$

From the compactness of A_1^* there is a such real number. Since the sequence of *-economies is a sequence of replicas of a pure exchange economy with quasi-concave utility functions, it follows that K is a per-capita bound for the games $\{(\overline{N}_r, V_r^*)_{r=1}^{\infty}\}$ induced by the *-economies. Obviously, since $V_r^0(\overline{N}_r) \subset V_r^*(\overline{N}_r)$, K is also a per-capita bound for the original sequence of games. □

The fundamental paper showing nonemptiness of approximate cores of games with a fixed distribution of consumer types under the assumption of per-capita boundedness is Wooders [36]; we rely heavily on results in that paper, especially Lemmas 1–7. We also use the notion of a balanced game (as in Scarf [30], and Shapley [31]).[20] We refer the reader to Wooders [36], or Kovalenkov and Wooders [22] for further discussion of the balanced cover of a game and properties related to those used below.

Before stating our two next key lemmas, we provide a brief, partially diagrammatic exposition of the relevant lemmas from Wooders [36] that will be used in the rest of the proof.

For any $\varepsilon \geqslant 0$ let $E^{\varepsilon}(r) \subset \mathbb{R}^T$ represent the set of equal treatment payoff vectors for the game $(\overline{N}_r, V_r^{\varepsilon})$ and let $\widetilde{E}^{\varepsilon}(r)$ represent the set of equal treatment payoff vectors for the balanced cover game derived from the game $(\overline{N}_r, V_r^{\varepsilon})$. From Scarf [30] it follows that the core of the balanced cover game is nonempty. It can be shown that there is an equal treatment payoff vector in the core of the balanced cover of a game; this payoff vector can be represented by an element of $\widetilde{E}^{\varepsilon}(r)$. (Of course this payoff vector may not be feasible for the original game, that is, it is not necessarily contained in $E^{\varepsilon}(r)$.)

From [36], given a fixed distribution of a finite number of player types in a sequence of games, per capita boundedness implies that, for all replication numbers r, $E^{\varepsilon}(r) \cap \mathbb{R}_+^T$ is contained in a compact set. It follows that, for all r, $\widetilde{E}^{\varepsilon}(r) \cap \mathbb{R}_+^T$ is contained in the same compact set. (This follows from [36, Lemma 5].)

From the properties of balanced cover games it holds that $\widetilde{E}^{\varepsilon}(r) \subset \widetilde{E}^{\varepsilon}(r+1)$ for all r [36, Lemma 7]. These two facts imply that the closed limit with respect to Hausdorff distance of the sequence of sets $\{\widetilde{E}^{\varepsilon}(r)\}$ exists; let $L(\varepsilon)$ denote this limit.[21]

These relationships are depicted in Fig. A.1.

From assumption (e), since we are dealing with a finite number of types of consumers, given any $\varepsilon > 0$, there exists a positive real number ρ such that $V_r^{\varepsilon}(S) + \rho\overline{1} \subset V_r^0(S)$ for all coalitions

[20] Given a game (N, V), the payoff sets for the *balanced cover game* $(N, \widetilde{V})$ are defined by

$$\widetilde{V}(S) = V(S) \quad \text{for all } S \subset N, \quad S \neq \emptyset, \quad S \neq N$$

and

$$\widetilde{V}(N) = \cup \bigcap_{S \in B} V(S),$$

where the union is taken over all balanced collections of subsets of N. A collection B of subsets of N is *balanced* if there exists weights w_S for $S \in B$ such that $\sum_{\substack{S \in B \\ i \in S}} w_s = 1$ for each $i \in N$. A balanced cover game is a balanced game, as defined in Scarf [30].

[21] See Hildenbrand ([17, p. 16]), for example, for a definition of the Hausdorff limit.

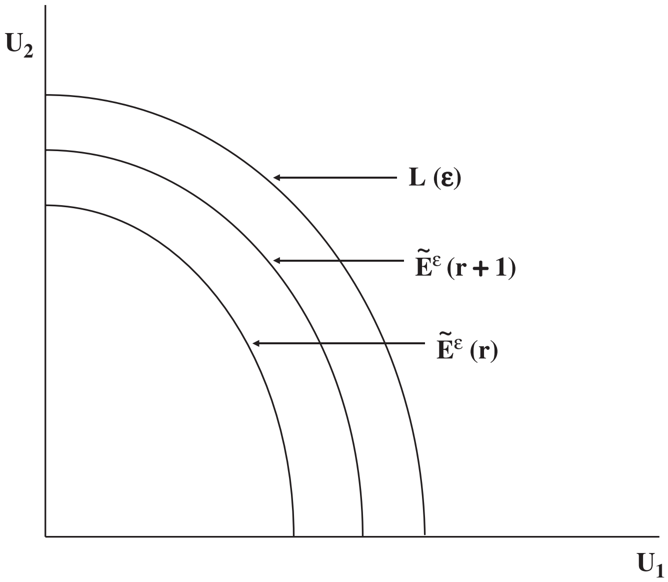

Fig. A.1.

$S \subset \overline{N}_r$ and for all r. From the definition of balanced cover games, it follows that $\widetilde{E}^{\varepsilon}(r) + \rho\overline{1} \subset \widetilde{E}^0(r)$, for all r. Taking the limits one obtains $L(\varepsilon) + \rho\overline{1} \subset L(0)$. Since the sequence of sets $\{\widetilde{E}^0(r)\}$ converges to $L(0)$ we have, for some $\widehat{r}$, $L(\varepsilon) \subset \widetilde{E}^0(\widehat{r})$.

Lemma 5 of [36] connects equal treatment payoffs of balanced cover games to equal treatment payoffs for larger replications of the game itself. Specifically, given the replication number $\widehat{r}$ there is an integer $m_{\widehat{r}}$ such that

$$\widetilde{E}^0(\widehat{r}) \subset E^0(m_{\widehat{r}}\widehat{r}),$$

a consequence of the fact that "minimal balanced collections" have rational weights. (See Shapley [31], where minimal balanced collections were introduced. The definition is also stated in [36, p. 290], and used in the proof of Lemma 5 of that paper.)

From superadditivity, for all positive integers ℓ we have $E^0(r) \subset E^0(\ell r)$ ([36, Lemma 3] applied to the sets of equal treatment payoffs). Thus, one obtains, for all positive integers ℓ,

$$L(\varepsilon) \subset \widetilde{E}^0(\widehat{r}) \subset E^0(\ell m_{\widehat{r}}\widehat{r}),$$

as depicted in Fig. A.2.

In the absence of assumption (e), it may be the case that there exists no positive real number $\rho > 0$ such that $L(\varepsilon) + \rho\overline{1} \subset L(0)$. Roughly, even though with monotonicity we are guaranteed the existence of $\rho_r > 0$ such that $\widetilde{E}^{\varepsilon}(r) + \rho_r\overline{1} \subset \widetilde{E}^0(r)$ in each rth replica of the economy, as the size of the economy becomes large, marginal utilities for private goods may go to zero so that ρ_r could converge to 0 as r becomes large.[22] If $L(\varepsilon) + \rho\overline{1} \subset L(0)$ does not hold, while we could demonstrate nonemptiness of approximate cores in terms of payoffs (or in other words, in utility space), we could not demonstrate nonemptiness of approximate cores in terms of states of the economy. However, when $L(\varepsilon) + \rho\overline{1} \subset L(0)$ for some $\rho > 0$ we have $L(\varepsilon) \subset E^0(\ell r^*)$ for some (finite) replication number r^* and for all positive integers ℓ. Therefore, we can find a state

[22] If there were only a finite set of sorts of clubs and if consumption sets for private goods were compact, then assumption (e) would be a consequence of monotonicity.

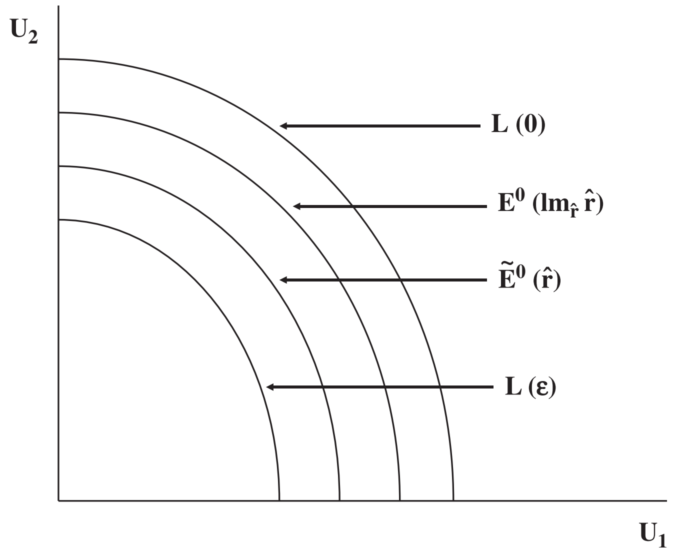

Fig. A.2.

of the economy for the ℓr^* replication with the property that any replication of this state cannot be $c(\varepsilon)$-improved upon by any coalition in the corresponding replicated economy and thus this state is a $c(\varepsilon)$-Edgeworth state of the economy.

Lemma 3. *There exists $u^* \in L(\frac{\varepsilon_0}{9})$ such that $u^* \notin \operatorname{int} V_r^{\varepsilon_0/9}(S)$ for all coalitions $S \subset \overline{N}_r$ and for all r.*

Proof. Let $u^r \in \mathbb{R}^T$ denote an equal treatment payoff vector in the core of the balanced cover game derived from the game $(\overline{N}_r, V_r^{\varepsilon_0/9})$ and let u^* denote the limit of any converging subsequence of the sequence $\{u^r\}$. Note that, given coalition formation costs, u^* cannot be improved upon by any coalition in any game where the consumers have the same attributes as those of consumers in $\overline{N}$, no matter how large the total consumer set; this is because, if some coalition could improve upon u^*, then "on the way to u^*" there would be an economy $\overline{N}_r$ containing that coalition and the coalition could improve upon the equal treatment core payoff vector u^r. □

Lemma 4. *There exist integers $v_0, \overline{r}$ and a state of the economy $(\overline{N}_{\overline{r}}, \overline{\gamma}_{\overline{r}})$, $(x^{\overline{N}_{\overline{r}}}, C(\overline{N}_{\overline{r}}))$, such that*

(i) $\left|N^{v_0} \setminus \overline{N}_{\overline{r}}\right| < \varepsilon_1 \left|N^{v_0}\right|$,

(ii) $(x^{\overline{N}_{\overline{r}}}, C(\overline{N}_{\overline{r}}))$ *is a $c(\frac{\varepsilon_0}{9})$-Edgeworth state of the economy $(\overline{N}_{\overline{r}}, \overline{\gamma}_{\overline{r}})$.*

Proof. As discussed prior to Lemma 3, from (e) and results for replication games, we can select $\widehat{r}$ sufficiently large so that $L(\frac{\varepsilon_0}{9}) \subset \widetilde{E}^0(\widehat{r})$. Moreover, we can select an integer $m_{\widehat{r}}$ such that $\widetilde{E}^0(\widehat{r}) \subset E^0(km_{\widehat{r}}\widehat{r})$ for all positive integers k.

For each v, let $r^v = k_1^v m_{\widehat{r}}\widehat{r} + k_2^v$, where $k_2^v < m_{\widehat{r}}\widehat{r}$. We choose v_0 sufficiently large so that $\frac{1}{\left|k_1^{v_0}\right|} < \frac{\varepsilon_1}{2}$.

Now, consider the economy (N^{v_0}, γ^{v_0}). The profile of the set of consumers in the economy can be written as

$$n^{v_0} = r^{v_0}\overline{n} + \ell^{v_0} = k_1^{v_0} m_{\widehat{r}}\widehat{r}\overline{n} + k_2^{v_0}\overline{n} + \ell^{v_0}.$$

Hence,

$$\frac{\left\|k_2^{v_0}\overline{n}+\ell^{v_0}\right\|}{\|n^{v_0}\|}\leqslant\frac{\|m_{\widehat{r}}\widehat{r}\overline{n}\|}{\left\|k_1^{v_0}m_{\widehat{r}}\widehat{r}\overline{n}\right\|}+\frac{\left\|\ell^{v_0}\right\|}{\|n^{v_0}\|}<\frac{\varepsilon_1}{2}+\frac{\varepsilon_1}{2}=\varepsilon_1.$$

We posit $\overline{r}=k_1^{v_0}m_{\widehat{r}}\widehat{r}$. It is clear from the above inequality that $\left|N^{v_0}\setminus\overline{N}_{\overline{r}}\right|<\varepsilon_1\left|N^{v_0}\right|$. Hence, we have proven (i) of the Lemma.

Moreover, since $u^*\in L(\frac{\varepsilon_0}{9})\subset E^0(\overline{r})$, there is a feasible state, $(x^{\overline{N}_{\overline{r}}},C(\overline{N}_{\overline{r}}))$, of the economy $(\overline{N}_{\overline{r}},\overline{\gamma}_{\overline{r}})$ satisfying:

$$u_t^*\leqslant u^{\overline{\gamma}_{\overline{r}}(i)}(x^{\overline{\gamma}_{\overline{r}}(i)},C[i;\overline{N}_{\overline{r}}]),$$

for all $i\in\overline{N}_{\overline{r}}$ such that $\overline{\gamma}_{\overline{r}}(i)=\omega_t$.

One can deduce from the previous lemma that $(x^{\overline{N}_{\overline{r}}},C(\overline{N}_{\overline{r}}))$ is a $c(\frac{\varepsilon_0}{9})$-Edgeworth state of the economy $(\overline{N}_{\overline{r}},\overline{\gamma}_{\overline{r}})$. □

Step 3: Let us consider the state $(x^{\overline{N}_{\overline{r}}},C(\overline{N}_{\overline{r}}))$ of the economy $(\overline{N}_{\overline{r}},\alpha^{v_0})$ where, for $i\in\overline{N}_{\overline{r}}$, $x^{\alpha^{v_0}(i)}=x^{\overline{\gamma}_{\overline{r}}(i)}+\frac{\varepsilon_0}{3}\overline{1}$. (Observe that we have returned to the original attribute function but with the subset of consumers $\overline{N}_{\overline{r}}$—that is, we have put aside the left-overs.)

Next we check that the state $(x^{\overline{N}_{\overline{r}}},C(\overline{N}_{\overline{r}}))$ is feasible in the economy, $(\overline{N}_{\overline{r}},\alpha^{v_0})$. Since $(x^{\overline{N}_{\overline{r}}},C(\overline{N}_{\overline{r}}))$ is feasible in the economy $(\overline{N}_{\overline{r}},\overline{\gamma}_{\overline{r}})$, it holds that

$$\sum_{i\in\overline{N}_{\overline{r}}}(x^{\overline{\gamma}_{\overline{r}}(i)}-e^{\overline{\gamma}_{\overline{r}}(i)})\leqslant\sum_{J_g\in C(\overline{N}_{\overline{r}})}z_{J_g^{\overline{\gamma}_{\overline{r}}}}.$$

From the continuity assumptions (g) and (h) one has

$$\sum_{i\in\overline{N}_{\overline{r}}}\left(\left(x^{\alpha^{v_0}(i)}-\frac{\varepsilon_0}{3}\overline{1}\right)-\left(e^{\alpha^{v_0}(i)}-\frac{4}{9}\varepsilon_0\overline{1}\right)-\frac{\varepsilon_0}{18}\overline{1}\right)\leqslant\sum_{J_g\in C(\overline{N}_{\overline{r}})}\left(z_{J_g^{\alpha^{v_0}}}+\frac{\varepsilon_0}{18M}\overline{1}\right).$$

Since $C(\overline{N}_{\overline{r}})$ has at most $M\left|\overline{N}_{\overline{r}}\right|$ clubs, one has

$$\sum_{i\in\overline{N}_{\overline{r}}}\left(x^{\alpha^{v_0}(i)}-e^{\alpha^{v_0}(i)}+\frac{\varepsilon_0}{18}\overline{1}\right)\leqslant\left(\sum_{J_g\in C(\overline{N}_{\overline{r}})}z_{J_g^{\alpha^{v_0}}}\right)+\frac{\varepsilon_0}{18}\left|\overline{N}_{\overline{r}}\right|\overline{1};$$

this implies that

$$\sum_{i\in\overline{N}_{\overline{r}}}(x^{\alpha^{v_0}(i)}-e^{\alpha^{v_0}(i)})\leqslant\sum_{J_g\in C(\overline{N}_{\overline{r}})}z_{J_g^{\alpha^{v_0}}}.$$

We claim that the state $(x^{\overline{N}_{\overline{r}}},C(\overline{N}_{\overline{r}}))$ is a $c(\varepsilon_0)$-Edgeworth state of the economy $(\overline{N}_{\overline{r}},\alpha^{v_0})$. Suppose not. Then there exist a replica number m, a coalition $S\subset\overline{N}_{m\overline{r}}$ and a $c(\varepsilon_0,S)$-feasible state of the economy $(\overline{N}_{m\overline{r}},\alpha_m^{v_0})$, $(y^S,C(S))$, such that for all consumers $i\in S$ it holds that

$$u^{\alpha_m^{v_0}(i)}(y^{\alpha_m^{v_0}(i)},C[i;S^{\alpha_m^{v_0}}])>u^{\alpha_m^{v_0}(i)}(x^{\alpha_m^{v_0}(i)},C[i;N^{\alpha_m^{v_0}}]).$$

Define $y^{\overline{\gamma}_{m\overline{r}}(i)} = y^{\alpha_m^{\nu_0}(i)} + \frac{\varepsilon_0}{3}\overline{1}$. From continuity assumption (f) one has

$$\begin{aligned} &u^{\overline{\gamma}_{m\overline{r}}(i)}(y^{\overline{\gamma}_{m\overline{r}}(i)}, C[i; S^{\overline{\gamma}_{m\overline{r}}}]) \\ &\quad = u^{\overline{\gamma}_{m\overline{r}}(i)}\left(y^{\alpha_m^{\nu_0}(i)} + \frac{\varepsilon_0}{3}\overline{1}, C[i; S^{\overline{\gamma}_{m\overline{r}}}]\right) \\ &\quad > u^{\alpha_m^{\nu_0}(i)}(y^{\alpha_m^{\nu_0}(i)}, C[i; S^{\alpha_m^{\nu_0}}]) \end{aligned}$$

and

$$\begin{aligned} &u^{\alpha_m^{\nu_0}(i)}(x^{\alpha_m^{\nu_0}(i)}, C[i; N^{\alpha_m^{\nu_0}}]) \\ &\quad > u^{\overline{\gamma}_{m\overline{r}}(i)}\left(x^{\alpha_m^{\nu_0}(i)} - \frac{\varepsilon_0}{3}\overline{1}, C[i; N^{\overline{\gamma}_{m\overline{r}}}]\right) \\ &\quad = u^{\overline{\gamma}_{m\overline{r}}(i)}(x^{\overline{\gamma}_{m\overline{r}}(i)}, C[i; N^{\overline{\gamma}_{m\overline{r}}}]). \end{aligned}$$

This implies that

$$u^{\overline{\gamma}_{m\overline{r}}(i)}(y^{\overline{\gamma}_{m\overline{r}}(i)}, C[i; S^{\overline{\gamma}_{m\overline{r}}}]) > u^{\overline{\gamma}_{m\overline{r}}(i)}(x^{\overline{\gamma}_{m\overline{r}}(i)}, C[i; N^{\overline{\gamma}_{m\overline{r}}}]).$$

From feasibility we have

$$\sum_{i\in S}(y^{\alpha_m^{\nu_0}(i)} - e^{\alpha_m^{\nu_0}(i)}) \leqslant \sum_{S_k\in C(S)} z_{S_k^{\alpha_m^{\nu_0}}} - \varepsilon_0|S|\overline{1}.$$

Then, it follows from continuity assumptions (g) and (h) that

$$\begin{aligned} &\sum_{i\in S}\left(\left(y^{\overline{\gamma}_{m\overline{r}}(i)} - \frac{\varepsilon_0}{3}\overline{1}\right) - \left(e^{\overline{\gamma}_{m\overline{r}}(i)} + \frac{4}{9}\varepsilon_0\overline{1}\right) - \frac{\varepsilon_0}{18}\overline{1}\right) \\ &\quad \leqslant \sum_{S_k\in C(S)}\left(z_{S_k^{\overline{\gamma}_{m\overline{r}}}} + \frac{\varepsilon_0}{18M}\overline{1}\right) - \varepsilon_0|S|\overline{1}. \end{aligned}$$

Therefore, by rearranging terms one obtains

$$\sum_{i\in S}(y^{\overline{\gamma}_{m\overline{r}}(i)} - e^{\overline{\gamma}_{m\overline{r}}(i)}) \leqslant \sum_{S_k\in C(S)} z_{S_k^{\overline{\gamma}_{m\overline{r}}}} - \frac{\varepsilon_0}{9}|S|\overline{1}.$$

This is a contradiction to the fact that $(x^{\overline{N}_{\overline{r}}}, C(\overline{N}_{\overline{r}}))$ is a $c(\frac{\varepsilon_0}{9})$-Edgeworth state of the economy $(\overline{N}_{\overline{r}}, \overline{\gamma}_{\overline{r}})$. □

A.3. Theorem 4

Theorem 4. *Let (N, α) be an economy and let $\varepsilon_1, \varepsilon_0 > 0$ be given. If $(x^N, C(N))$ is an ε_1-remainder $c(\varepsilon_0)$-Edgeworth state of the economy then there exists a price system for private goods p and participation price system Π with the property that $((x^N, C(N)), p, \Pi)$ is an ε_1-remainder $c(\varepsilon_0)$-equilibrium.*

Proof. As already noted, the proof of the theorem is an extension of proofs of convergence of the core to equilibrium states due to Debreu and Scarf [10] and existence proofs of Foley [14] and Wooders [37]. Without any loss of generality we can assume that there exists a set of consumers $N^0 \subset N$ such that $\frac{|N\backslash N^0|}{|N|} < \varepsilon_1$ and $(x^{N^0}, C(N^0))$ in the $c(\varepsilon_0)$-core of the economy for all replications of the economy. Let $\{S_1, \ldots, S_k, \ldots, S_K\}$ denote the set of all clubs in N^0 and let $C(N^0) = \{J_1, \ldots, J_g, \ldots, J_G\}$.

Preliminaries. We first consider the following space $A = \mathbb{R}^{KN^0}$ where N^0 is the set of consumers and K is the number of all possible clubs in N^0. Let $a = (a^1, \ldots, a^i, \ldots, a^{N^0})$ be a vector where, for each i, $a^i = (a^i_1, \ldots, a^i_k, \ldots, a^i_K)$ and for each $k = 1, \ldots, K$, $a^i_k \in \mathbb{R}$. Let A_i be the set of elements in $\mathbb{R}^K$ defined by

$$A_i = \{a \in \mathbb{R}^{KN^0} : a^{i'}_k = 0 \text{ if } i' \neq i \text{ or if } i \notin S_k\}.$$

For a given $C[i; S] \in \mathbf{C}[i; N]$, we represent $C[i; S]$ in A_i by $a \in A_i$ where a^i_k equals one if S_k belongs to $C[i; S]$ and equals zero otherwise. Observe that we can represent the total consumption $(x^i, C[i; N^0])$ of each consumer $i \in N^0$ by $(x^i, \tilde{a}^i) \in \mathbb{R}^{L+KN^0}$.

We next define a "virtual" production set in the extended commodity space. For each k define $b[k] \in \mathbb{R}^{KN^0}$ as a vector having the properties that

(i) $b[k]^i_{k'} = 0$ if $k \neq k'$ or if $\imath \notin S_k$,
(ii) for any i in S_k, $b[k]^i_k = 1$.

Define the virtual production set Y as the convex cone generated by the $\{(z_{S_k}, b[k]) : k = 1, \ldots, K\}$, where z_{S_k} is the input required to form the club S_k. The set Y is precisely the set of all positive linear combinations of $\{(z_{S_k}, b[k]) : k = 1, \ldots, K\}$.

Step 1: *The sets of preferred allocations* Γ_i. Let $\Gamma_i = \{(y^i - e^i + \varepsilon_0\overline{1}, a^i) \in X^i \times A_i :$ for every club structure $C(S)$ with the property that $C[i; S] = \{S_k | a^i_k = 1\}$, we have $u^i(y^i, C[i; S]) > u^i(x^i, C[i; N^0])\}$.

The set $\Gamma_i \subset \mathbb{R}^{L+KN^0}$ describes the set of net trades of private goods (plus communication costs) and club memberships for consumer i that are strictly preferred to his situation in the given state of the economy $(x^{N^0}, C(N^0))$. It is clear that Γ_i is not necessarily convex.

Step 2: *The preferred set* Γ. Let Γ denote the convex hull of the union of the sets Γ_i, $i = 1, \ldots, N^0$. We now show, in the remainder of Step 2, that

$$\Gamma \cap Y = \emptyset.$$

Suppose, on the contrary, that $(y, a) \in \Gamma \cap Y$. Then, by the definition of Γ, there exist an integer J and $\lambda \in \mathbb{R}^J$ such that $(y, a) = \sum_{j=1}^J \lambda_j (y^j, a^j)$ with $\lambda_j > 0$, $\sum \lambda_j = 1$.

From the definition of Y there exist a $K' \in \{1, \ldots, K\}$ and $\mu \in \mathbb{R}^{K'}_{++}$ such that

$$(y, a) = \sum_{k \in K'} \mu_k (z_{S_k}, b[k]).$$

Let us consider $J[i] = \{j | (y^j, a^j) \in \Gamma_i\}$. Then, it follows from

$$\sum_{j=1}^J \lambda_j (y^j, a^j) = \sum_{k \in K'} \mu_k (z_{S_k}, b[k])$$

that for each $k \in K'$ and each $i \in S_k$ we have

$$\sum_{j \in J[i]} \lambda_j a^{j,i}_k = \mu_k.$$

For given (y^j, a^j) in Γ_i let $\{(\beta^n)\}_n$ be a sequence of real numbers satisfying $\beta^n \geqslant 1$ for each n and $\{(\beta^n)\}_n$ converges to one as n goes to infinity. From continuity of preferences, for all n

sufficiently large, $(\beta^n y^j, a^j) \in \Gamma_i$. We now will show that, from the supposition that $\Gamma \cap Y \neq \emptyset$, for some sufficiently large replication we can form a blocking coalition. We will use the following lemma.

Lemma 5. *There exists a sequence of rational numbers $(\lambda_1^n, \ldots, \lambda_j^n, \ldots, \lambda_J^n)$ converging to $(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_J)$ and having the properties that*

(i) $\lambda_j^n \leqslant \lambda_j$,
(ii) *for any k, and for any $i, i' \in S_k$ we have*

$$\sum_{j \in J[i]} \lambda_j^n a_k^{j,i} = \sum_{j \in J[i']} \lambda_j^n a_k^{j,i'}.$$

Proof. Let us consider the compact set $\Pi_j[0, \lambda_j]$ in $\mathbb{R}^J$. From convexity it follows that, for any $\alpha \in \Pi_j[0, \lambda_j]$, for any k and for any $i, i' \in S_k$ we have

$$\sum_{j \in J[i]} \alpha_j a_k^{j,i} = \sum_{j \in J[i']} \alpha_j a_k^{j,i'}.$$

But, we know that Q^J, where Q is the set of rational numbers, is dense in $\mathbb{R}^J$. Hence, $Q^J \cap \Pi_j[0, \lambda_j]$ is dense in $\Pi_j[0, \lambda_j]$ and therefore we can choose a sequence satisfying (i) and (ii). $\square$

Let us consider the sequence $(\lambda_1^n, \ldots, \lambda_j^n, \ldots, \lambda_J^n)$ defined above, and let us select a positive integer n (that will eventually tend to infinity). For each j define $y^{jn} = \frac{\lambda_j}{\lambda_j^n} y^j$. From the concluding paragraph of the last step, for all n sufficiently large it holds that $(y^{jn}, a^j) \in \Gamma_i$. Let n satisfy the property that $(y^{jn}, a^j) \in \Gamma_i$ for each i. Recall that λ_j^n is a rational number.

Now, let us define $\mu_k^n = \sum_{j \in J[i]} \lambda_j^n a_n^{j,i'}$. Since

$$\sum_{j=1}^{J} \lambda_j^n y^{jn} = \sum_{k \in K'} \mu_k z_k \quad \text{and} \quad \mu_k^n \leqslant \mu_k \ \text{ for all } n$$

and $z_k \in -\mathbb{R}_+^L$, it follows that

$$\sum_{j=1}^{J} \lambda_j^n y^{jn} \leqslant \sum_{k \in K'} \mu_k^n z_k.$$

Let r' be a replication number such that $r'\lambda_j^n$ is an integer for all j. Let $\delta_j = r'\lambda_j^n$ and $\gamma_k = \sum_{j \in J[i]} \delta_j$. It holds that

$$\sum_{j=1}^{J} \delta_j y^{jn} \leqslant \sum_{k \in K'} \gamma_k z_k.$$

Let $\widehat{r}$ be an integer sufficiently large so that there are γ_k copies of the club S_k, for each k, contained in the set $N_{\widehat{r}}^0$, the $\widehat{r}$th replication of N^0, and so that this does not hold for any $r < \widehat{r}$, that is, $\widehat{r}$ is minimal. This implies that there is a state of the economy for a coalition $S \subset N_{\widehat{r}}^0$

that can $c(\varepsilon_0)$-improve upon the initially given state of the economy $(x^{N^0}, C(N^0))$. The state of the economy for S described by the consumption plans (y^{jn}, a^j), for δ_j consumers, for each $j \in J$ is $c(\varepsilon_0)$-feasible and preferred by all members of the replication of the initially given state of the economy $(x^{N^0}, C(N^0))$. Consequently, S can $c(\varepsilon_0)$-improve upon the $\widehat{r}$th replication of $(x^{N^0}, C(N^0))$, which is a contradiction. Therefore $\Gamma \cap Y = \emptyset$.

Step 3: *Prices*. From the Minkowski Separating Hyperplane Theorem, there is a hyperplane with normal $(p, \pi) \neq 0$, where $p \in \mathbb{R}^L$ and $\pi \in \mathbb{R}^{KN^0}$, such that, for some constant C,

$$p \cdot x + \pi \cdot a \geqslant C \quad \text{for all } (x, a) \in \Gamma \quad \text{and}$$

$$p \cdot z + \pi \cdot b \leqslant C \quad \text{for all } (z, b) \in Y.$$

Since Y is a closed convex cone with vertex zero, it follows that we can choose $C = 0$. Then for each (y^i, a^i) such that $u^i(y^i, a^i) > u^i(x^i, \tilde{a}^i)$, it follows that

$$p \cdot (y^i - e^i + \varepsilon_0 \overline{1}) + \sum_{\{k | a^i_k = 1\}} \pi^i(S_k) \geqslant 0,$$

and for each club $S_k \subset N_0$ we have

$$p \cdot z_{S_k} + \sum_{i \in S_k} \pi^i(S_k) \leqslant 0.$$

Recall that $(x^{N^0}, C(N^0))$ is a $c(\varepsilon_0)$-core state of the economy relative to the club structure $C(N^0) = \{J_1, \ldots, J_G\}$ of N^0. From monotonicity it follows that $p \geqslant 0$. Suppose that $p = 0$. Therefore, from the separating hyperplane it follows that for each S_k we have

$$\sum_{i \in S_k} \pi^i(S_k) \leqslant 0,$$

and for each $i \in S_k$ we have $\pi^i(S_k) \geqslant 0$. Thus $\pi^i(S_k) = 0$, for each S_k and each $i \in S_k$, which is a contradiction to the fact that $(p, \pi) \neq 0$.

Since, for each i, $(x^i - e^i + \varepsilon_0 \overline{1}, \tilde{a}^i)$ is in the closure of Γ_i, it holds that

$$p \cdot (x^i - e^i + \varepsilon_0 \overline{1}) + \sum_{\{g | i \in J_g\}} \pi^i(J_g) \geqslant 0.$$

Moreover, for each club J_g we have

$$p \cdot z_{J_g} + \sum_{\{i \in J_g\}} \pi^i(J_g) \leqslant 0.$$

Summing the above inequalities over consumers one obtains

$$p \cdot \sum_{i \in N^0} (x^i - e^i + \varepsilon_0 \overline{1}) + \sum_g \sum_{\{i \in J_g\}} \pi^i(J_g) \geqslant 0,$$

and summing over clubs one obtains

$$\sum_g p \cdot z_{J_g} + \sum_g \sum_{\{i \in J_g\}} \pi^i(J_g) \leqslant 0.$$

Since $p \in \mathbb{R}^L_+ \setminus \{0\}$ and $\sum_{i \in N^0} (x^i - e^i) \leqslant \sum_g z_{J_g}$ it follows that

$$p \cdot \sum_{i \in N^0} (x^i - e^i) \leqslant p \cdot \sum_g z_{J_g}.$$

Then, from the above inequalities it follows that

$$-\varepsilon_0 \sum_{i \in N^0} p \cdot \bar{1} \leqslant \sum_g p \cdot z_{J_g} + \sum_g \sum_{i \in J_g} \pi^i(J_g) \leqslant 0$$

and

$$-\varepsilon_0 \sum_{i \in N^0} p \cdot \bar{1} \leqslant \sum_{i \in N^0} p \cdot (x^i - e^i) + \sum_{i \in N^0} \sum_g \pi^i(J_g) \leqslant 0.$$

Now, we claim that $((x^{N^0}, C(N^0)), p, \Pi)$ is a $c(\varepsilon_0)$-equilibrium. Checking the proof so far, it remains only to show that individual consumers are optimizing, i.e., that the prices p, Π and the state $(x^{N^0}, C(N^0))$ satisfy condition (iii) of the definition of an equilibrium.

Suppose that for some consumer i, and some consumption (y^i, a^i),

$$u^i(y^i, a^i) > u^i(x^i, \tilde{a}^i) \quad \text{and}$$

$$p \cdot (y^i - e^i + \varepsilon_0 \bar{1}) + \sum_{\{k | a^i_k = 1\}} \pi^i(S_k) \leqslant 0.$$

From desirability of endowment, assumption (d), there is a consumption $y^0 \in X^i$ such that

$$p \cdot (y^0 - e^i + \varepsilon_0 \bar{1}) + \sum_{\{k | a^i_k = 1\}} \pi^i(S_k) < 0.$$

It follows that for some $\bar{y}^i$ in the segment $[y^0, y^i]$ one has

$$u^i(\bar{y}^i, a^i) > u^i(x^i, \tilde{a}^i)$$

and

$$p \cdot (\bar{y}^i - e^i + \varepsilon_0 \bar{1}) + \sum_{\{k | a^i_k = 1\}} \pi^i(S_k) < 0,$$

which is a contradiction. □

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